Part B Electromagntism: Problem Sheet 1

[*The material on this problem sheet covers section 1 of the lecture notes.*]

1. In Cartesian coordinates on $\mathbb{R}^3$ we define $\mathbf{r} = (x_1, x_2, x_3)$ and $r = |\mathbf{r}|$, as usual.

   (a) Show that $\nabla \cdot \mathbf{r} = 3$ and that, on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, $\nabla r = \mathbf{r}/r$. Deduce that on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$

   $$\nabla^2 f(r) = f'' + \frac{2}{r} f' = \frac{1}{r}(rf)'' ,$$

   where a prime denotes derivative with respect to r.

   (b) We define $\mathbf{E} \equiv -\nabla(\kappa/r)$ where κ is a constant. Show that on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ we have both $\nabla \wedge \mathbf{E} = \mathbf{0}$ and $\nabla \cdot \mathbf{E} = 0$, while

   $$\int_\Sigma \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 4\pi\kappa ,$$

   holds for any closed surface Σ that bounds a region R containing the origin.
   [*Hint*: First take Σ to be a sphere centred on the origin, then use the divergence theorem to obtain the result for more general Σ.]

2. Consider a spherically symmetric distribution of charge in which the charge density ρ is zero everywhere except for the region $0 \leq a \leq r \leq b$, in which it is a constant ρ_0. Use the symmetry of the problem to argue that the electric field $\mathbf{E}$ must be radial, and hence determine the electric field $\mathbf{E}$ everywhere using Gauss' law.

   Using your answer, deduce that:

   (a) The electric field *outside* a ball of constant charge density and total charge Q is the same as that generated by a point charge Q at the centre of the ball.

   (b) In the limit $a \to b$, $\rho_0 \to \infty$, with $(b-a)\rho_0 = \sigma$ kept fixed, the normal component of the electric field is discontinuous across the resulting spherical shell of charge, jumping by σ/ϵ_0.

3. Consider two point charges $\pm q$, with $-q$ placed at the origin and $+q$ placed at position $\mathbf{d}$.

   (a) Write down the electrostatic potential and electric field for this configuration. Roughly sketch the field lines.

   (b) Define $\mathbf{p} \equiv q\,\mathbf{d}$, and consider the limit in which $q \to \infty$ and $d \equiv |\mathbf{d}| \to 0$, with $\mathbf{p}$ kept fixed. Show that the electrostatic potential in this limit is

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} .$$

   Such a configuration is called an *electric dipole*.

   (c) From the electric dipole potential compute the corresponding electric field. What do you expect the net charge of the configuration to be? Check this result using Gauss' law for a sphere centred on the dipole.

4. Consider an infinite cylinder of radius a and uniform charge density ρ_0. Find the electrostatic potential at all points in space. [*Hint*: Introduce cylindrical polar coordinates, and use the symmetry of the problem.]

5. In cylindrical polar coordinates (s, φ, z) consider the charge density

$$\rho(s, \varphi, z) = \frac{Q}{2\pi s}\, \delta(s-a)\, \delta(z) \ ,$$

where Q is a constant. What does this charge density describe? Compute the electrostatic potential for all points in space with $s = 0$. Can you compare this with anything you have seen before? [Note: we use (s, φ) for polar coordinates in the (x, y)-plane, so $x = s\cos\varphi$, $y = s\sin\varphi$.]

6. Show that the electrostatic energy of a sphere of radius b and constant charge density is

$$W = \frac{3}{5}\, \frac{Q^2}{4\pi\epsilon_0\, b} \ ,$$

where Q is the total charge. [*Hint*: use results from question 2]. The energy of a point charge, obtained by taking $b \to 0$, is thus infinite!

7. (*Optional*) The time-averaged electric potential of a neutral hydrogen atom is given by

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\, \frac{\mathrm{e}^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right) \ ,$$

where $r = |\mathbf{r}|$, and q, α are positive constants. Find the distribution of charge that generates this potential, and interpret your result physically.

Please send comments and corrections to `sparks@maths.ox.ac.uk`.