Part B Electromagntism: Problem Sheet 2

[*The material on this problem sheet covers section 2 of the lecture notes.*]

1. Two infinite, grounded, conducting planes are located at $x = a/2$ and $x = -a/2$, with $a > 0$. A point charge q is placed between the planes at the point (x_0, y_0, z_0), so that $-a/2 < x_0 < a/2$.

   Use the method of images to satisfy the boundary conditions on the potential. Hence write down the Dirichlet Green's function $G_D(\mathbf{r}, \mathbf{r}')$ for the region between the conducting planes.

   [*Hint*: The original charge will have an image for each conducting plane, but those images will also have images ... have you ever looked at two mirrors facing each other?]

2. A conducting grounded sphere of radius a is placed in a uniform electric field E_0 along the z-axis direction.

   (a) Use the method of images to show that the potential outside the sphere is
   $$\phi(x, y, z) \;=\; -E_0\, z + \frac{E_0 a^3\, z}{(x^2+y^2+z^2)^{3/2}} \;.$$
   [*Hint*: A uniform electric field along the z-axis can be generated by starting with two opposite charges $\pm q$, at positions $z = \pm d/2$ on the z-axis. In the limit in which $d \to \infty$, with an appropriate scaling of q one obtains a uniform electric field.]

   (b) Hence determine the induced surface charge density on the sphere.

3. Consider a sphere of radius a, centred on the origin, that is made of an insulator. The potential in the upper hemisphere is kept at $\phi = +V$ while the potential at the lower hemisphere is kept at $\phi = -V$. Use the Dirichlet Green's function for the sphere found in lectures to compute the potential for all points on the z-axis with $z > a$.

4. A hollow cube has conducting walls defined by the six planes $x = y = z = 0$ and $x = y = z = a$. The walls at $z = 0$ and $z = a$ are held at constant potential $\phi = V$. The other four sides are at zero potential.

   Using orthonormal functions find the potential $\phi(x, y, z)$ at any point inside the cube.

5. Use orthonormal functions to find the potential at all points outside a sphere of radius a, where the potential on the sphere is fixed in spherical polar coordinates to be
   $$\phi(a, \theta, \varphi) \;=\; \frac{1}{\sqrt{4\pi}} \sin^2\theta \;.$$

6. The Dirichlet Green's function in the region between two spherical shells of radius a and $b > a$ is given by
   $$G(\mathbf{r}, \mathbf{r}') \;=\; 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\left(r_<^{\ell} - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(r_>^{-(\ell+1)} - \frac{r_>^{\ell}}{b^{2\ell+1}}\right)}{(2\ell+1)\left(1 - \frac{a^{2\ell+1}}{b^{2\ell+1}}\right)} \overline{Y_{\ell,m}(\theta', \varphi')} Y_{\ell,m}(\theta, \varphi) \;,$$
   with notation as in the lecture notes. Use this to compute the potential in this region, if the potential at the boundaries is fixed to be $\phi(r = a, \theta, \varphi) = 0$ and $\phi(r = b, \theta, \varphi) = 1$.

   Could you have solved this problem in a simpler way?

Please send comments and corrections to `sparks@maths.ox.ac.uk`.