Part B Electromagntism: Problem Sheet 3

[*The material on this problem sheet covers sections 3 and 4 of the lecture notes.*]

1. A point particle of mass m and charge q moves with velocity $\dot{\mathbf{r}} = \mathbf{v}$ in a constant magnetic field $\mathbf{B} = B\,\mathbf{e}_3$, where $\mathbf{e}_3$ is the unit vector parallel to the z-axis and B is a constant.

   (a) Write down Newton's second law for the particle, assuming the force on the particle is $q\,\mathbf{v} \wedge \mathbf{B}$, and show that $\mathbf{v} \cdot \mathbf{e}_3$ and $v \equiv |\mathbf{v}|$ are constants of the motion.

   (b) Show that if $\mathbf{v} \cdot \mathbf{e}_3 = 0$ then the particle follows a circular path in a plane of constant z, with speed v and radius a related by $mv = aq\,|\mathbf{B}|$. What is the angular frequency of this circular motion?

   (c) What is the path if $\mathbf{v} \cdot \mathbf{e}_3 \neq 0$?

2. Consider a circular loop of wire of radius a carrying a steady current I. If the loop is placed in the (x, y)-plane $\{z = 0\}$, centred at the origin, show using the integral form of the Biot-Savart law that the magnetic field at the point $(0, 0, z)$ on the axis of symmetry is $\mathbf{B} = (0, 0, B(z))$, where

$$B(z) = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} .$$

3. Consider an infinitely long cylindrical conductor of radius a, its axis of symmetry being the z-axis, carrying a uniform current in the z-direction of constant density J. Using Ampère's law, and assuming that the magnetic field is of the form $\mathbf{B} = B(s)\,\mathbf{e}_\varphi$ in cylindrical polar coordinates (s, φ, z), show that

$$B(s) = \begin{cases} \dfrac{\mu_0 J s}{2} , & s \leq a \\ \dfrac{\mu_0 J a^2}{2s} , & s \geq a . \end{cases}$$

   Hence show that the magnetic field outside the conductor is the same as that (derived in lectures) for an infinitely long wire carrying the same current I.

4. (a) In Cartesian coordinates in *two dimensions* we define $V_x \equiv -\partial G/\partial y$, $V_y \equiv \partial G/\partial x$, where $G = G(x, y)$. Show using *Green's theorem* that

$$\int_C V_x \,\mathrm{d}x + V_y \,\mathrm{d}y = \int_D \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \mathrm{d}x \,\mathrm{d}y ,$$

   where C is a positively oriented simple closed curve bounding a region $D \subset \mathbb{R}^2$. On $\mathbb{R}^2 \setminus \{(0, 0)\}$ we next define

$$G(x, y) \equiv \frac{1}{2\pi} \log \sqrt{x^2 + y^2} .$$

   Show that G satisfies the Laplace equation on $\mathbb{R}^2 \setminus \{(0, 0)\}$, but also

$$\int_C V_x \,\mathrm{d}x + V_y \,\mathrm{d}y = 1 ,$$

   where C is a circle of radius $a > 0$ centred on the origin. Deduce that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) G = \delta(x)\,\delta(y) .$$

(b) Now in *three dimensions* define the magnetostatic potential $\mathbf{A} \equiv (0, 0, -k\,G(x, y))$, with $G(x, y)$ given in part (a). Show that $\mathbf{A}$ satisfies the Lorenz gauge condition, together with the magnetostatic Maxwell equation

$$\nabla^2 \mathbf{A} = -\mu_0 \, \mathbf{J} \ ,$$

where you should determine the current $\mathbf{J}$. Have you seen this problem before? Show that the magnetic field $\mathbf{B} \equiv \nabla \wedge \mathbf{A} = k\,(V_x, V_y, 0)$, and hence show that

$$\int_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = k \ ,$$

for any closed curve C which winds once around the z-axis anticlockwise.

[*Hint*: First take C to be a circle in the plane $\{z = 0\}$ centred at the origin, then use Stokes' theorem to obtain the general result.]

5. Assume $r \neq 0$ throughout this question. Define the vector field $\mathbf{A}$ by

$$\mathbf{A} \equiv \frac{1}{r^3} \mathbf{k} \wedge \mathbf{r} \ ,$$

where $\mathbf{k}$ is a constant vector, and $r = |\mathbf{r}|$, as usual.

(a) Show that $\mathbf{A}$ can also be written as $\mathbf{A} = \nabla\phi \wedge \mathbf{k}$ with $\phi = 1/r$, and hence show that $\nabla \cdot \mathbf{A} = 0$.

(b) Defining the magnetic field $\mathbf{B} \equiv \nabla \wedge \mathbf{A}$, show that $\mathbf{B}$ can be written as

$$\mathbf{B} = \nabla \left[\mathbf{k} \cdot \nabla(1/r) \right] \ .$$

Deduce that $\nabla \cdot \mathbf{B} = 0$, and that $\nabla \wedge \mathbf{B} = \mathbf{0}$.

(c) Show that

$$\mathbf{B} = \frac{3\,\mathbf{k} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{k}}{r^3} \ ,$$

which thus describes the magnetic field of a *magnetic dipole*, with $\mathbf{k} = (\mu_0/4\pi)\,\mathbf{m}$ where $\mathbf{m}$ is the magnetic dipole moment. Hence show that

$$\lim_{r\to\infty} \int_{\Sigma_r} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \ ,$$

where Σ_r is a sphere of radius r centred at the origin. Using these results, explain why the magnetic flux through *any* closed surface Σ is zero. (The magnetic dipole field hence has zero magnetic charge, even at the origin where it is singular.)

6. (*Optional*) Show that an electric dipole at the origin, with dipole moment $\mathbf{p}_0$, exerts a force $\mathbf{F}$ on another electric dipole at position $\mathbf{r}$, with dipole moment $\mathbf{p}$, given by

$$\mathbf{F} = \frac{3}{4\pi\epsilon_0\, r^4} \left[(\mathbf{p} \cdot \hat{\mathbf{r}})\, \mathbf{p}_0 + (\mathbf{p_0} \cdot \hat{\mathbf{r}})\, \mathbf{p} + (\mathbf{p}_0 \cdot \mathbf{p})\, \hat{\mathbf{r}} - 5(\mathbf{p_0} \cdot \hat{\mathbf{r}})(\mathbf{p} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} \right] \ .$$

Here as usual $\hat{\mathbf{r}} \equiv \mathbf{r}/r$, with $r \equiv |\mathbf{r}|$.

(The same formula holds for a pair of magnetic dipoles *e.g.* a pair of iron filings, replacing electric dipole moments by magnetic dipole moments, and ϵ_0 by $1/\mu_0$.)

Please send comments and corrections to `sparks@maths.ox.ac.uk`.