Part B Electromagntism: Problem Sheet 4

[*The material on this problem sheet covers sections 5 and 6 of the lecture notes.*]

1. (a) Starting from the general time-dependent Maxwell equations with sources explain how to introduce the potentials, and what a gauge transformation is. Assuming that a solution ψ to the following wave equation with source

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = F ,$$

exists for any function F which you encounter, show that it is possible by means of a gauge transformation to impose the *Lorenz gauge condition*

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0 ,$$

on the potentials ϕ and $\mathbf{A}$.

(b) The following solution to the time-dependent Maxwell equations

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{r}'\in\mathbb{R}^3} \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' ,$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{\mathbf{r}'\in\mathbb{R}^3} \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' ,$$

was derived in lectures. Show that these potentials satisfy the Lorenz gauge condition. [*Hint*: Use the continuity equation, and an integration by parts.]

2. Show that if $\mathbf{E}$ and $\mathbf{B}$ satisfy the source-free ($\rho = 0$, $\mathbf{J} = \mathbf{0}$) Maxwell equations, then so do

$$\mathbf{E}' = \cos\alpha\,\mathbf{E} - \sin\alpha\, c\,\mathbf{B} , \qquad \mathbf{B}' = \frac{1}{c}\sin\alpha\,\mathbf{E} + \cos\alpha\,\mathbf{B} ,$$

for any constant angle α (this is called a *duality rotation*).

3. An electromagnetic wave has electric field $\mathbf{E}$ defined in Cartesian coordinates (x, y, z) and time t by

$$\mathbf{E} = E\left(\cos\left[\omega\left(t - \frac{z}{c}\right)\right] , \sin\left[\omega\left(t - \frac{z}{c}\right)\right] , 0\right) ,$$

where E, ω and c are constants, c being the speed of light in vacuum.

(a) Find the corresponding magnetic field strength $\mathbf{B}$, and verify that the source-free Maxwell equations are satisfied. What does it mean to say that this electromagnetic wave is *circularly polarized*?

(b) Calculate the energy density $\mathcal{E} \equiv \frac{1}{2}\epsilon_0(|\mathbf{E}|^2 + c^2|\mathbf{B}|^2)$ and Poynting vector $\boldsymbol{\mathcal{P}} \equiv \frac{1}{\mu_0}\mathbf{E}\wedge\mathbf{B}$.

(c) A particle of mass m and charge q moves with velocity $\mathbf{v}$ in this electromagnetic field. What forces act on the particle? If the particle is confined to a plane of constant z, on which it can otherwise move freely, show that the magnetic field does not influence the motion and that the particle can move in circles, whose radius you should find.

4. (a) For a complex electromagnetic plane wave with electric field

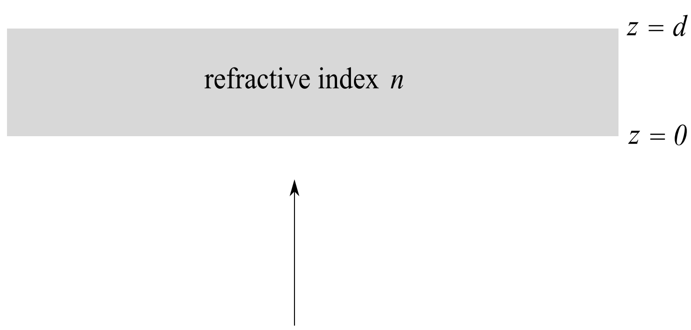
$$\mathbf{E}_{\mathbb{C}} \equiv \mathbf{E}_0 \, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} ,$$

the *time-averaged Poynting vector* is defined as

$$\langle \boldsymbol{\mathcal{P}} \rangle \equiv \frac{1}{T} \int_{t=0}^{T} \boldsymbol{\mathcal{P}} \, \mathrm{d}t ,$$

where T is the period. Show that $\langle \boldsymbol{\mathcal{P}} \rangle = \frac{1}{\mu_0 c} \mathbf{E}_0 \cdot \overline{\mathbf{E}}_0 \, \mathbf{e}$, where $\mathbf{e}$ is a unit vector in the direction of propagation of the wave.

(b) A plane wave moving in a medium with refractive index 1 is incident from below on a layered interface of refractive index n and thickness d, as shown in the Figure.



(i) We take the electric field for $z < 0$ to be

$$\mathbf{E}_{\mathbb{C}} = E \, \mathbf{e}_1 \, \mathrm{e}^{\mathrm{i}(kz-\omega t)} + E' \, \mathbf{e}_1 \, \mathrm{e}^{\mathrm{i}(-kz-\omega t)} ,$$

corresponding to an incident ray and reflected ray, respectively, travelling in the $\pm$ z-axis direction, polarized in the x-axis direction, with angular frequency ω. Compute the corresponding magnetic field.

(ii) Introduce corresponding expressions in the layered interface region $0 < z < d$ and transmitted region $z > d$, where you may assume all waves have the same angular frequency and polarization. What are the boundary conditions at $z = 0$ and $z = d$?

(iii) Assuming the layered interface has magnetic permeability 1, show that the *transmission coefficient* $\mathcal{T}$ above the layer is

$$\mathcal{T} = \frac{4}{4\cos^2\left(\frac{n\omega d}{c}\right) + \left(n + \frac{1}{n}\right)^2 \sin^2\left(\frac{n\omega d}{c}\right)} ,$$

where this is defined as the ratio of the transmitted time-averaged Poynting flux $|\langle \boldsymbol{\mathcal{P}}_{\mathrm{transmitted}} \rangle|$ to the incident time-averaged Poynting flux $|\langle \boldsymbol{\mathcal{P}}_{\mathrm{incident}} \rangle|$.

Please send comments and corrections to `sparks@maths.ox.ac.uk`.