Part A Electromagnetism

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$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \wedge \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$

Contents

0 Introduction

0.1 About these notes

This is a working set of lecture notes for the Part A Electromagnetism course, which is part of the mathematics syllabus at the University of Oxford. I have attempted to put together a concise set of notes that describes the basics of electromagnetic theory to an audience of undergraduate mathematicians. In particular, therefore, many of the important physical applications are not covered. I claim no great originality for the content: for example, section 4.2 closely follows the treatment and notation in Woodhouse's book (see section 0.3), while section 4.4 is based on lecture notes of Prof Paul Tod.

Please send any questions/corrections/comments to `sparks@maths.ox.ac.uk`.

0.2 Preamble

In this course we take a first look at the *classical* theory of electromagnetism. Historically, this begins with Coulomb's inverse square law force between stationary point charges, dating from 1785, and culminates (for us, at least) with Maxwell's formulation of electromagnetism in his 1864 paper, *A Dynamical Theory of the Electromagnetic Field*. It was in the latter paper that the electromagnetic wave equation was first written down, and in which Maxwell first proposed that *"light is an electromagnetic disturbance propagated through the field according to electromagnetic laws"*. Maxwell's equations, which appear on the front of these lecture notes, describe an astonishing number of physical phenomena, over an absolutely enormous range of scales. For example, the electromagnetic force[1] holds the negatively charged electrons in orbit around the positively charged nucleus of an atom. Interactions between atoms and molecules are also electromagnetic, so that chemical forces are really electromagnetic forces. The electromagnetic force is essentially responsible for almost all physical phenomena encountered in day-to-day experience, with the exception of gravity: friction, electricity, electric motors, permanent magnets, electromagnets, lightning, electromagnetic radiation (radiowaves, microwaves, X-rays, *etc*, as well as visible light), . . . it's all electromagnetism.

[1] Quantum mechanics also plays an important role here, which I'm suppressing.

0.3 Bibliography

This is a short, introductory course on electromagnetism, focusing more on the mathematical formalism than on physical applications. Those who wish (and have time!) to learn more about the physics are particularly encouraged to dip into some of the references below, which are in no particular order:

- W. J. Duffin, *Electricity and Magnetism*, McGraw-Hill, fourth edition (2001), chapters 1-4, 7, 8, 13.
- N. M. J. Woodhouse, *Special Relativity*, Springer Undergraduate Mathematics, Springer Verlag (2002), chapters 2, 3.
- R. P. Feynman, R. B. Leighton, M. Sands, *The Feynman Lectures on Physics, Volume 2: Electromagnetism*, Addison-Wesley.
- B. I. Bleaney, B. Bleaney, *Electricity and Magnetism*, OUP, third edition, chapters 1.1-4, 2 (except 2.3), 3.1-2, 4.1-2, 4.4, 5.1, 8.1-4.
- J. D. Jackson, *Classical Electrodynamics*, John Wiley, third edition (1998), chapters 1, 2, 5, 6, 7 (this is more advanced).

0.4 Preliminary comments

As described in the course synopsis, classical electromagnetism is an application of the three-dimensional vector calculus you learned in Moderations: div, grad, curl, and the Stokes and divergence theorems. Since this is only an 8 lecture course, I won't have time to revise this before we begin. However, I've included a brief appendix which summarizes the main definitions and results. Please, *please*, ***please*** take a look at this after the first lecture, and make sure you're happy with everything there.

We'll take a usual, fairly historical, route, by starting with Coulomb's law in electrostatics, and eventually building up to Maxwell's equations on the front page. The disadvantage with this is that you'll begin by learning special cases of Maxwell's equations – having learned one equation, you will later find that more generally there are other terms in it. On the other hand, simply starting with Maxwell's equations and then deriving everything else from them is probably too abstract, and doesn't really give a feel for where the equations have come from. My advice is that at the end of each lecture you should take another look at the equations on the front cover – each time you should find that you understand better what they mean. Starred paragraphs are not examinable, either because they are slightly off-syllabus, or because they are more difficult.

There are 2 problem sheets for the course, for which solution sets are available.

1 Electrostatics

1.1 Point charges and Coulomb's law

It is a fact of nature that elementary particles have a property called *electric charge*. In SI units[2] this is measured in *Coulombs* C, and the electron and proton carry equal and opposite charges $\mp q$, where $q = 1.6022 \times 10^{-19}\,\mathrm{C}$. Atoms consist of electrons orbiting a nucleus of protons and neutrons (with the latter carrying charge 0), and thus all charges in stable matter, made of atoms, arise from these electron and proton charges.

Electrostatics is the study of charges *at rest*. We model space by $\mathbb{R}^3$, or a subset thereof, and represent the position of a stationary point charge q by the position vector $\mathbf{r} \in \mathbb{R}^3$.

Given two such charges, q_1, q_2 at positions $\mathbf{r}_1$, $\mathbf{r}_2$, respectively, the first charge experiences[3] an *electrical force* $\mathbf{F}_1$ due to the second charge given by

$$\mathbf{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_1 - \mathbf{r}_2) \ . \tag{1.1}$$

Note this only makes sense if $\mathbf{r}_1 \neq \mathbf{r}_2$, which we thus assume. The constant ϵ_0 is called the *permittivity of free space*, which in SI units takes the value $\epsilon_0 = 8.8542 \times 10^{-12}\,\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}$. Without loss of generality, we might as well put the second charge at the origin $\mathbf{r}_2 = \mathbf{0}$, denote $\mathbf{r}_1 = \mathbf{r}$, $q_2 = q$, and equivalently rewrite (1.1) as

$$\mathbf{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q}{r^2} \hat{\mathbf{r}} \tag{1.2}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector and $r = |\mathbf{r}|$. This is *Coulomb's law* of electrostatics, and is an experimental fact. Note that:

E1: *The force is proportional to the product of the charges, so that opposite (different sign) charges attract, while like (same sign) charges repel.*

E2: *The force acts in the direction of the vector joining the two charges, and is inversely proportional to the square of the distance of separation.*

The above two statements are equivalent to Coulomb's law.

The final law of electrostatics says what happens when there are more than just two charges:

E3: *Electrostatic forces obey the* Principle of Superposition.

[2]where, for example, distance is measured in metres, time is measured in seconds, force is measured in Newtons.

[3]Of course, if this is the *only* force acting on the first charge, by Newton's second law of motion it will necessarily begin to move, and we are no longer dealing with statics.

That is, if we have N charges q_i at positions $\mathbf{r}_i$, $i = 1, \ldots, N$, then an additional charge q at position $\mathbf{r}$ experiences a force

$$\mathbf{F} \;=\; \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{q q_i}{|\mathbf{r}-\mathbf{r}_i|^3} \left(\mathbf{r}-\mathbf{r}_i\right) \; . \tag{1.3}$$

So, to get the total force on charge q due to all the other charges, we simply *add up* (superpose) the Coulomb force (1.1) from each charge q_i.

1.2 The electric field

The following definition looks completely trivial at first sight, but in fact it's an ingenious shift of viewpoint. Given a particular distribution of charges, as above, we define the *electric field* $\mathbf{E} = \mathbf{E}(\mathbf{r})$ to be the force on a *unit test charge* (*i.e.* $q = 1$) at position $\mathbf{r}$. Here the nomenclature "test charge" indicates that the charge is not regarded as part of the distribution of charges that it is "probing". The force in (1.3) is thus

$$\mathbf{F} \;=\; q\,\mathbf{E} \tag{1.4}$$

where by definition

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|^3} \left(\mathbf{r}-\mathbf{r}_i\right) \tag{1.5}$$

is the electric field produced by the N charges. It is a vector field[4], depending on position $\mathbf{r}$. As we have defined it, the electric field is just a mathematically convenient way of describing the force a unit test charge would feel if placed in some position in a fixed background of charges. In fact, the electric field will turn out to be a fundamental object in electromagnetic theory. Notice that $\mathbf{E}$ also satisfies the Principle of Superposition, and that it is measured in $\mathrm{N\,C}^{-1}$.

1.3 Gauss' law

From (1.2), the electric field of a point charge q at the origin is

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\,\hat{\mathbf{r}} \;=\; \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3} \; . \tag{1.6}$$

Since on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ we have $\nabla\cdot\mathbf{r} = 3$ and $\nabla r = \mathbf{r}/r$, it follows that in this domain

$$\nabla\cdot\mathbf{E} \;=\; \frac{q}{4\pi\epsilon_0}\left(\frac{3}{r^3} - \frac{3\mathbf{r}\cdot\mathbf{r}}{r^5}\right) \;=\; 0 \; . \tag{1.7}$$

[4] defined on $\mathbb{R}^3 \setminus \{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$.

It immediately follows from the divergence theorem A.2 that if R is any region of $\mathbb{R}^3$ that does *not* contain the origin, then

$$\int_{\Sigma=\partial R} \mathbf{E}\cdot\mathbf{n}\,\mathrm{d}S \;=\; \int_R \nabla\cdot\mathbf{E}\,\mathrm{d}V \;=\; 0\;. \tag{1.8}$$

Here $\mathbf{n}$ is the outward unit normal vector to $\Sigma = \partial R$. One often uses the notation $\mathrm{d}\mathbf{S}$ for $\mathbf{n}\,\mathrm{d}S$. The integral $\int_\Sigma \mathbf{E}\cdot\mathrm{d}\mathbf{S}$ is called the *flux* of the electric field $\mathbf{E}$ through Σ.

Consider instead a sphere Σ_a of radius a, centred on the origin. Since the outward unit normal to Σ_a is $\mathbf{n} = \mathbf{r}/r$, from (1.6) we have

$$\int_{\Sigma_a} \mathbf{E}\cdot\mathrm{d}\mathbf{S} \;=\; \frac{q}{4\pi a^2\epsilon_0}\int_{\Sigma_a}\mathrm{d}S \;=\; \frac{q}{\epsilon_0}\;. \tag{1.9}$$

Here we have used the fact that a sphere of radius a has surface area $4\pi a^2$.

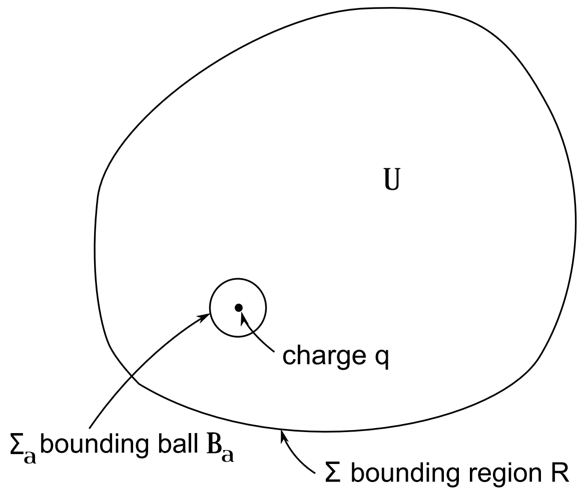


Figure 1: Region R, with boundary Σ, containing a point charge q. We divide R into a small ball B_a of radius a, with boundary sphere Σ_a, and a region U with boundaries Σ and Σ_a.

It is now straightforward to combine these two calculations to prove that for *any* closed surface Σ bounding a region R

$$\int_\Sigma \mathbf{E}\cdot\mathrm{d}\mathbf{S} \;=\; \begin{cases} \dfrac{q}{\epsilon_0} & \text{if the charge is contained in the region } R \text{ bounded by } \Sigma \\ 0 & \text{otherwise}\;. \end{cases} \tag{1.10}$$

To see this, notice that if R contains the charge, we may write the former as the union of a small ball B_a centred on the charge, with boundary sphere Σ_a of radius a, and a region U not containing the charge, with boundaries Σ and Σ_a – see Figure 1. We then sum the results above:

$$\begin{aligned}\int_\Sigma \mathbf{E}\cdot\mathrm{d}\mathbf{S} \;&=\; \int_{\Sigma_a}\mathbf{E}\cdot\mathrm{d}\mathbf{S} + \int_U \nabla\cdot\mathbf{E}\,\mathrm{d}V \qquad \text{(divergence theorem)}\\ &=\; \int_{\Sigma_a}\mathbf{E}\cdot\mathrm{d}\mathbf{S} \;=\; \frac{q}{\epsilon_0}\;. \end{aligned} \tag{1.11}$$

This is easily extended to the case of a distribution of multiple point charges. Note first that the electric field (1.5) has zero divergence on $\mathbb{R}^3 \setminus \{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$ – the ith term in the sum has zero divergence on $\mathbb{R}^3 \setminus \{\mathbf{r}_i\}$ by the same calculation as (1.7). Suppose that a region R contains the charges $q_1, \ldots, q_m$, with $m \leq N$. Then one may similarly write R as the union of m small balls, each containing one charge, and a region containing no charges. For the ith ball the electric flux calculation (1.9) proceeds in exactly the same way for the ith term in the sum in (1.5); on the other hand, the remaining terms in the sum have zero divergence in this ball. Thus essentially the same calculation[5] as that above proves

Gauss' law: *For any closed surface Σ bounding a region R,*

$$\int_{\Sigma} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0} \sum_{i=1}^{m} q_i = \frac{Q}{\epsilon_0} \tag{1.12}$$

where R contains the point charges $q_1, \ldots, q_m$, and Q is the total charge *in R.*

Note that this extension from a single charge to many charges is an application of the Principle of Superposition, ***E3***.

1.4 Charge density and Gauss' law

For many problems it is not convenient to deal with point charges. If the point charges we have been discussing are, say, electrons, then a macroscopic[6] object will consist of an absolutely enormous number of electrons, each with a very tiny charge. We thus introduce the concept of *charge density* $\rho(\mathbf{r})$, which is a function giving the *charge per unit volume.* This means that, by definition, the total charge $Q = Q(R)$ in a region R is

$$Q = \int_R \rho \, \mathrm{d}V \ . \tag{1.13}$$

We shall always assume that the function ρ is sufficiently well-behaved, for example at least continuous (although see the starred section below). For the purposes of physical arguments, we shall often think of the Riemann integral as the limit of a sum (which is what it is). Thus, if $\delta R \subset \mathbb{R}^3$ is a small region centred around a point $\mathbf{r} \in \mathbb{R}^3$ in such a sum, that region contributes a charge $\rho(\mathbf{r})\delta V$, where δV is the volume of δR.

With this definition, the obvious limit of the sum in (1.5), replacing a point charge q' at position $\mathbf{r}'$ by $\rho(\mathbf{r}')\delta V'$, becomes a volume integral

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{r}' \in R} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') \, \mathrm{d}V' \ . \tag{1.14}$$

[5] Make sure that you understand this by writing out the argument carefully.

[6] * A different issue is that, at the microscopic scale, the charge of an electron is not pointlike, but rather is effectively smeared out into a smooth distribution of charge. In fact in quantum mechanics the precise position of an electron cannot be measured *in principle*!

Here $\{\mathbf{r} \in \mathbb{R}^3 \mid \rho(\mathbf{r}) \neq 0\} \subset R$, so that all charge is contained in the (usually bounded) region R. We shall come back to this formula in the next subsection. Similarly, the limit of (1.12) becomes

$$\int_\Sigma \mathbf{E} \cdot \mathrm{d}\mathbf{S} \;=\; \frac{1}{\epsilon_0} \int_R \rho \, \mathrm{d}V \tag{1.15}$$

for any region R with boundary $\partial R = \Sigma$. Using the divergence theorem A.2 we may rewrite this as

$$\int_R \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\epsilon_0} \right) \mathrm{d}V \;=\; 0 \; . \tag{1.16}$$

Since this holds for all R, we conclude from Lemma A.3 another version of *Gauss' law*

$$\nabla \cdot \mathbf{E} \;=\; \frac{\rho}{\epsilon_0} \; . \tag{1.17}$$

We have derived the first of Maxwell's equations, on the front cover, from the three simple laws of electrostatics. In fact this equation holds in general, *i.e.* even when there are magnetic fields and time dependence.

* If (1.17) is to apply for the density ρ of a *point charge*, say at the origin, notice that (1.7) implies that $\rho = 0$ on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, but still the integral of ρ over a neighbourhood of the origin is non-zero. This is odd behaviour for a function. In fact ρ for a point charge is a *Dirac delta function* centred at the point, which is really a "distribution", or a "generalized function", rather than a function. Unfortunately we will not have time to discuss this further here – curious readers may consult page 26 of the book by Jackson, listed in the bibliography. When writing a charge density ρ we shall generally assume it is at least continuous. Thus if one has a continuous charge distribution described by such a ρ, but also point charges are present, then one must add the point charge contribution (1.5) to (1.14) to obtain the total electric field.

1.5 The electrostatic potential and Poisson's equation

Returning to our point charge (1.6) again, note that

$$\mathbf{E} \;=\; -\nabla \phi \tag{1.18}$$

where

$$\phi(\mathbf{r}) \;=\; \frac{q}{4\pi\epsilon_0 r} \; . \tag{1.19}$$

Since the curl of a gradient is identically zero, we have

$$\nabla \wedge \mathbf{E} \;=\; \mathbf{0} \; . \tag{1.20}$$

Equation (1.20) is another of Maxwell's equations from the front cover, albeit only in the special case where $\partial \mathbf{B}/\partial t = 0$ (the magnetic field is time-independent).

Strictly speaking, we have shown this is valid only on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ for a point charge. However, for a bounded continuous charge density $\rho(\mathbf{r})$, with support $\{\mathbf{r} \in \mathbb{R}^3 \mid \rho(\mathbf{r}) \neq 0\} \subset R$ with R a bounded region, we may define

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{r}' \in R} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V' \ . \tag{1.21}$$

Theorem A.4 then implies that ϕ is differentiable with $-\nabla\phi = \mathbf{E}$ given by (1.14). The proof of this goes as follows. Notice first that

$$\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \ , \qquad \mathbf{r} \neq \mathbf{r}' \ . \tag{1.22}$$

Hence[7]

$$\begin{aligned} -\nabla\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_{\mathbf{r}' \in R} \left[-\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] \rho(\mathbf{r}') \, \mathrm{d}V' \\ &= \frac{1}{4\pi\epsilon_0} \int_{\mathbf{r}' \in R} \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}V' = \mathbf{E}(\mathbf{r}) \ . \end{aligned} \tag{1.23}$$

Theorem A.4 also states that if $\rho(\mathbf{r})$ is differentiable then $\nabla\phi = -\mathbf{E}$ is differentiable. Since $\mathbf{E}$ is a gradient its curl is identically zero, and we thus deduce that the Maxwell equation (1.20) is valid on $\mathbb{R}^3$.

* In fact $\nabla \wedge \mathbf{E} = 0$ *implies* that the vector field $\mathbf{E}$ is the gradient of a function (1.18), *provided* the domain of definition has simple enough topology. For example, this is true in $\mathbb{R}^3$ or in an open ball. For other domains, such as $\mathbb{R}^3$ minus a line (say, the z-axis), it is not always possible to write a vector field $\mathbf{E}$ with zero curl as a gradient. A systematic discussion of this is certainly beyond this course. The interested reader can find a proof for an open ball in appendix B of the book by Prof Woodhouse listed in the Bibliography. From now on we always assume that $\mathbf{E}$ is a gradient (1.18), which in particular will be true if we work on the whole of $\mathbb{R}^3$ or an open ball.

The function ϕ is called the *electrostatic potential*. Recall from Moderations that forces $\mathbf{F}$ which are gradients are called *conservative forces*. Since $\mathbf{F} = q\,\mathbf{E}$, we see that the electrostatic force is conservative. The *work done* against the electrostatic force in moving a charge q along a curve C is then the line integral

$$W = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -q \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} = q \int_C \nabla\phi \cdot \mathrm{d}\mathbf{r} = q \left(\phi(\mathbf{r}_1) - \phi(\mathbf{r}_0) \right) \ . \tag{1.24}$$

Here the curve C begins at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$. The work done is of course independent of the choice of curve connecting the two points, because the force is conservative. Notice that one may add a constant to ϕ without changing $\mathbf{E}$. It is only the *difference* in values of ϕ that is physical, and this is called the *voltage*. If we fix some arbitrary point $\mathbf{r}_0$ and choose

[7] One should worry about what happens when $\mathbf{r} \in R$, since $1/|\mathbf{r} - \mathbf{r}'|$ diverges at $\mathbf{r}' = \mathbf{r}$. However, the steps above are nevertheless correct as stated – see the proof of Theorem A.4 in the lecture course "Calculus in Three Dimensions and Applications" for details.

$\phi(\mathbf{r}_0) = 0$, then $\phi(\mathbf{r})$ has the interpretation of work done against the electric field in moving a unit charge from $\mathbf{r}_0$ to $\mathbf{r}$. Note that ϕ in (1.19) is zero "at infinity". From the usual relation between work and energy, ϕ is also the *potential energy* per unit charge.

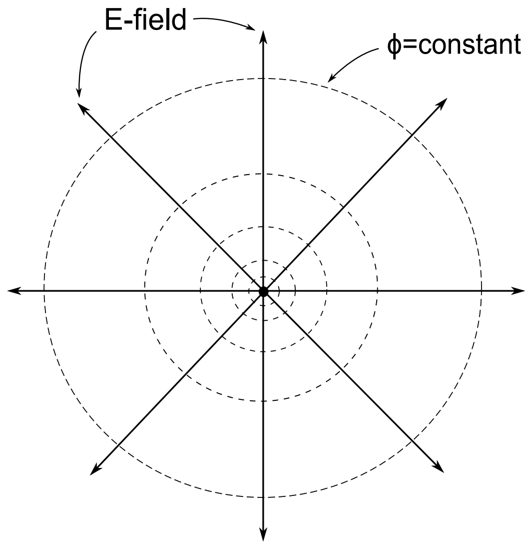


Figure 2: The field lines, which represent the direction of the electric field, and equipotentials around a positive point charge.

The surfaces of constant ϕ are called *equipotentials.* Notice that the electric field is always *normal* to such a surface. For, if $\mathbf{t}$ is a vector at a fixed point $\mathbf{r}$ that is *tangent* to the equipotential for $\phi(\mathbf{r})$, then $\mathbf{t} \cdot \nabla\phi = 0$ at this point. This means that $\nabla\phi = -\mathbf{E}$ is normal to a surface of constant ϕ.

From Gauss' law (1.17) we immediately have (*cf.* (1.21) and Theorem A.4)

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \qquad (1.25)$$

which is called *Poisson's equation.* It reduces to Laplace's equation when $\rho = 0$.

1.6 Boundary conditions and surface charge

We have now reduced electrostatics to a single equation for the electrostatic potential ϕ, (1.25), for a given charge density ρ. The solution for ϕ in terms of ρ is given by (1.21), as in Theorem A.4. Recall here that we are always free to add a constant to ϕ, so (1.21) corresponds to a particular choice of this constant. For a continuous charge density ρ in a bounded region R, (1.21) implies that $\phi = O(1/r)$ as $r \to \infty$. The proof of the asymptotics of ϕ is on Problem Sheet 1.

More interesting is when the distribution of charge is not described by a continuous charge density. We have already encountered point charges. For many problems it is useful to introduce the concepts of *surface charge density* σ on a surface S, say for a charge distribution on a thin metal sheet, and also *line charge density* λ on a curve C, say for a charge distribution in a thin wire. These will be taken to be appropriately well-behaved functions on S and C, representing charge per unit area and charge per unit length, respectively.

In fact the concept of surface charge density doesn't require a thin metal sheet to be useful, for the following reason. An *electrical conductor* is a material where some of the electrons ("conduction electrons") are free to move in the presence of an electric field. In a *static* situation, the electric field *inside* the conducting material must be zero. Why? Because if it weren't, then the conduction electrons in the interior would experience a force, and thus move by Newton's second law. Imagine what happens if we now switch on an external electric field: a conduction electron will move in the opposite direction to the field (because it is negatively charged), until either (a) it gets to the boundary of the material, or (b) the electric field inside the material has relaxed to its equilibrium of zero. This way, one ends up with lots of electrons at, or very near, the surface of the material; their distribution (and the distribution of other immobile charges) throughout the material produces an electric field which precisely *cancels* the external field *inside* the material. Since $\rho = 0$ inside the material by (1.17), the charge must be described by a surface charge density.

Surface and line charge densities of course contribute to the total charge and electric field via surface and line integrals, respectively. For example, a surface S with surface charge density σ gives rise to an electric field

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int_{\mathbf{r}'\in S}\frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,(\mathbf{r}-\mathbf{r}')\,\mathrm{d}S' \;. \tag{1.26}$$

Notice that for $\mathbf{r}$ in $\mathbb{R}^3\setminus S$ and σ smooth the integrand is smooth. However, it turns out that $\mathbf{E}$ is *not continuous* across S! One can prove using (1.26) that the components of $\mathbf{E}$ tangent to S are continuous across S, but that the *normal* component of $\mathbf{E}$ is not. More precisely, if $\mathbf{n}$ is a unit normal vector field to the surface pointing into what we'll call the "+ side", then

$$\mathbf{E}^+\cdot\mathbf{n} \,-\, \mathbf{E}^-\cdot\mathbf{n} \;=\; \frac{\sigma}{\epsilon_0} \tag{1.27}$$

at every point on the surface. A general rigorous proof of this would take a little too long, so we will content ourselves with the following argument, which is the one usually found in most textbooks.

Consider a surface S which has a surface charge density σ. Consider the cylindrical region R on left hand side of Figure 3, of height ε and cross-sectional area δA. Gauss' law gives us

$$\int_{\partial R}\mathbf{E}\cdot\mathrm{d}\mathbf{S} \;=\; \frac{1}{\epsilon_0}\,(\text{total charge in}\,R)\;\;. \tag{1.28}$$

In the limit $\varepsilon \to 0$ the left hand side becomes $(\mathbf{E}^+ \cdot \mathbf{n} - \mathbf{E}^- \cdot \mathbf{n})\delta A$ for small δA, where $\mathbf{E}^\pm$ are the electric fields on the two sides of S and the unit normal $\mathbf{n}$ points into the $+$ side. The right hand side, on the other hand, tends to $\sigma\delta A/\epsilon_0$. Thus there is necessarily a discontinuity in the component of $\mathbf{E}$ normal to S given by (1.27).

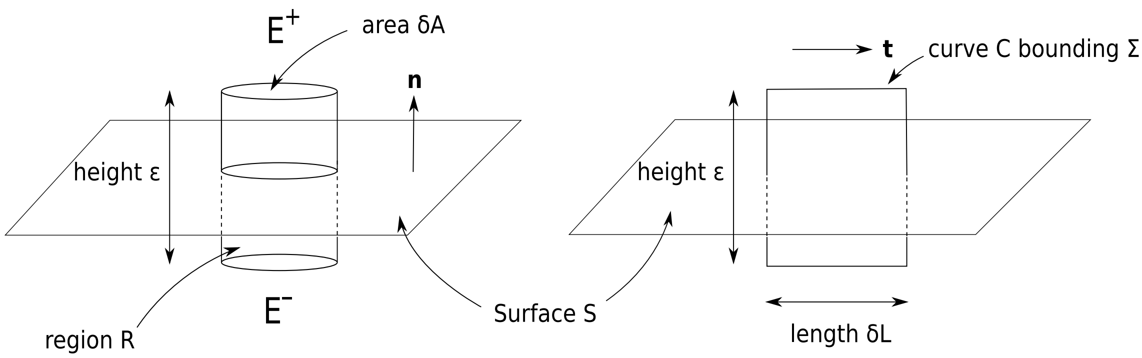


Figure 3: The surface S.

Consider, instead, the rectangular loop C on the right hand side of Figure 3, of height ε and length δL, bounding the rectangular surface Σ. By Stokes' theorem A.1 we have

$$\int_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} \;=\; \int_\Sigma (\nabla \wedge \mathbf{E}) \cdot \mathrm{d}\mathbf{S} \;=\; 0 \tag{1.29}$$

where we have used the electrostatic Maxwell equation (1.20). If $\mathbf{t}$ denotes a unit tangent vector along C on the $+$ side, then in the limit $\varepsilon \to 0$ we obtain

$$\left(\mathbf{E}^+ \cdot \mathbf{t} - \mathbf{E}^- \cdot \mathbf{t}\right) \delta L \;=\; 0 \tag{1.30}$$

for small δL. Thus the components of $\mathbf{E}$ tangent to S are continuous across S

$$\mathbf{E}^+ \cdot \mathbf{t} \;=\; \mathbf{E}^- \cdot \mathbf{t}\ . \tag{1.31}$$

1.7 Electrostatic energy

In this subsection we derive a formula for the *energy* of an electrostatic configuration as an integral of a *local energy density*. We shall return to this subject again in section 3.5.

We begin with a point charge q_1 at $\mathbf{r}_1$. This has potential

$$\phi^{(1)}(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{r} - \mathbf{r}_1|}\ . \tag{1.32}$$

Consider now moving a charge q_2 from infinity to the point $\mathbf{r}_2$. From (1.24), the work done against the electric field in doing this is

$$W_2 \;=\; q_2\,\phi^{(1)}(\mathbf{r}_2) \;=\; \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{|\mathbf{r}_2 - \mathbf{r}_1|}\ . \tag{1.33}$$

Next move in another charge q_3 from infinity to the point $\mathbf{r}_3$. We must now do work against the electric fields of *both* q_1 and q_2. By the Principle of Superposition, this work done is

$$W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_3 q_1}{|\mathbf{r}_3 - \mathbf{r}_1|} + \frac{q_3 q_2}{|\mathbf{r}_3 - \mathbf{r}_2|}\right) . \tag{1.34}$$

The total work done so far is thus $W_2 + W_3$.

Obviously, we may continue this process and inductively deduce that the *total* work done in assembling charges $q_1, \ldots, q_N$ at $\mathbf{r}_1, \ldots, \mathbf{r}_N$ is

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j<i}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{2}\cdot\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j\neq i}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} . \tag{1.35}$$

This is also the *potential energy* of the collection of charges.

We now rewrite (1.35) as

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i\,\phi_i \tag{1.36}$$

where we have defined

$$\phi_i = \frac{1}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} . \tag{1.37}$$

This is simply the electrostatic potential produced by all but the ith charge, evaluated at position $\mathbf{r}_i$. In the usual continuum limit, (1.36) becomes

$$W = \frac{1}{2}\int_R \rho\,\phi\,\mathrm{d}V \tag{1.38}$$

where $\phi(\mathbf{r})$ is given by (1.21). Now, using Gauss' law (1.17) we may write

$$\phi\,\frac{\rho}{\epsilon_0} = \phi\,\nabla\cdot\mathbf{E} = \nabla\cdot(\phi\,\mathbf{E}) - \nabla\phi\cdot\mathbf{E} = \nabla\cdot(\phi\,\mathbf{E}) + \mathbf{E}\cdot\mathbf{E} , \tag{1.39}$$

where in the last step we used (1.18). Inserting this into (1.38), we have

$$W = \frac{\epsilon_0}{2}\left[\int_{\Sigma=\partial R}\phi\,\mathbf{E}\cdot\mathrm{d}\mathbf{S} + \int_R \mathbf{E}\cdot\mathbf{E}\,\mathrm{d}V\right] \tag{1.40}$$

where we have used the divergence theorem on the first term. Taking R to be a very large ball of radius r, enclosing all charge, the surface Σ is a sphere. For boundary conditions[8] $\phi = O(1/r)$ as $r \to \infty$, this surface term is zero in the limit that the ball becomes infinitely large, and we deduce the important formula

$$W = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3}\mathbf{E}\cdot\mathbf{E}\,\mathrm{d}V . \tag{1.41}$$

When this integral exists the configuration is said to have *finite energy.* The formula (1.41) suggests that the energy is stored in a local *energy density*

$$\mathcal{U} = \frac{\epsilon_0}{2}\,\mathbf{E}\cdot\mathbf{E} = \frac{\epsilon_0}{2}\,|\mathbf{E}|^2 . \tag{1.42}$$

[8]See Problem Sheet 1, question 3.

2 Magnetostatics

2.1 Electric currents

So far we have been dealing with stationary charges. In this subsection we consider how to describe charges in motion.

Consider, for example, an electrical conductor. In such a material there are electrons (the "conduction electrons") which are free to move when an external electric field is applied. Although these electrons move around fairly randomly, with typically large velocities, in the presence of a macroscopic electric field there is an induced *average drift velocity* $\mathbf{v} = \mathbf{v}(\mathbf{r})$. This is the average velocity of a particle at position $\mathbf{r}$. In fact, we might as well simply ignore the random motion, and regard the electrons as moving through the material with velocity vector field $\mathbf{v}(\mathbf{r})$.

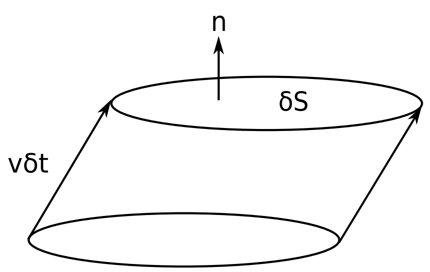


Figure 4: The current flow through a surface element $\delta\Sigma$ of area δS.

Given a distribution of charge with density ρ and velocity vector field $\mathbf{v}$, the *electric current density* $\mathbf{J}$ is defined as

$$\mathbf{J} = \rho\mathbf{v} . \tag{2.1}$$

To interpret this, imagine a small surface $\delta\Sigma$ of area δS at position $\mathbf{r}$, as shown in Figure 4. Recall that we define $\delta\mathbf{S} = \mathbf{n}\,\delta S$, where $\mathbf{n}$ is the unit normal vector to the surface $\delta\Sigma$. The volume of the oblique cylinder in Figure 4 is $\mathbf{v}\,\delta t \cdot \delta\mathbf{S}$, which thus contains the charge $\rho\,\mathbf{v}\,\delta t \cdot \delta\mathbf{S} = \mathbf{J} \cdot \delta\mathbf{S}\,\delta t$. In the time δt this is the total charge passing through $\delta\Sigma$. Thus $\mathbf{J}$ is a vector field in the direction of the flow, and its magnitude is the amount of charge flowing per unit time per unit perpendicular cross-section to the flow.

The *electric current* $I = I(\Sigma)$ through a surface Σ is defined to be

$$I = \int_\Sigma \mathbf{J} \cdot \mathrm{d}\mathbf{S} . \tag{2.2}$$

This is the rate of flow of charge through Σ. The units of electric current are $\mathrm{C\,s^{-1}}$, which is also called the *Ampère* A.

2.2 The continuity equation

An important property of electric charge is that it is *conserved*, *i.e.* it is neither created nor destroyed. There is a differential equation that expresses this experimental fact called the *continuity equation*.

Suppose that Σ is a closed surface bounding a region R, so $\partial R = \Sigma$. From the discussion of current density $\mathbf{J}$ in the previous subsection, we see that the rate of flow of electric charge passing *out* of Σ is given by the current (2.2) through Σ. On the other hand, the total charge in R is

$$Q = \int_R \rho \, \mathrm{d}V \ . \tag{2.3}$$

If electric charge is conserved, then the rate of charge passing out of Σ must equal *minus* the rate of change of Q:

$$\int_\Sigma \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_R \frac{\partial \rho}{\partial t} \, \mathrm{d}V \ . \tag{2.4}$$

Here we have allowed time dependence in $\rho = \rho(\mathbf{r}, t)$. Using the divergence theorem A.2 this becomes

$$\int_R \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) \mathrm{d}V = 0 \tag{2.5}$$

for all R, and thus

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \ .} \tag{2.6}$$

This is the *continuity equation*.

In magneto*statics* we shall impose $\partial \rho / \partial t = 0$, and thus

$$\nabla \cdot \mathbf{J} = 0 \ . \tag{2.7}$$

Such currents are called *steady currents*.

2.3 The Lorentz force and the magnetic field

The force on a point charge q at *rest* in an electric field $\mathbf{E}$ is simply $\mathbf{F} = q\,\mathbf{E}$. We used this to *define* $\mathbf{E}$ in fact.

When the charge is moving the force law is more complicated. From experiments one finds that if q at position $\mathbf{r}$ is moving with velocity $\mathbf{u} = \mathrm{d}\mathbf{r}/\mathrm{d}t$ it experiences a force

$$\boxed{\mathbf{F} = q\,\mathbf{E}(\mathbf{r}) + q\,\mathbf{u} \wedge \mathbf{B}(\mathbf{r}) \ .} \tag{2.8}$$

Here $\mathbf{B} = \mathbf{B}(\mathbf{r})$ is a vector field, called the *magnetic field*, and we may similarly regard the *Lorentz force* $\mathbf{F}$ in (2.8) as defining $\mathbf{B}$. The magnetic field is measured in SI units in *Teslas*, which is the same as $\mathrm{N\,s\,m^{-1}\,C^{-1}}$.

Since (2.8) may look peculiar at first sight, it is worthwhile discussing it a little further. The magnetic component may be written as

$$\mathbf{F}_{\mathrm{mag}} = q\,\mathbf{u} \wedge \mathbf{B} \,. \qquad (2.9)$$

In experiments, the magnetic force on q is found to be proportional to q, proportional to the magnitude $|\mathbf{u}|$ of $\mathbf{u}$, and is perpendicular to $\mathbf{u}$. Note this latter point means that the magnetic force *does no work* on the charge. One also finds that the magnetic force at each point is perpendicular to a particular fixed direction at that point, and is also proportional to the sine of the angle between $\mathbf{u}$ and this fixed direction. The vector field that describes this direction is called the magnetic field $\mathbf{B}$, and the above, rather complicated, experimental observations are summarized by the simple formula (2.9).

In practice (2.8) was deduced not from moving test charges, but rather from *currents* in *test wires*. A current of course consists of moving charges, and (2.8) was deduced from the forces on these test wires.

2.4 The Biot-Savart law

If electric charges produce the electric field, what produces the magnetic field? The answer is that *electric currents* produce magnetic fields! Note carefully the distinction here: currents produce magnetic fields, but, by the Lorentz force law just discussed, magnetic fields exert a force on moving charges, and hence currents.

The usual discussion of this involves currents in wires, since this is what Ampère actually did in 1820. One has a wire with steady current I flowing through it. Here the latter is defined in terms of the current density via (2.2), where Σ is any cross-section of the wire. This is independent of the choice of cross-section, and thus makes sense, because[9] of the steady current condition (2.7). One finds that another wire, the test wire, with current I' experiences a force. This force is conveniently summarized by introducing the concept of a magnetic field: the first wire produces a magnetic field, which generates a force on the second wire via the Lorentz force (2.8) acting on the charges that make up the current I'.

Rather than describe this in detail, we shall instead simply note that if currents produce magnetic fields, then fundamentally it is *charges in motion* that produce magnetic fields. One may summarize this by an analogous formula to the Coulomb formula (1.6) for the electric

[9] use the divergence theorem for a cylindrical region bounded by any two cross-sections and the surface of the wire.

field due to a point charge. A charge q at the origin with velocity $\mathbf{v}$ produces a magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 q}{4\pi} \frac{\mathbf{v} \wedge \mathbf{r}}{r^3} . \tag{2.10}$$

Here the constant μ_0 is called the *permeability of free space*, and takes the value $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N\,s^2\,C^{-2}}$. Compare (2.10) to (1.6).

Importantly, we also have that

The magnetic field obeys the Principle of Superposition.

We may easily compute the magnetic field due to the steady current I in a wire C by summing contributions of the form (2.10):

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_C \frac{\mathrm{d}\mathbf{r}' \wedge (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} . \tag{2.11}$$

To see this, imagine dividing the wire into segments, with the segment $\delta\mathbf{r}'$ at position $\mathbf{r}'$. Suppose this segment contains a charge $q(\mathbf{r}')$ with velocity vector $\mathbf{v}(\mathbf{r}')$ – notice here that $\mathbf{v}(\mathbf{r}')$ points in the same direction as $\delta\mathbf{r}'$. From (2.10) this segment contributes

$$\delta\mathbf{B}(r) = \frac{\mu_0 q}{4\pi} \frac{\mathbf{v} \wedge (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} . \tag{2.12}$$

to the magnetic field. Now by definition of the current I we have $I\,\delta\mathbf{r}' = \rho\,\mathbf{v} \cdot \delta\mathbf{S}\,\delta\mathbf{r}'$, where we may take $\delta\Sigma$ of area δS to be a perpendicular cross-section of the wire. But the total charge q in the cylinder of cross-sectional area δS and length $|\delta\mathbf{r}'|$ is $q = \rho\,\delta S\,|\delta\mathbf{r}'|$. We thus deduce that $I\,\delta\mathbf{r}' = q\,\mathbf{v}$ and hence that (2.11) holds.

The formula (2.11) for the magnetic field produced by the steady current in a wire is called the *Biot-Savart law*. In fact (2.10) is also often called the Biot-Savart law.

Example: the magnetic field of an infinite straight line current

As an example, let's compute the magnetic field due to a steady current I in an infinitely long straight wire. Place the wire along the z-axis, and let $P = \mathbf{r}$ be a point in the $x - y$ plane at distance s from the origin O, as in Figure 5. Let $Q = \mathbf{r}'$ be a point on the wire at distance z from the origin. Then in the Biot-Savart formula we have $\mathbf{r}' = \mathbf{e}_z\, z$, where $\mathbf{e}_z$ is a unit vector in the z-direction (*i.e.* along the wire), and so $\mathrm{d}\mathbf{r}' = \mathbf{e}_z\,\mathrm{d}z$. The vector $\mathrm{d}\mathbf{r}' \wedge (\mathbf{r} - \mathbf{r}')$ points in a direction that is independent of $Q = \mathbf{r}'$: it is always tangent to the circle of radius s in the $x - y$ plane. We thus simply have to compute its magnitude, which is $|QP|\,\mathrm{d}z\,\sin\theta = s\,\mathrm{d}z$, where θ is the angle between OQ and QP, as shown. Thus from (2.11) we compute the magnitude $B(s) = |\mathbf{B}|$ of the magnetic field to be[10]

$$\begin{aligned} B(s) &= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{s}{(s^2 + z^2)^{3/2}}\,\mathrm{d}z \\ &= \frac{\mu_0 I}{2\pi s} . \end{aligned} \tag{2.13}$$

[10] *Hint*: put $z = s \tan\varphi$ to do the integral.

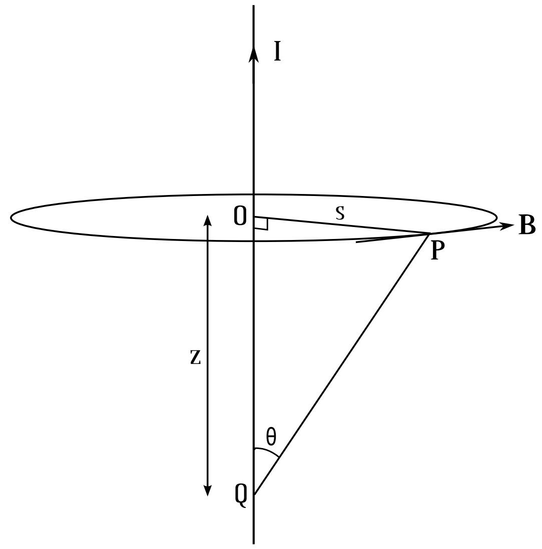


Figure 5: Computing the magnetic field $\mathbf{B}$ around a long straight wire carrying a steady current I.

2.5 Magnetic monopoles?

Returning to the point charge at the origin, with magnetic field (2.10), note that since $\mathbf{r}/r^3 = -\nabla(1/r)$ we have

$$\nabla \cdot \left(\frac{\mathbf{v} \wedge \mathbf{r}}{r^3}\right) = \mathbf{v} \cdot \mathrm{curl}\left(\nabla\left(\frac{1}{r}\right)\right) = 0 \ . \tag{2.14}$$

Here we have used the identity (A.12) and the fact that the curl of a gradient is identically zero. Thus we have shown that $\nabla \cdot \mathbf{B} = 0$, except at the origin $\mathbf{r} = \mathbf{0}$. However, unlike the case of the electric field and Gauss' law, the integral of the magnetic field around the point charge is zero. To see this, let Σ_a be a sphere of radius a centred on the charge, and compute

$$\int_{\Sigma_a} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \frac{\mu_0 \, q}{4\pi} \int_{\Sigma_a} \left(\frac{\mathbf{v} \wedge \mathbf{r}}{r^3}\right) \cdot \left(\frac{\mathbf{r}}{r}\right) \mathrm{d}S = 0 \ . \tag{2.15}$$

By the divergence theorem, it follows that

$$\int_R \nabla \cdot \mathbf{B} \, \mathrm{d}V = 0 \tag{2.16}$$

for *any* region R, and thus

$$\boxed{\nabla \cdot \mathbf{B} = 0 \ .} \tag{2.17}$$

This is another of Maxwell's equations on the front cover. It says that there are *no magnetic (mono)poles* that generate magnetic fields, analogous to the way that electric charges generate electric fields. Instead magnetic fields are produced by electric currents. Although we have only deduced (2.17) above for the magnetic field of a moving point charge, the general case follows from the Principle of Superposition.

You might wonder what produces the magnetic field in *permanent* magnets, such as bar magnets. Where is the electric current? It turns out that, in certain magnetic materials, the macroscopic magnetic field is produced by the alignment of tiny atomic currents generated by the electrons in the material (in others it is due to the alignment of the "spins" of the electrons, which is a quantum mechanical property).

* Mathematically, it is certainly possible to allow for *magnetic monopoles* and *magnetic currents* in Maxwell's equations. In fact the equations then become completely symmetric under the interchange of $\mathbf{E}$ with $-c\mathbf{B}$, $c\mathbf{B}$ with $\mathbf{E}$, and corresponding interchanges of electric with magnetic sources and currents. Here $c^2 = 1/\epsilon_0\mu_0$. There are also theoretical reasons for introducing magnetic monopoles. For example, the quantization of electric charge – that all electric charges are an *integer* multiple of some fixed fundamental quantity of charge – may be understood in quantum mechanics using magnetic monopoles. This is a beautiful argument due to Dirac. However, no magnetic monopoles have ever been observed in nature. If they existed, (2.17) would need correcting.

2.6 Ampère's law

There is just one more static Maxwell equation to discuss, and that is the equation involving the curl of $\mathbf{B}$. This is *Ampère's law*. In many treatments of magnetostatics, this is often described as an additional experimental result, and/or is derived for special symmetric configurations, having first solved for $\mathbf{B}$ using the Biot-Savart law (2.10) or (2.11). However, it is possible to *derive* Ampère's law directly from (2.10).

We first use (2.10) and the Principle of Superposition to write $\mathbf{B}$ generated by a current density $\mathbf{J}$ as a volume integral

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathbf{r}'\in R}\frac{\mathbf{J}(\mathbf{r}')\wedge(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}V' \ . \tag{2.18}$$

This follows directly from the definition $\mathbf{J} = \rho\,\mathbf{v}$ in (2.1) and taking the limit of a sum of terms of the form (2.10). R is by definition a region containing the set of points with $\mathbf{J}\neq 0$.

We next define the vector field

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_R\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' \ . \tag{2.19}$$

One easily computes (*cf.* Theorem A.4 with $f = \mu_0 J_i\,/\,4\pi$)

$$\frac{\partial A_i}{\partial x_j}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\int_R\frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,(x_j - x_j')\,\mathrm{d}V' \tag{2.20}$$

and thus

$$\mathbf{B}(\mathbf{r}) = \nabla\wedge\mathbf{A}(\mathbf{r}) \ . \tag{2.21}$$

Note that (2.21) directly implies $\nabla \cdot \mathbf{B} = 0$.

We next take the curl of (2.21). Using

$$\nabla \wedge (\nabla \wedge \mathbf{A}) \;=\; \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \;, \tag{2.22}$$

which holds for any vector field $\mathbf{A}$, we obtain

$$\nabla \wedge \mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi} \nabla \left[\int_R \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \mathrm{d}V' \right] \;-\; \nabla^2 \mathbf{A}(\mathbf{r}) \;. \tag{2.23}$$

We first deal with the first term. It is important that the integrand is in fact integrable over a neighbourhood of $\mathbf{r} = \mathbf{r}'$. One can see this using spherical polar coordinates about the point $\mathbf{r}$ – see the proof of Theorem A.4. Now using

$$\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \;=\; -\nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \;, \qquad \mathbf{r} \neq \mathbf{r}' \;, \tag{2.24}$$

where ∇' denotes derivative with respect to $\mathbf{r}'$, the term in square brackets is

$$\begin{aligned} \int_R \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \mathrm{d}V' \;&=\; -\int_R \nabla' \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) \mathrm{d}V' \\ &\quad + \int_R \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left(\nabla' \cdot \mathbf{J}(\mathbf{r}') \right) \mathrm{d}V' \;. \end{aligned} \tag{2.25}$$

The second term on the right hand side of equation (2.25) is zero for steady currents (2.7). Moreover, we may use the divergence theorem on the first term to obtain a surface integral on ∂R. But by assumption $\mathbf{J}$ vanishes on this boundary, so this term is also zero.

Thus $\nabla \cdot \mathbf{A} = 0$ and we have shown that

$$\nabla \wedge \mathbf{B} \;=\; -\nabla^2 \mathbf{A} \;. \tag{2.26}$$

We now use Theorem A.4, again with $f = \mu_0 J_i \,/\, 4\pi$ for each component J_i of $\mathbf{J}$, to deduce

$$\boxed{\nabla \wedge \mathbf{B} \;=\; \mu_0 \, \mathbf{J} \;.} \tag{2.27}$$

This is *Ampère's law* for magnetostatics. It is the final Maxwell equation on the front cover, albeit in the special case where the electric field is independent of time, so $\partial \mathbf{E}/\partial t = 0$. Note this equation is consistent with the steady current assumption (2.7).

We may equivalently rewrite (2.27) using Stokes' theorem as

Ampère's law: *For any simple closed curve $C = \partial\Sigma$ bounding a surface Σ*

$$\int_{C=\partial\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{r} \;=\; \mu_0 \int_\Sigma \mathbf{J} \cdot \mathrm{d}\mathbf{S} \;=\; \mu_0 I \tag{2.28}$$

where I is the current through Σ.

As an example, notice that integrating the magnetic field $\mathbf{B}$ given by (2.13) around a circle C in the $x - y$ plane of radius s and centred on the z-axis indeed gives $\mu_0 I$.

2.7 The magnetostatic vector potential

In magnetostatics it is possible to introduce a *magnetic vector potential* $\mathbf{A}$ analogous to the electrostatic potential ϕ in electrostatics. In fact we have already introduced such a vector potential in (2.21) and (2.19):

$$\mathbf{B} = \nabla \wedge \mathbf{A} \tag{2.29}$$

where $\mathbf{A}$ is the *magnetic vector potential*.

* Notice that (2.29) is a sufficient condition for the Maxwell equation (2.17). It is also *necessary* if we work in a domain with simple enough topology, such as $\mathbb{R}^3$ or an open ball. A domain where not every vector field $\mathbf{B}$ with zero divergence may be written as a curl is $\mathbb{R}^3$ with a point removed. Compare this to the corresponding starred paragraph in section 1.5. Again, a proof for an open ball is contained in appendix B of Prof Woodhouse's book.

Notice that $\mathbf{A}$ in (2.29) is far from unique: since the curl of a gradient is zero, we may add $\nabla\psi$ to $\mathbf{A}$, for any function ψ, without changing $\mathbf{B}$:

$$\mathbf{A} \rightarrow \widehat{\mathbf{A}} = \mathbf{A} + \nabla\psi \ . \tag{2.30}$$

This is called a *gauge transformation* of $\mathbf{A}$. We may fix this *gauge freedom* by imposing additional conditions on $\mathbf{A}$. For example, suppose we have chosen a particular $\mathbf{A}$ satisfying (2.29). Then by choosing ψ to be a solution of the Poisson equation

$$\nabla^2\psi = -\nabla \cdot \mathbf{A} \tag{2.31}$$

it follows that $\widehat{\mathbf{A}}$ in (2.30) satisfies

$$\nabla \cdot \widehat{\mathbf{A}} = 0 \ . \tag{2.32}$$

This is called the *Lorenz gauge* (Mr Lorenz and Mr Lorentz were two different people). As you learned[11] in the Moderations course on vector calculus, the solution to Poisson's equation (2.31) is unique for fixed boundary conditions.

Many equations simplify with this choice for $\mathbf{A}$. For example, Ampère's law (2.27) becomes

$$\mu_0 \mathbf{J} = \nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{2.33}$$

so that in Lorenz gauge $\nabla \cdot \mathbf{A} = 0$ this becomes

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \ . \tag{2.34}$$

Compare with Poisson's equation (1.25) in electrostatics. Notice that we showed in the previous subsection that $\mathbf{A}$ in (2.19) is in Lorenz gauge.

* Gauge invariance and the vector potential $\mathbf{A}$ play a fundamental role in more advanced formulations of electromagnetism. The magnetic potential also plays an essential *physical* role in the *quantum* theory of electromagnetism.

[11] Corollary 7.3.

3 Electrodynamics and Maxwell's equations

3.1 Maxwell's displacement current

Let's go back to Ampère's law (2.28) in magnetostatics

$$\int_{C=\partial\Sigma} \mathbf{B}\cdot \mathrm{d}\mathbf{r} \;=\; \mu_0 \int_\Sigma \mathbf{J}\cdot \mathrm{d}\mathbf{S}\ . \tag{3.1}$$

Here $C = \partial\Sigma$ is a simple closed curve bounding a surface Σ. Of course, one may use *any* such surface spanning C on the right hand side. If we pick a different surface Σ', with $C = \partial\Sigma'$, then

$$\begin{aligned} 0 \;&=\; \int_\Sigma \mathbf{J}\cdot \mathrm{d}\mathbf{S} - \int_{\Sigma'} \mathbf{J}\cdot \mathrm{d}\mathbf{S} \\ &=\; \int_S \mathbf{J}\cdot \mathrm{d}\mathbf{S}\ . \end{aligned} \tag{3.2}$$

Here S is the *closed* surface obtained by gluing Σ and Σ' together along C. Thus the flux of $\mathbf{J}$ through any closed surface is zero. We may see this in a different way if we assume that $S = \partial R$ bounds a region R, since then

$$\int_S \mathbf{J}\cdot \mathrm{d}\mathbf{S} \;=\; \int_R \nabla\cdot\mathbf{J}\,\mathrm{d}V \;=\; 0 \tag{3.3}$$

and in the last step we have used the steady current condition (2.7).

But in general, (2.7) should be replaced by the continuity equation (2.6). The above calculation then changes to

$$\begin{aligned} \int_\Sigma \mathbf{J}\cdot \mathrm{d}\mathbf{S} - \int_{\Sigma'} \mathbf{J}\cdot \mathrm{d}\mathbf{S} \;&=\; \int_S \mathbf{J}\cdot \mathrm{d}\mathbf{S} \\ &=\; \int_R \nabla\cdot\mathbf{J}\,\mathrm{d}V \\ &=\; -\int_R \frac{\partial\rho}{\partial t}\,\mathrm{d}V \\ &=\; -\epsilon_0 \int_R \frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{E}\right)\,\mathrm{d}V \\ &=\; -\epsilon_0 \int_{S=\partial R} \frac{\partial \mathbf{E}}{\partial t}\cdot \mathrm{d}\mathbf{S} \\ &=\; -\epsilon_0 \int_\Sigma \frac{\partial \mathbf{E}}{\partial t}\cdot \mathrm{d}\mathbf{S} \;+\; \epsilon_0 \int_{\Sigma'} \frac{\partial \mathbf{E}}{\partial t}\cdot \mathrm{d}\mathbf{S}\ . \end{aligned} \tag{3.4}$$

Notice we have used Gauss' law (1.17), and that we now regard $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ as a vector field depending on time. This shows that

$$\int_\Sigma \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)\cdot \mathrm{d}\mathbf{S} \;=\; \int_{\Sigma'} \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)\cdot \mathrm{d}\mathbf{S} \tag{3.5}$$

for any two surfaces Σ, Σ' spanning C, and thus suggests replacing Ampère's law (2.27) by

$$\boxed{\nabla\wedge\mathbf{B} \;=\; \mu_0\left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)\ .} \tag{3.6}$$

This is indeed the correct time-dependent Maxwell equation on the front cover. The additional term $\partial \mathbf{E}/\partial t$ is called the *displacement current*. It says that a *time-dependent* electric field also produces a magnetic field.

3.2 Faraday's law

The electrostatic equation (1.20) is also modified in the time-dependent case. We can motivate how precisely by the following argument. Consider the electromagnetic field generated by a set of charges all moving with constant velocity $\mathbf{v}$. The charges generate both an $\mathbf{E}$ and a $\mathbf{B}$ field, the latter since the charges are in motion. However, consider instead an observer who is *also* moving at the same constant velocity $\mathbf{v}$. For this observer, the charges are at *rest*, and thus he/she will measure *only* an electric field $\mathbf{E}'$ from the Lorentz force law (2.8) on one of the charges! Since (or assuming) the two observers must be measuring the same force on a given charge, we conclude that

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \wedge \mathbf{B} . \tag{3.7}$$

Now since the field is electrostatic for the moving observer,

$$\begin{aligned} 0 &= \nabla \wedge \mathbf{E}' \\ &= \nabla \wedge \mathbf{E} + \nabla \wedge (\mathbf{v} \wedge \mathbf{B}) \\ &= \nabla \wedge \mathbf{E} + \mathbf{v}(\nabla \cdot \mathbf{B}) - (\mathbf{v} \cdot \nabla)\, \mathbf{B} \\ &= \nabla \wedge \mathbf{E} - (\mathbf{v} \cdot \nabla)\, \mathbf{B} . \end{aligned} \tag{3.8}$$

Here we have used the identity (A.11), and in the last step we have used $\nabla \cdot \mathbf{B} = 0$. Now, for the original observer the charges are all moving with velocity $\mathbf{v}$, so the magnetic field at position $\mathbf{r} + \mathbf{v}\tau$ and time $t + \tau$ is the same as that at position $\mathbf{r}$ and time t:

$$\mathbf{B}(\mathbf{r} + \mathbf{v}\tau, t + \tau) = \mathbf{B}(\mathbf{r}, t) . \tag{3.9}$$

This implies the partial differential equation

$$(\mathbf{v} \cdot \nabla)\, \mathbf{B} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3.10}$$

and we deduce from (3.8) that

$$\boxed{\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} .} \tag{3.11}$$

This is *Faraday's law*, and is another of Maxwell's equations. The above argument raises issues about what happens in general to our equations when we change to a moving frame. This is not something we want to pursue further here, because it leads to Einstein's theory of *Special Relativity*! This will be studied further in the Part B course on Special Relativity and Electromagnetism.

As usual, the equation (3.11) may be expressed as an integral equation as

Faraday's law: *For any simple closed curve $C = \partial\Sigma$ bounding a* fixed *surface Σ*

$$\int_{C=\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\mathbf{r} \;=\; -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \;. \tag{3.12}$$

This says that a *time-dependent* magnetic field produces an electric field. For example, if one moves a bar magnet through a loop of conducting wire C, the resulting electric field from (3.11) induces a current in the wire via the Lorentz force. This is what Faraday did, in fact, in 1831. The integral $\int_\Sigma \mathbf{B} \cdot \mathrm{d}\mathbf{S}$ is called the *magnetic flux through* Σ.

* The current in the wire then *itself* produces a magnetic field of course, via Ampère's law. However, the signs are such that this magnetic field is in the *opposite* direction to the change in the magnetic field that created it. This is called *Lenz's law*. The whole setup may be summarized as follows:

$$\text{changing } \mathbf{B} \;\overset{\text{Faraday}}{\longrightarrow}\; \mathbf{E} \;\overset{\text{Lorentz}}{\longrightarrow}\; \text{current} \;\overset{\text{Ampère}}{\longrightarrow}\; \mathbf{B} \;. \tag{3.13}$$

3.3 Maxwell's equations

We now summarize the full set of Maxwell equations.

There are two scalar equations, namely Gauss' law (1.17) from electrostatics, and the equation (2.17) from magnetostatics that expresses the absence of magnetic monopoles:

$$\nabla \cdot \mathbf{E} \;=\; \frac{\rho}{\epsilon_0} \tag{3.14}$$

$$\nabla \cdot \mathbf{B} \;=\; 0 \;. \tag{3.15}$$

Although we discussed these only in the time-independent case, they are in fact true in general.

The are also two vector equations, namely Faraday's law (3.11) and Maxwell's modification (3.6) of Ampère's law (2.27) from magnetostatics:

$$\nabla \wedge \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t} \tag{3.16}$$

$$\nabla \wedge \mathbf{B} \;=\; \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \;. \tag{3.17}$$

Together with the Lorentz force law

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{u} \wedge \mathbf{B})$$

which governs the mechanics, this is *all* of electromagnetism. Everything else we have discussed may in fact be derived from these equations.

Maxwell's equations, for given ρ and $\mathbf{J}$, are 8 equations for 6 unknowns. There must therefore be two consistency conditions. To see what these are, we compute

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = -\nabla \cdot (\nabla \wedge \mathbf{E}) = 0\ , \tag{3.18}$$

where we have used (3.16). This is clearly consistent with (3.15). We get something non-trivial by instead taking the divergence of (3.17), which gives

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{J} + \epsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) \\ &= \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\ , \end{aligned} \tag{3.19}$$

where we have used (3.14). Thus the continuity equation arises as a *consistency condition* for Maxwell's equations: if ρ and $\mathbf{J}$ do not satisfy (3.19), there is no solution to Maxwell's equations for this choice of charge density and current.

3.4 Electromagnetic potentials and gauge invariance

In the general time-dependent case one can introduce electromagnetic potentials in a similar way to the static cases. We work in a suitable domain in $\mathbb{R}^3$, such as $\mathbb{R}^3$ itself or an open ball therein, as discussed in previous sections. Since $\mathbf{B}$ has zero divergence (3.15), we may again introduce a vector potential

$$\mathbf{B} = \nabla \wedge \mathbf{A}\ , \tag{3.20}$$

where now $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$. It follows from Faraday's law that

$$0 = \nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right)\ . \tag{3.21}$$

Thus we may introduce a scalar potential $\phi = \phi(\mathbf{r}, t)$ via

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi\ . \tag{3.22}$$

Thus

$$\mathbf{B} = \nabla \wedge \mathbf{A} \tag{3.23}$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}\ . \tag{3.24}$$

Notice that we similarly have the *gauge transformations*

$$\mathbf{A} \to \mathbf{A} + \nabla\psi, \qquad\qquad \phi \to \phi - \frac{\partial\psi}{\partial t} \tag{3.25}$$

leaving (3.23) and (3.24) invariant. Again, one may fix this non-uniqueness of $\mathbf{A}$ and ϕ by imposing certain gauge choices. This will be investigated further in Problem Sheet 2. Note that, by construction, with (3.23) and (3.24) the Maxwell equations (3.15) and (3.16) are automatically satisfied.

3.5 Electromagnetic energy and Poynting's theorem

Recall that in section 1.7 we derived a formula for the *electrostatic energy density* $\mathcal{U}_{\text{electric}} = \epsilon_0\,|\mathbf{E}|^2\,/\,2$ in terms of the electric field $\mathbf{E}$. The electrostatic energy of a given configuration is the integral of this density over space (1.41). One can motivate the similar formula $\mathcal{U}_{\text{magnetic}} = |\mathbf{B}|^2\,/\,2\mu_0$ in magnetostatics, although unfortunately we won't have time to derive this here. A natural candidate for the *electromagnetic energy density* in general is thus

$$\mathcal{U} = \frac{\epsilon_0}{2}\,|\mathbf{E}|^2 + \frac{1}{2\mu_0}\,|\mathbf{B}|^2\,. \tag{3.26}$$

Consider taking the partial derivative of (3.26) with respect to time:

$$\begin{aligned} \frac{\partial\mathcal{U}}{\partial t} &= \epsilon_0\,\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\,\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} \\ &= \frac{1}{\mu_0}\,\mathbf{E}\cdot(\nabla\wedge\mathbf{B} - \mu_0\,\mathbf{J}) - \frac{1}{\mu_0}\,\mathbf{B}\cdot(\nabla\wedge\mathbf{E}) \\ &= -\nabla\cdot\left(\frac{1}{\mu_0}\,\mathbf{E}\wedge\mathbf{B}\right) - \mathbf{E}\cdot\mathbf{J}\,. \end{aligned} \tag{3.27}$$

Here after the first step we have used the Maxwell equations (3.17) and (3.16), respectively. The last step uses the identity (A.12). If we now define the *Poynting vector* $\mathbf{P}$ to be

$$\mathbf{P} = \frac{1}{\mu_0}\,\mathbf{E}\wedge\mathbf{B} \tag{3.28}$$

then we have derived *Poynting's theorem*

$$\boxed{\frac{\partial\mathcal{U}}{\partial t} + \nabla\cdot\mathbf{P} = -\mathbf{E}\cdot\mathbf{J}\,.} \tag{3.29}$$

Notice that, in the absence of a source current, $\mathbf{J} = 0$, this takes the form of a *continuity equation*, analogous to the continuity equation (2.6) that expresses conservation of charge. It is thus natural to interpret (3.29) as a *conservation of energy* equation, and so identify the Poynting vector $\mathbf{P}$ as some kind of rate of energy flow density. One can indeed justify this by examining the above quantities in various physical applications.

Integrating (3.29) over a region R with boundary Σ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R \mathcal{U}\,\mathrm{d}V = -\int_\Sigma \mathbf{P}\cdot\mathrm{d}\mathbf{S} - \int_R \mathbf{E}\cdot\mathbf{J}\,\mathrm{d}V\,. \tag{3.30}$$

Given our discussion of $\mathcal{U}$, the left hand side is clearly the *rate of increase of energy* in R. The first term on the right hand side is the *rate of energy flow into* the region R. When $\mathbf{J} = 0$, this is precisely analogous to our discussion of charge conservation in section 2.2.

The final term in (3.30) is interpreted as (minus) the *rate of work by the field on the sources*. To see this, remember that the force on a charge q moving at velocity $\mathbf{v}$ is $\mathbf{F} = q\,(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})$. This force does *work* at a rate given by $\mathbf{F}\cdot\mathbf{v} = q\,\mathbf{E}\cdot\mathbf{v}$. Recalling the definition (2.1) of $\mathbf{J} = \rho\,\mathbf{v}$, we see that the force does work on the charge $q = \rho\,\delta V$ in a small volume δV at a rate $\mathbf{F}\cdot\mathbf{v} = \mathbf{E}\cdot\mathbf{J}\,\delta V$.

4 Electromagnetic waves

4.1 Source-free equations and electromagnetic waves

We begin by writing down Maxwell's equations in *vacuum*, where there is no electric charge or current present:

$$\nabla\cdot\mathbf{E} = 0 \tag{4.1}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{4.2}$$

$$\nabla\wedge\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0} \tag{4.3}$$

$$\nabla\wedge\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mathbf{0} \tag{4.4}$$

where we have defined

$$c = \sqrt{\frac{1}{\epsilon_0\,\mu_0}}\ . \tag{4.5}$$

If you look back at the units of ϵ_0 and μ_0, you'll see that $\epsilon_0\,\mu_0$ has units $(\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2})\cdot(\mathrm{N}\,\mathrm{s}^2\,\mathrm{C}^{-2}) = (\mathrm{m}\,\mathrm{s}^{-1})^{-2}$. Thus c is a *speed*. The great insight of Maxwell was to realise it is the *speed of light* in vacuum.

Taking the curl of (4.3) we have

$$\begin{aligned}\mathbf{0} &= \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} + \nabla\wedge\frac{\partial\mathbf{B}}{\partial t}\\ &= -\nabla^2\mathbf{E} + \frac{\partial}{\partial t}(\nabla\wedge\mathbf{B})\\ &= -\nabla^2\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\ .\end{aligned} \tag{4.6}$$

Here after the first step we have used (4.1), and in the last step we have used (4.4). It follows that each component of $\mathbf{E}$ satisfies the *wave equation*

$$\Box\, u = 0 \tag{4.7}$$

where $u = u(\mathbf{r}, t)$, and we have defined the *d'Alembertian* operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 . \tag{4.8}$$

You can easily check that $\mathbf{B}$ also satisfies $\Box\,\mathbf{B} = \mathbf{0}$.

The equation (4.7) governs the propagation of waves of speed c in three-dimensional space. It is the natural generalization of the one-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 , \tag{4.9}$$

which you met in the Moderations courses "Fourier Series and Two Variable Calculus", and "Partial Differential Equations in Two Dimensions and Applications". Recall that this has particular solutions of the form $u_\pm(x, t) = f(x \mp ct)$, where f is any function which is twice differentiable. In this case, the waves look like the graph of f travelling at constant velocity c in the direction of increasing/decreasing x, respectively. The *general*[12] solution to (4.9) is $u(x, t) = f(x - ct) + g(x + ct)$.

This naturally generalizes to the three-dimensional equation (4.7), by writing

$$u(\mathbf{r}, t) = f(\mathbf{e} \cdot \mathbf{r} - ct) \tag{4.10}$$

where $\mathbf{e}$ is a fixed *unit vector*, $|\mathbf{e}|^2 = 1$. Indeed, note that $\nabla^2 u = \mathbf{e} \cdot \mathbf{e}\, f''$, $\partial^2 u/\partial t^2 = c^2 f''$. Solutions of the form (4.10) are called *plane-fronted waves*, since at any constant time, u is constant on the planes $\{\mathbf{e} \cdot \mathbf{r} = \text{constant}\}$ orthogonal to $\mathbf{e}$. As time t increases, these plane wavefronts propagate in the direction of $\mathbf{e}$ at speed c. However, unlike the one-dimensional equation, we *cannot* write the general solution to (4.7) as a sum of two plane-fronted waves travelling in opposite directions.

4.2 Monochromatic plane waves

An important special class of plane-fronted waves (4.10) are given by the *real harmonic waves*

$$u(\mathbf{r}, t) = \alpha \cos \Omega(\mathbf{r}, t) + \beta \sin \Omega(\mathbf{r}, t) \tag{4.11}$$

where α, β are constants, and we define

$$\Omega(\mathbf{r}, t) = \frac{\omega}{c}(ct - \mathbf{e} \cdot \mathbf{r}) \tag{4.12}$$

where $\omega > 0$ is the constant *frequency* of the wave. In fact it is a result of Fourier analysis that *every* solution to the wave equation (4.7) is a linear combination (in general involving an integral) of these harmonic waves.

[12] see the Mods course on PDEs referred to above.

Since the components of $\mathbf{E}$ (and $\mathbf{B}$) satisfy (4.7), it is natural to look for solutions of the harmonic wave form

$$\mathbf{E}(\mathbf{r},t) \;=\; \boldsymbol{\alpha}\,\cos\Omega(\mathbf{r},t) \;+\; \boldsymbol{\beta}\,\sin\Omega(\mathbf{r},t) \tag{4.13}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are constant vectors, and $\Omega = \Omega(\mathbf{r},t)$ is again given by (4.12). This of course satisfies the wave equation, but we must ensure that we satisfy *all* of the Maxwell equations in vacuum. The first (4.1) implies

$$0 \;=\; \nabla\cdot\mathbf{E} \;=\; \frac{\omega}{c}\left(\mathbf{e}\cdot\boldsymbol{\alpha}\,\sin\Omega \;-\; \mathbf{e}\cdot\boldsymbol{\beta}\,\cos\Omega\right) \tag{4.14}$$

which implies that

$$\mathbf{e}\cdot\boldsymbol{\alpha} \;=\; \mathbf{e}\cdot\boldsymbol{\beta} \;=\; 0\;, \tag{4.15}$$

i.e. the $\mathbf{E}$-field is *orthogonal* to the direction of propagation $\mathbf{e}$. Next we look at (4.3):

$$-\frac{\partial\mathbf{B}}{\partial t} \;=\; \nabla\wedge\mathbf{E} \;=\; \frac{\omega}{c}\left(\mathbf{e}\wedge\boldsymbol{\alpha}\,\sin\Omega \;-\; \mathbf{e}\wedge\boldsymbol{\beta}\,\cos\Omega\right)\;. \tag{4.16}$$

We may satisfy this equation by taking

$$\mathbf{B}(\mathbf{r},t) \;=\; \frac{1}{c}\,\mathbf{e}\wedge\mathbf{E}(\mathbf{r},t) \;=\; \frac{1}{c}\,\left[\mathbf{e}\wedge\boldsymbol{\alpha}\,\cos\Omega(\mathbf{r},t) \;+\; \mathbf{e}\wedge\boldsymbol{\beta}\,\sin\Omega(\mathbf{r},t)\right]\;. \tag{4.17}$$

It is now simple to verify that the last two Maxwell equations (4.2), (4.4) are satisfied by (4.13), (4.17) – the calculation is analogous to that above, so I leave this as a short exercise. Notice that the $\mathbf{B}$-field (4.17) is orthogonal to both $\mathbf{E}$ and the direction of propagation.

The solution to Maxwell's equations with $\mathbf{E}$ and $\mathbf{B}$ given by (4.13), (4.17) is called a *monochromatic electromagnetic plane wave*. It is specified by the constant direction of propagation $\mathbf{e}$, $|\mathbf{e}|^2 = 1$, two constant vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ orthogonal to this direction, $\mathbf{e}\cdot\boldsymbol{\alpha} = \mathbf{e}\cdot\boldsymbol{\beta} = 0$, and the frequency ω. Again, using Fourier analysis one can show that the general vacuum solution is a combination of these monochromatic plane waves.

Notice from the first equality in (4.17) that the Poynting vector (3.28) for a monochromatic plane wave is

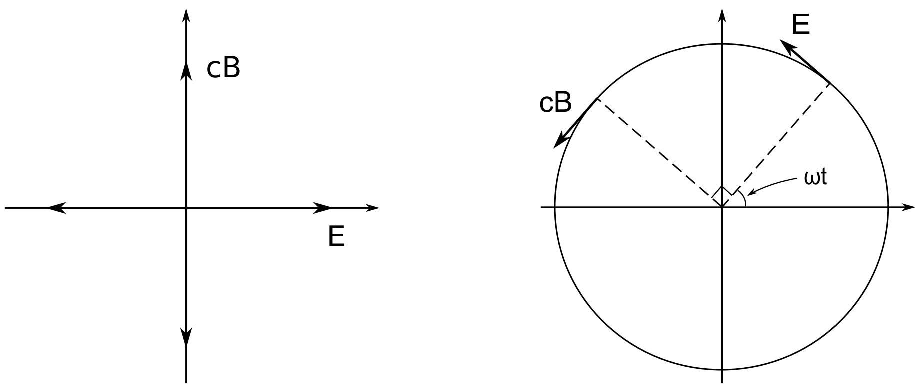
$$\mathbf{P} \;=\; \frac{1}{\mu_0 c}\,|\mathbf{E}|^2\,\mathbf{e} \;=\; \sqrt{\frac{\epsilon_0}{\mu_0}}\;|\boldsymbol{\alpha}\,\cos\Omega \;+\; \boldsymbol{\beta}\,\sin\Omega|^2\;\mathbf{e} \tag{4.18}$$

which is in the direction of propagation of the wave. Thus electromagnetic waves carry energy, a fact which anyone who has made a mircowave pot noodle can confirm.

4.3 Polarization

Consider a monochromatic plane wave with $\mathbf{E}$ given by (4.13). If we fix a particular point in space, say the origin $\mathbf{r} = 0$, then

$$\mathbf{E}(0,t) \;=\; \boldsymbol{\alpha}\,\cos(\omega t) \;+\; \boldsymbol{\beta}\,\sin(\omega t)\;. \tag{4.19}$$



(a) Linear polarization (b) Circular polarization

Figure 6: Polarizations of monochromatic electromagnetic plane waves, viewed from the direction of propagation.

As t varies, this[13] sweeps out an *ellipse* in the plane spanned by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

As a simple example, taking $\boldsymbol{\alpha} = \alpha\,\mathbf{e}_1$, $\boldsymbol{\beta} = \beta\,\mathbf{e}_2$ where $\mathbf{e}_1$, $\mathbf{e}_2$ are orthonormal vectors, so that $\mathbf{e} = \mathbf{e}_3$ is the direction of propagation, then the 1 and 2 components of $\mathbf{E}$ in (4.19) are

$$E_1 = \alpha \cos(\omega t) \tag{4.20}$$

$$E_2 = \beta \sin(\omega t) \tag{4.21}$$

and thus

$$\frac{E_1^2}{\alpha^2} + \frac{E_2^2}{\beta^2} = 1 \ . \tag{4.22}$$

This is an ellipse with semi-major(minor) axis length α, semi-minor(major) axis length β, centred on the origin.

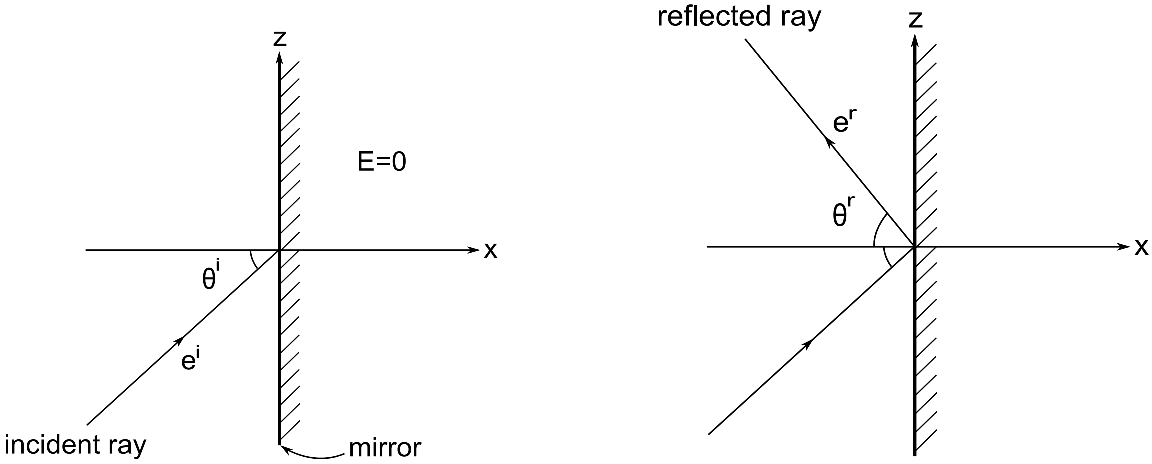
There are two special choices of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which have names:

Linear polarization: If $\boldsymbol{\alpha}$ is proportional to $\boldsymbol{\beta}$, so that the ellipse degenerates to a *line*, then the monochromatic plane wave is said to be *linearly polarized*. In this case, $\mathbf{E}$ and $\mathbf{B}$ oscillate in two fixed orthogonal directions – see Figure 6a.

Circular polarization: If $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$ (as in the example above) and also $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, so that the ellipse is a *circle*, then the monochromatic plane wave is said to be *circularly polarized*. In this case, $\mathbf{E}$ and $\mathbf{B}$ rotate at constant angular velocity about the direction of propagation – see Figure 6b. Moreover, a circularly polarized wave is said to be *right-handed* or *left-handed*, depending on whether $\boldsymbol{\beta} = \mathbf{e} \wedge \boldsymbol{\alpha}$ or $\boldsymbol{\beta} = -\mathbf{e} \wedge \boldsymbol{\alpha}$, respectively.

In general, electromagnetic waves are combinations of waves with different polarizations.

[13]Similar remarks apply to the **B**-field.



(a) Incident ray in direction $\mathbf{e}^i$ (b) Reflected ray in direction $\mathbf{e}^r$

Figure 7: Incident and reflected rays, striking a mirror at $x = 0$.

4.4 Reflection

In this final subsection we derive the *laws of reflection* from what we've learned so far. The argument is straightfoward, but a little fiddly.

Suppose we have a plane mirror at $x = 0$, as shown in Figure 7a, and consider a linearly polarized incident wave of the form

$$\mathbf{E}^i = \alpha^i \mathbf{j} \cos \Omega^i \tag{4.23}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the standard orthonormal basis vectors for $\mathbb{R}^3$ in Cartesian coordinates (x, y, z), and

$$\Omega^i = \frac{\omega^i}{c}\left(ct - \mathbf{e}^i \cdot \mathbf{r}\right) . \tag{4.24}$$

This is polarized in the direction $\mathbf{j}$, which is normal to Figure 7a, and has frequency $\omega^i > 0$. Since $\mathbf{E}^i$ is orthogonal to the unit vector direction of propagation $\mathbf{e}^i$ we have $\mathbf{e}^i \cdot \mathbf{j} = 0$ and thus

$$\mathbf{e}^i = (\cos\theta^i, 0, \sin\theta^i) \tag{4.25}$$

where θ^i is the *angle of incidence*. It follows from (4.24) that $\mathbf{E}^i$ is independent of the coordinate y.

We suppose that this wave arrives from $x < 0$ and that the mirror is a conductor[14], so that $\mathbf{E} = \mathbf{0}$ for $x > 0$. We suppose that the reflected wave takes the monochromatic form

$$\mathbf{E}^r = \boldsymbol{\alpha}^r \cos\Omega^r + \boldsymbol{\beta}^r \sin\Omega^r \tag{4.26}$$

[14]See the discussion in section 1.6.

where

$$\Omega^r = \frac{\omega^r}{c}\left(ct - \mathbf{e}^r \cdot \mathbf{r}\right) \ . \tag{4.27}$$

The total electric field is thus

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^r \ . \tag{4.28}$$

From the boundary conditions deduced in section 1.6, we know[15] that the components of $\mathbf{E} = \mathbf{E}^i + \mathbf{E}^r$ *parallel* to the boundary surface at $x = 0$ are continuous across the surface. Thus at $x = 0$

$$\left(\mathbf{E}^i + \mathbf{E}^r\right) \cdot \mathbf{j} \ |_{x=0} = 0 \tag{4.29}$$

$$\left(\mathbf{E}^i + \mathbf{E}^r\right) \cdot \mathbf{k} \ |_{x=0} = 0 \ . \tag{4.30}$$

Notice from (4.29) that $\mathbf{E}^r \cdot \mathbf{j} \neq 0$ identically, as otherwise from (4.23) $\mathbf{E}^i = \mathbf{0}$. Since $\mathbf{E}^i$ is independent of y, it thus follows from (4.29) that so is $\mathbf{E}^r$, and hence $\mathbf{e}^r \cdot \mathbf{j} = 0$. That is,

R1: *The incident ray, normal, and reflected ray are coplanar.*

This is the first law of reflection. We may now write

$$\mathbf{e}^r = (-\cos\theta^r, \, 0 \, , \, \sin\theta^r) \tag{4.31}$$

with θ^r the *angle of reflection*, as shown in Figure 7b.

We next define $\alpha^r = \boldsymbol{\alpha}^r \cdot \mathbf{j}$, $\beta^r = \boldsymbol{\beta}^r \cdot \mathbf{j}$, so that

$$\mathbf{E} \cdot \mathbf{j} = \alpha^i \cos\Omega^i + \alpha^r \cos\Omega^r + \beta^r \sin\Omega^r \ . \tag{4.32}$$

By (4.29), this must vanish at $x = 0$. In fact, if we further put $z = 0$ then $\Omega^i = \omega^i\, t$, $\Omega^r = \omega^r\, t$, and thus

$$f(t) \equiv \mathbf{E} \cdot \mathbf{j} \ |_{x=z=0} = \alpha^i \cos\omega^i\, t + \alpha^r \cos\omega^r\, t + \beta^r \sin\omega^r\, t = 0 \ . \tag{4.33}$$

This is to hold for all t. Hence

$$f(0) = 0 \quad \Rightarrow \quad \alpha^i + \alpha^r = 0 \tag{4.34}$$

$$\dot{f}(0) = 0 \quad \Rightarrow \quad \beta^r = 0 \tag{4.35}$$

$$\ddot{f}(0) = 0 \quad \Rightarrow \quad (\omega^i)^2\, \alpha^i + (\omega^r)^2\, \alpha^r = 0 \ . \tag{4.36}$$

Combining (4.34) and (4.36) implies $\omega^i = \omega^r \equiv \omega$, and thus

[15] The alert reader will notice that the derivation in section 1.6 was for electro*statics*. However, the same formula (1.31) holds in general. To see this, one notes that the additional surface integral of $\partial\mathbf{B}/\partial t$ in (1.29) tends to zero as $\varepsilon \to 0$.

The reflected frequency (colour) is the same as the incident frequency (colour).

We have now reduced (4.29) to

$$\alpha\left[\cos\frac{\omega}{c}\left(ct - z\sin\theta^i\right) - \cos\frac{\omega}{c}\left(ct - z\sin\theta^r\right)\right] = 0\ , \tag{4.37}$$

where $\alpha \equiv \alpha^i$. Thus $\sin\theta^i = \sin\theta^r$. Hence $\theta^i = \theta^r \equiv \theta$ and we have shown

R2: *The angle of incidence is equal to the angle of reflection.*

This is the second law of reflection.

Finally, note that (4.30) is simply $\mathbf{E}^r \cdot \mathbf{k} = 0$. Since $\mathbf{E}^r \cdot \mathbf{e}^r = 0$, from (4.31) we see that $\mathbf{E}^r$ is parallel[16] to $\mathbf{j}$. Hence

The reflected wave has the same polarization as the incident wave.

The final form of $\mathbf{E}$ (for $x < 0$) is then computed to be

$$\mathbf{E} = 2\alpha\sin\left[\frac{\omega}{c}\left(ct - z\sin\theta\right)\right]\sin\left[\frac{\omega}{c}x\cos\theta\right]\mathbf{j}\ . \tag{4.38}$$

[16]Note this follows only if $\theta \neq \pi/2$. If $\theta = \pi/2$ the waves are propagating *parallel* to the mirror, in the z-direction.

A Summary: vector calculus

The following is a summary of some of the key results of the Moderations course "Calculus in Three Dimensions and Applications". As in the main text, all functions and vector fields are assumed to be sufficiently well-behaved in order for formulae to make sense. For example, one might take everything to be *smooth* (derivatives to all orders exist). Similar remarks apply to (the parametrizations of) curves and surfaces in $\mathbb{R}^3$.

A.1 Vectors in $\mathbb{R}^3$

We work in $\mathbb{R}^3$, or a domain therein, in Cartesian coordinates. If $\mathbf{e}_1 = \mathbf{i} = (1,0,0)$, $\mathbf{e}_2 = \mathbf{j} = (0,1,0)$, $\mathbf{e}_3 = \mathbf{k} = (0,0,1)$ denote the standard orthonormal basis vectors, then a position vector is

$$\mathbf{r} = \sum_{i=1}^{3} x_i \, \mathbf{e}_i \tag{A.1}$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$ are the Cartesian coordinates in this basis. We denote the Euclidean length of $\mathbf{r}$ by

$$|\mathbf{r}| = r = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{A.2}$$

so that $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector for $\mathbf{r} \neq \mathbf{0}$. A vector field $\mathbf{f} = \mathbf{f}(\mathbf{r})$ may be written in this basis as

$$\mathbf{f}(\mathbf{r}) = \sum_{i=1}^{3} f_i(\mathbf{r}) \, \mathbf{e}_i \ . \tag{A.3}$$

The *scalar product* of two vectors $\mathbf{a}$, $\mathbf{b}$ is denoted by

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_i \tag{A.4}$$

while their *vector product* is the vector

$$\mathbf{a} \wedge \mathbf{b} = (a_2 b_3 - a_3 b_2) \, \mathbf{e}_1 + (a_3 b_1 - a_1 b_3) \, \mathbf{e}_2 + (a_1 b_2 - a_2 b_1) \, \mathbf{e}_3 \ . \tag{A.5}$$

A.2 Vector operators

The *gradient* of a function $\psi = \psi(\mathbf{r})$ is the vector field

$$\mathbf{grad} \, \psi = \nabla \psi = \sum_{i=1}^{3} \frac{\partial \psi}{\partial x_i} \mathbf{e}_i \ . \tag{A.6}$$

The *divergence* of a vector field $\mathbf{f} = \mathbf{f}(\mathbf{r})$ is the function (scalar field)

$$\text{div } \mathbf{f} = \nabla \cdot \mathbf{f} = \sum_{i=1}^{3} \mathbf{e}_i \cdot \frac{\partial \mathbf{f}}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} \tag{A.7}$$

while the *curl* is the vector field

$$\begin{aligned}\mathbf{curl}\,\mathbf{f} &= \nabla\wedge\mathbf{f} = \sum_{i=1}^{3}\mathbf{e}_i\wedge\frac{\partial\mathbf{f}}{\partial x_i}\\ &= \left(\frac{\partial f_3}{\partial x_2}-\frac{\partial f_2}{\partial x_3}\right)\mathbf{e}_1+\left(\frac{\partial f_1}{\partial x_3}-\frac{\partial f_3}{\partial x_1}\right)\mathbf{e}_2+\left(\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\right)\mathbf{e}_3\ .\end{aligned} \tag{A.8}$$

Two important identities are

$$\nabla\wedge(\nabla\psi) = \mathbf{0} \tag{A.9}$$

$$\nabla\cdot(\nabla\wedge\mathbf{f}) = 0\ . \tag{A.10}$$

Two more identities we shall need are

$$\nabla\wedge(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}\ , \tag{A.11}$$

$$\nabla\cdot(\mathbf{a}\wedge\mathbf{b}) = \mathbf{b}\cdot(\nabla\wedge\mathbf{a}) - \mathbf{a}\cdot(\nabla\wedge\mathbf{b})\ . \tag{A.12}$$

The second order operator ∇^2 defined by

$$\nabla^2\psi = \nabla\cdot(\nabla\psi) = \sum_{i=1}^{3}\frac{\partial^2\psi}{\partial x_i^2} \tag{A.13}$$

is called the *Laplacian*. We shall also use the identity

$$\nabla\wedge(\nabla\wedge\mathbf{f}) = \nabla(\nabla\cdot\mathbf{f}) - \nabla^2\mathbf{f}\ . \tag{A.14}$$

A.3 Integral theorems

Definition (Line integral) Let C be a curve in $\mathbb{R}^3$, parametrized by $\mathbf{r}:[t_0,t_1]\to\mathbb{R}^3$, or $\mathbf{r}(t)$ for short. Then the *line integral* of a vector field $\mathbf{f}$ along C is

$$\int_C \mathbf{f}\cdot\mathrm{d}\mathbf{r} = \int_{t_0}^{t_1}\mathbf{f}(\mathbf{r}(t))\cdot\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\,\mathrm{d}t\ . \tag{A.15}$$

Note that $\boldsymbol{\tau}(t) = \mathrm{d}\mathbf{r}/\mathrm{d}t$ is the *tangent vector* to the curve – a vector field on C. The value of the integral is independent of the choice of *oriented* parametrization (proof uses the chain rule).

A curve is *simple* if $\mathbf{r}:[t_0,t_1]\to\mathbb{R}^3$ is injective (then C is non-self-intersecting), and is *closed* if $\mathbf{r}(t_0)=\mathbf{r}(t_1)$ (then C forms a loop).

Definition (Surface integral) Let Σ be a surface in $\mathbb{R}^3$, parametrized by $\mathbf{r}(u,v)$, with $(u,v)\in D\subset\mathbb{R}^2$. The *unit normal* $\mathbf{n}$ to the surface is

$$\mathbf{n} = \frac{\mathbf{t}_u\wedge\mathbf{t}_v}{|\mathbf{t}_u\wedge\mathbf{t}_v|} \tag{A.16}$$

where

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u}, \qquad \mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} \tag{A.17}$$

are the two *tangent vectors* to the surface. These are all vector fields defined on Σ. The *surface integral* of a function ψ over Σ is

$$\int_\Sigma \psi \, \mathrm{d}S \; = \; \iint_D \psi(\mathbf{r}(u,v)) \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u \, \mathrm{d}v \; . \tag{A.18}$$

The *sign* of $\mathbf{n}$ in (A.16) is not in general independent of the choice of parametrization. Typically, the whole of a surface cannot be parametrized by a single domain D; rather, one needs to cover Σ with several parametrizations using domains $D_I \subset \mathbb{R}^2$, where I labels the domain. The surface integral (A.18) is then defined in the obvious way, as a sum of integrals in $D_I \subset \mathbb{R}^2$. However, in doing this it might not be possible to define a continuous $\mathbf{n}$ over the whole of Σ (*e.g.* the Möbius strip):

Definition (Orientations) A surface Σ is *orientable* if there is a choice of continuous unit normal vector field $\mathbf{n}$ on Σ. If an orientable Σ has boundary $\partial\Sigma$, a simple closed curve, then the normal $\mathbf{n}$ induces an orientation of $\partial\Sigma$: we require that $\boldsymbol{\tau} \wedge \mathbf{n}$ points *away* from Σ, where $\boldsymbol{\tau}$ denotes the oriented tangent vector to $\partial\Sigma$.

We may now state

Theorem A.1 *(Stokes) Let Σ be an orientable surface in $\mathbb{R}^3$, with unit normal vector $\mathbf{n}$ and boundary curve $\partial\Sigma$. If $\mathbf{f}$ is a vector field then*

$$\int_\Sigma (\nabla \wedge \mathbf{f}) \cdot \mathbf{n} \, \mathrm{d}S \; = \; \int_{\partial\Sigma} \mathbf{f} \cdot \mathrm{d}\mathbf{r} \; . \tag{A.19}$$

Definition (Volume integral) The integral of a function ψ in a (bounded) region R in $\mathbb{R}^3$ is

$$\int_R \psi \, \mathrm{d}V \; = \; \iiint_R \psi(\mathbf{r}) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \; . \tag{A.20}$$

Theorem A.2 *(Divergence) Let R be a bounded region in $\mathbb{R}^3$ with boundary surface ∂R. If $\mathbf{f}$ is a vector field then*

$$\int_R \nabla \cdot \mathbf{f} \, \mathrm{d}V \; = \; \int_{\partial R} \mathbf{f} \cdot \mathbf{n} \, \mathrm{d}S \tag{A.21}$$

where $\mathbf{n}$ is the outward *unit normal vector to ∂R.*

Note that the surface Σ in Stokes' theorem has a boundary $\partial\Sigma$, whereas the surface ∂R in the divergence theorem does not (it is itself the boundary of the region R).

We finally state two more results proved in the Moderations calculus course that we shall need:

Lemma A.3 *(Lemma 8.1 from "Calculus in Three Dimensions and Applications") If f is a continuous function such that*

$$\int_R f \, \mathrm{d}V = 0 \tag{A.22}$$

for all bounded regions R, then $f \equiv 0$.

Theorem A.4 *(Theorem 10.1 from "Calculus in Three Dimensions and Applications") Let $f(\mathbf{r})$ be a bounded continuous function with support $\{\mathbf{r} \in \mathbb{R}^3 \mid f(\mathbf{r}) \neq 0\} \subset R$ contained in a bounded region R, and define*

$$F(\mathbf{r}) = \int_{\mathbf{r}' \in R} \frac{f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V' \, . \tag{A.23}$$

Then F is differentiable on $\mathbb{R}^3$ with

$$\nabla F(\mathbf{r}) = - \int_{\mathbf{r}' \in R} \frac{f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, (\mathbf{r} - \mathbf{r}') \, \mathrm{d}V' \, . \tag{A.24}$$

Both F and ∇F are continuous and tend to zero as $r \to \infty$. Moreover, if f is differentiable then ∇F is differentiable with

$$\nabla^2 F = -4\pi f \, . \tag{A.25}$$