B7.1 Classical Mechanics

Sheet 1 — MT26

Section A: *introductory questions, with solutions provided*
Section B: *core questions to be handed in for marking, to be discussed in classes, solutions not provided for students*
Section C: *optional extension questions, with solutions provided*

Students are strongly encouraged not to consult the solutions for parts A & C before properly attempting the problems.

Section A

1. Consider a closed system consisting of N point particles with masses m_I, position vectors $\mathbf{r}_I$ in an inertial frame $\mathcal{S}$, such that particle J exerts a force $\mathbf{F}_{IJ}$ on particle I for $I \neq J$.

   (a) Explain why Newton's third law and Galilean invariance mean there exists an inertial frame $\mathcal{S}_0$ where the centre of mass

   $$\mathbf{R} = \frac{\sum_{I=1}^{N} m_I \mathbf{r}_I}{\sum_{I=1}^{N} m_I}$$

   is at rest at the origin.

   (b) Suppose now that there exist functions $V_{IJ} = V_{JI} = V_{IJ}(|\mathbf{r}_I - \mathbf{r}_J|)$, depending only on the distances $|\mathbf{r}_I - \mathbf{r}_J|$ between pairs of particles, such that $\mathbf{F}_{IJ} = -\partial_{\mathbf{r}_I} V_{IJ}$. Show that the total angular momentum and total energy

   $$E = \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 + \sum_{I<J} V_{IJ}(|\mathbf{r}_I - \mathbf{r}_J|)$$

   are conserved.

2. Consider the one-dimensional harmonic oscillator with action $S[q(t)] = \int_0^{2\pi} L(q, \dot{q})\, \mathrm{d}t$, Lagrangian

   $$L = L(q, \dot{q}) \ = \ \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m q^2 \ ,$$

   and with boundary conditions $q(0) = q(2\pi) = 1$. Determine the Lagrange equation of motion, and show that a solution is $q(t) = \cos t$. Is this solution unique? By considering the two paths $q_1(t) = 1$, $q_2(t) = \cos 2t$ show that the critical function $q(t)$ is neither a maximum nor a minimum of the action S.

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Section B

3. Consider the change of generalized coordinates $\mathbf{q} = \mathbf{q}(\tilde{\mathbf{q}}, t)$. Show that

$$\dot{q}_a = \sum_{b=1}^{n} \frac{\partial q_a}{\partial \tilde{q}_b} \dot{\tilde{q}}_b + \frac{\partial q_a}{\partial t}.$$

Defining

$$\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) \equiv L(\mathbf{q}(\tilde{\mathbf{q}}, t), \dot{\mathbf{q}}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t), t) \ ,$$

show that the Lagrange equations in the two coordinate systems are related via

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_a}\right) - \frac{\partial \tilde{L}}{\partial \tilde{q}_a} = \sum_{b=1}^{n} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_b}\right) - \frac{\partial L}{\partial q_b}\right] \frac{\partial q_b}{\partial \tilde{q}_a} \ .$$

Hence conclude that the Lagrange equations take the same form in all coordinate systems.

4. Consider the purely kinetic Lagrangian

$$L \ = \ T \ = \ \frac{1}{2} \sum_{a,b=1}^{n} g_{ab}(\mathbf{q}) \dot{q}_a \dot{q}_b \ ,$$

where we assume that the symmetric matrix $g_{ab} = g_{ba}$ is positive definite at each point and depends on the generalized coordinates $\mathbf{q}$ and is invertible at each point in configuration space. Show that Lagrange's equations take the form

$$\ddot{q}_a + \sum_{b,c=1}^{n} \Gamma^a_{bc} \dot{q}_b \dot{q}_c \ = \ 0 \ , \qquad a \ = \ 1, \ldots, n \ ,$$

where

$$\Gamma^a_{bc} \ \equiv \ \frac{1}{2} \sum_{d=1}^{n} (g^{-1})^{ad} \left(\frac{\partial g_{bd}}{\partial q_c} + \frac{\partial g_{cd}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_d} \right) \ .$$

The matrix of functions $g_{ab}(\mathbf{q})$ defines a *metric* on configuration space, and these Lagrange equations are known as the *geodesic equations*.

5. Consider a system of two light rods of equal length, smoothly jointed together, with the other two ends of the rods fixed at two points in a horizontal plane. A mass m is attached to the point where the rods are jointed but otherwise moves in three dimensions under the influence of gravity. Introduce an appropriate generalized coordinate for the system, and determine the Lagrangian.

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6. A *double pendulum* consists of a simple pendulum of mass m_1 and length l_1 pivoted at the origin, together with another simple pendulum of mass m_2 and length l_2, pivoted at the mass m_1. The whole system moves freely in a vertical plane under gravity. If θ_1 and θ_2 denote the angles each pendulum makes with the vertical, show that the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right] \\ &\quad + m_1 g l_1\cos\theta_1 + m_2 g(l_2\cos\theta_2 + l_1\cos\theta_1)\ . \end{aligned}$$

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Section C

7. Consider the central inverse-square-law force Lagrangian

$$L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + \frac{\kappa}{r} ,$$

where $r = |\mathbf{r}|$ and $\kappa > 0$ is a constant.

(a) Show by direct computation that the vector

$$\mathbf{A} = \mathbf{p} \wedge \mathbf{L} - m\kappa\frac{\mathbf{r}}{r}$$

is conserved, where $\mathbf{p} = m\dot{\mathbf{r}}$ and $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ are momentum and angular momentum about the origin, respectively.

[*Hint*: You may find it helpful to write $\dot{\mathbf{p}} = h(r)\frac{\mathbf{r}}{r}$ and show $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} \wedge \mathbf{L}) = -mh(r)r^2\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathbf{r}}{r})$.]

(b) Show that $\mathbf{A} \cdot \mathbf{L} = 0$ and $|\mathbf{A}|^2 = 2mE|\mathbf{L}|^2 + m^2\kappa^2$, where E is the conserved energy. Explain why $\mathbf{A}$ is a constant vector in the plane of the orbit.

(c) By taking the dot product $\mathbf{A} \cdot \mathbf{r}$ and assuming $\mathbf{L} \neq \mathbf{0}$ derive the orbit equation

$$\frac{1}{r} = \frac{m\kappa}{|\mathbf{L}|^2}\left(1 + \frac{|\mathbf{A}|}{m\kappa}\cos\theta\right) ,$$

where θ denotes the angle between $\mathbf{r}$ and $\mathbf{A}$. Notice we have found the orbit without solving any differential equation! The eccentricity is $|\mathbf{A}|/m\kappa$.

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