Quantum Theory

Sheet 4 — MT21

(Questions on sections 7–9 of the lecture notes)

1. A particle of mass m and charge e moving in a plane perpendicular to a magnetic field B has Hamiltonian

$$H = \frac{1}{2m}\left((P_1 + \tfrac{1}{2}eBX_2)^2 + (P_2 - \tfrac{1}{2}eBX_1)^2\right) ,$$

where we suppose that $eB \neq 0$ is constant. Show that the energy levels have the form

$$E = E_n = \frac{|eB|\hbar}{m}(n + \tfrac{1}{2}) ,$$

where n is a non-negative integer.

[*Hint: Introduce new operators P and X, proportional to $P_1 + \frac{1}{2}eBX_2$ and $P_2 - \frac{1}{2}eBX_1$, and show that the given Hamiltonian takes the same form as the harmonic oscillator Hamiltonian for a suitable choice of angular frequency ω.*]

2. The *spin representation* of angular momentum has $j = 1/2$, with angular momentum matrices $J_i = S_i$ given by

$$S_1 = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad S_2 = \frac{1}{2}\hbar\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad S_3 = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Here $J_3 = S_3$ has eigenvalues $\pm\frac{1}{2}\hbar$, with corresponding eigenstates $\psi_+ = (1,0)^T$, $\psi_- = (0,1)^T$, which are called *spin up* and *spin down* states. Introducing the unit vector $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in spherical polar coordinates, we define the operator $S_{\mathbf{n}} \equiv \sum_{i=1}^{3} n_i S_i$, which is the component of spin angular momentum in the direction $\mathbf{n}$.

(a) Show that

$$S_{\mathbf{n}} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix} .$$

(b) Hence, or otherwise, show that $(S_{\mathbf{n}})^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and deduce that $S_{\mathbf{n}}$ has eigenvalues $\pm\frac{\hbar}{2}$. Verify that the corresponding normalized eigenstates of $S_{\mathbf{n}}$ are

$$\psi_{\mathbf{n},+} = \cos\tfrac{\theta}{2}\,\psi_+ + \sin\tfrac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}\,\psi_- ,$$
$$\psi_{\mathbf{n},-} = \sin\tfrac{\theta}{2}\,\psi_+ - \cos\tfrac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}\,\psi_- .$$

(c) The spin angular momentum of S_3 is measured for a quantum system, obtaining the value $\frac{1}{2}\hbar$ (so that this is "spin up" along the z-axis). The spin angular momentum of $S_{\mathbf{n}}$ is now measured. Find the probabilities for obtaining the values $\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$.

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3. Recall that the orbital angular momentum operators L_i are in spherical polar coordinates given by

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right) , \qquad L_3 = -i\hbar\frac{\partial}{\partial\phi} ,$$

where $L_\pm = L_1 \pm iL_2$ are raising and lowering operators.

   (a) Using the fact that $L_+ Y_{\ell,\ell}(\theta,\phi) = 0$, hence show that $Y_{\ell,\ell}(\theta,\phi) = a_\ell(\sin\theta\, e^{i\phi})^\ell$, where a_ℓ is a normalization constant.

   (b) By applying the lowering operator L_- appropriately, hence find $Y_{\ell,m}(\theta,\phi)$ for $(\ell,m) = (1,1),(1,0),(1,-1),(2,2),(2,1)$ and $(2,0)$, up to overall normalization constants that you may ignore.

4. In a two-dimensional model of the hydrogen atom, the stationary state Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi ,$$

where (r,ϕ) are polar coordinates.

   (a) By separating the equation via $\psi(r,\phi) = R(r)\Phi(\phi)$, show that $\Phi(\phi)$ is a constant linear combination of $e^{i\ell\phi}$ and $e^{-i\ell\phi}$, where ℓ is a non-negative integer. [*Hint: use the fact that* $\Phi(\phi+2\pi) = \Phi(\phi)$.]

   (b) By further substituting $R(r) = f(r)e^{-\kappa r}$, where $\kappa = \sqrt{-2mE}/\hbar$, show that the radial equation becomes

$$f'' + \left(\frac{1}{r} - 2\kappa\right)f' - \left(\frac{\ell^2}{r^2} + \frac{\kappa-\beta}{r}\right)f = 0 ,$$

   where β is a constant you should identify.

   (c) By substituting a generalized power series expansion for f, of the form $f(r) = \sum_{k=0}^{\infty} a_k r^{k+c}$, argue that $c = \ell$ for a non-singular wave function, and in this case hence deduce the recurrence relation

$$a_k = \frac{2\kappa(k+\ell) - \kappa - \beta}{(k+\ell)^2 - \ell^2} a_{k-1} .$$

   (d) Hence or otherwise show that the energy levels are of the form $E_n = -\nu/(2n+1)^2$, where ν is a positive constant and n is a non-negative integer. [*Hint: appeal to normalizability to make the series terminate.*] What is the degeneracy of each energy level?

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5. (a) (i) Making use of any results you need from the lecture notes, show that the $(N, \ell, m) = (2, 1, 0)$ state of the hydrogen atom has wave function

$$\psi(r, \theta, \phi) = B\, r\, \mathrm{e}^{-r/2a} \cos\theta \ ,$$

where a is the Bohr radius and B is a normalization constant.

(ii) By normalizing $\psi = \psi(r, \theta, \phi)$, show that we may take $B = \sqrt{\frac{1}{32\pi a^5}}$.

(iii) Compute $\mathbb{E}_\psi(r)$, the expected value of the distance of the electron from the nucleus in this excited state.

(b) Recall that the normalized ground state wave function for an electron in a hydrogen-like atom with Z protons and any number A neutrons in the nucleus is

$$\psi = \sqrt{\frac{Z^3}{\pi a^3}}\, \mathrm{e}^{-Zr/a} \ ,$$

where a is the Bohr radius and r is distance from the nucleus.

An electron is in the ground state of tritium ($Z = 1$, $A = 2$). A nuclear reaction (β-*decay*) instantaneously changes the nucleus into a helium-3 ion $^3\mathrm{He}^+$ ($Z = 2$, $A = 1$). Show that the probability of measuring the electron to be in the ground state of $^3\mathrm{He}^+$ is $\frac{512}{729}$.

[*In both parts you may use the integral* $\int_0^\infty r^n\, \mathrm{e}^{-r/b}\, \mathrm{d}r = b^{n+1} n!$, *where* $b > 0$ *and* $n \in \mathbb{N}$.]

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