Introduction to Gauge Theory and Chern-Weil Theory

§1 Yang-Mills gauge theory

X: connected, closed, oriented n-dimensional Riemannian manifold.

$$\begin{array}{c} G \to P \\ \downarrow \pi \\ X \end{array}$$

G: compact Lie group
P: principal G-bundle.

Motivation

Standard model : $G = U(1) \times SU(2) \times SU(3)$ (electromagnetic, weak, strong)
(Not quantised in this talk $\Rightarrow$ classical.
Whole system Wick rotated $\Rightarrow$ Riemannian metric)
Not quite QFT, but the classical system is interesting enough.
(Instantons are (semi) classical objects.)

idea of field theory.
$\updownarrow$
Basic setup in terms of differential geometry, topology.
$\Rightarrow$ theory of connection, Chern-Weil theory.

Most important example is $X = S^4$, $G = SU(2)$.

Def.

(gauge field / potential.)
A connection 1-form ω is a $\mathfrak{g}(=\mathrm{Lie}(G))$-valued 1-form on P s.t.

(C1) $R_a^* \omega = \mathrm{Ad}_{a^{-1}}(\omega)$ $\quad \forall a \in G$
(C2) $\omega(A^*) = A$ $\quad \forall A \in \mathfrak{g}$
where A^* is the fundamental vector field defined as
$$A^*u = \frac{d}{dt}\Big|_{t=0} u \cdot \exp(tA)$$
(meaning tangent vector at $t=0$.)

(C1) : invariance

~~[illegible]~~

(C2) defines an isomorphism Vertical subspace $\simeq \mathfrak{g}$

$\Rightarrow$ defines the <u>horizontal space</u> H as $\ker(\omega)$.

Conceptually, ω gives a G-invariant splitting to

$$0 \to \mathfrak{g} \to T_uP \to H \to 0 \quad \left(\begin{array}{c}\text{noncanonical} \\ \text{without } \omega\end{array}\right)$$

ω, $H = \ker(\omega)$, $u \in P$.

<u>Curvature form</u> Ω is defined by

$$\Omega := d\omega + \tfrac{1}{2}[\omega, \omega] \quad \left(\mathfrak{g}\text{-valued 2-form on } P.\right)$$

$\{U_\alpha\}$: trivialising open cover on X, $\{t_{\alpha\beta}\}$: transition functions

Let $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha) \cong U_\alpha \times G$

$x \longmapsto (x, \mathrm{id}_G)$.

$\Rightarrow \Omega_\alpha := \sigma_\alpha^* \Omega$ is a $\mathfrak{g}$-valued form on $U_\alpha \subseteq X$.

(C1) $\Rightarrow \Omega_\beta = \mathrm{Ad}_{t_{\alpha\beta}^{-1}}(\Omega_\alpha)$.

<u>Prop.</u>

$\{\Omega_\alpha\}$ defines a $P \times_{\mathrm{Ad}} \mathfrak{g}$-valued 2-form on X, written $R \in A^2(P \times_{\mathrm{Ad}} \mathfrak{g})$ (<u>curvature</u> field strength tensor (observables)),

where $P \times_{\mathrm{Ad}} \mathfrak{g} := P \times \mathfrak{g} / [u, A] \sim [ug, \mathrm{Ad}_{g^{-1}}(A)]$

called the <u>adjoint bundle</u>

(vector bundle associated to P with fibre $\mathfrak{g}$)

Now let $\mathcal{C}$ be the set of all connections on P.

Recall the Hodge star operator $* : \Lambda^p T^*X \to \Lambda^{n-p} T^*X$

defined by
$$\alpha \wedge * \beta = g(\alpha, \beta)\, \underbrace{\mathrm{vol}(X)}_{\text{volume form}}$$
$\alpha, \beta \in A^p(X)$,

Def.

Yang-Mills functional $\mathcal{YM}: \mathcal{C} \to \mathbb{R}$ is a ~~[illegible]~~ functional defined by

$$\mathcal{YM}(\omega) := \frac{1}{2}\int_X \langle R(\omega) \wedge * R(\omega) \rangle$$

where $\langle , \rangle$ is an Ad_G-invariant inner product on $\mathfrak{g}$.
(exists since G is compact $\Rightarrow {}^\exists$ Haar measure
If G is semisimple, $-\langle , \rangle =$ killing form)

$\langle R \wedge * R \rangle$ means: $R = \frac{1}{2}\sum_{i,\alpha,\beta} F^i_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta \otimes B_i$

$\{\theta^\alpha\}$: o.n.b. for $T^*_x X$
$\{B_i\}$: o.n.b. for $\mathfrak{g}$ w.r.t. $\langle , \rangle$.

$$\Rightarrow \langle R \wedge * R \rangle = \sum_{i,\alpha,\beta} |F^i_{\alpha\beta}|^2 * 1.$$

(C1), (C2) imply: $\omega_0, \omega_1 \in \mathcal{C} \Rightarrow t\omega_0 + (1-t)\omega_1 \in \mathcal{C}$.
i.e. $\mathcal{C}$ is an affine space.
$\Rightarrow$ Fix $\omega \in \mathcal{C}$ once and for all and regard $\mathcal{C}$ as a vector space with "origin" ω.
Then,
$$\omega' = \omega + \alpha \qquad \underline{\alpha \in A^1(P \times_{Ad} \mathfrak{g})}$$
for ${}^\forall \omega' \in \mathcal{C}$.

$\Updownarrow$

$\boxed{\alpha \in A^1(P) \text{ s.t. } R_a^* \alpha = Ad_{a^{-1}}(\alpha),\ \alpha(A^*) = 0\ \forall A \in \mathfrak{g}}$ ((C1), (C2))

We compute the 1st variation of $\mathcal{YM}$. ~~[illegible]~~

Since $\mathcal{C}$ is a vector space, it is enough to set
$\omega_t := \omega + t\alpha \quad \alpha \in A^1(P \times_{Ad} \mathfrak{g})$ and compute

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}(\omega_t) = \int_X \langle D^\omega \alpha \wedge * R(\omega) \rangle$$

(complicated calculation)

$$= \int_X \langle \alpha \wedge * (\underbrace{D^{\omega *}}_{\text{formal adjoint}} R(\omega)) \rangle = 0 \quad \forall \alpha.$$

(Note $\langle \cdot \wedge * \cdot \rangle$ defines an inner product.)

($D^{\omega *} = * D^{\omega} *$ on 2-forms.)

Thus we get the Yang-Mills equation

$$D^{\omega *} R(\omega) = 0.$$

As for the 2^{nd} variation, we say ω is weakly stable if (ω satisfies YM eq. and)

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{YM}(\omega_t) \geq 0 \qquad \forall \alpha \in A^1(P \times_{Ad} \mathfrak{g})$$

$$\left(\overset{\shortparallel}{\int_X \langle D^{\omega *} D^{\omega} \alpha + 2 * (* \Omega \wedge \alpha) \wedge * \alpha \rangle} \right)$$

Calculation in local coordinates shows

Theorem (Bourguignon - Lawson - Simons, 1981).

If $X = S^n$, $n \geq 5$ no solution of Yang-Mills equation is weakly stable (unless $R = 0$).
($\Rightarrow$ energy functional unbounded from below.)

Def.

$$\mathcal{G}(P) := \left\{ \varphi \in \mathrm{Diff}(P) \;\middle|\; \begin{array}{ll} \varphi(pg) = \varphi(p) g & p \in P \\ \pi \circ \varphi = \pi & g \in G \end{array} \right\}$$

is called the group of gauge transformations.

Action of $\mathcal{G}$: trivialise over $U \subseteq X$,
$\pi^{-1}(U) \simeq U \times G$.
Then, $\varphi \in \mathcal{G}$ acts as

$$\varphi|_{\pi^{-1}(U)}(x, g) = (x, f(x) g) \in U \times G$$

for some G-valued function $f : U \to G$.

$\mathcal{G}(P)$ locally "acts like a transition function".

Recall that the coordinate change formula for R was

$$\Omega_\beta = \mathrm{Ad}_{\psi_{\alpha\beta}^{-1}}(\Omega_\alpha)$$

$$\Rightarrow \varphi^*|_u \Omega_u = \mathrm{Ad}_{\underbrace{f(x)^{-1}}_{\text{inverse in } G}}(\Omega_u)$$

$\Rightarrow$ Since $\langle , \rangle$ is Ad_G-invariant, $\mathcal{YM}$ is invariant under $\mathcal{G}$. (even before integration $\int_X$)

$\Rightarrow$ $\mathcal{G}$ acts on the space of solutions of YM equation.

$\Rightarrow$ moduli space ...

§2 Chern-Weil theory

Main Theorem of Chern-Weil Theory

$$G \to P \xrightarrow{\pi} X$$

Let $f : \mathrm{Sym}^k \mathfrak{g} \to \mathbb{R}$ be an Ad_G-invariant symmetric multilinear form on $\mathfrak{g}$.
Then, there exists a differential $2k$-form on X, written $w(f)$ (Weil homom), s.t.

$$f(\Omega, \dots, \Omega) = \pi^* w(f)$$

$$\overset{\|}{f(\Omega)}$$

($\mathfrak{g}$-valued 2-form)

Moreover, $w(f)$ is closed and its cohomology class $[w(f)] \in H^{2k}(X; \mathbb{R})$ is independent of connection used to define Ω.

Examples are all for matrix Lie group G.

$\Rightarrow$ Note $w(f)$ is a homogeneous poly in R.

$\because$ Apply $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha) \cong U_\alpha \times G$, $x \mapsto (x, \mathrm{id.})$

$$\Rightarrow \sigma_\alpha^* f(\Omega) = \sigma_\alpha^* \pi^* w(f) = w(f)$$

$\overset{\|}{f(R_\alpha)}$ Locally true. Globally true because of Ad_G-invariance. //

(makes sense since R is a 2-form with values in matrix Lie algebra $\mathfrak{g}$.)

(So corresponding to each Ad_G-invariant <u>homogeneous polynomial in R</u> of degree k, we find a characteristic class of a principal G-bundle P.)

Examples

$$\det(I+tX) = 1 + t\sigma_1(X) + t^2\sigma_2(X) + \cdots$$

$$S_k(X) = \mathrm{tr}(X^k)$$

$$E = P \times_{\text{natural}} \begin{matrix}\mathbb{C}^r \\ \mathbb{R}^r\end{matrix} \text{ for each } G.$$

(associated vector bundle.)

1. $G = U(r)$.

   k-th <u>Chern class</u>: $c_k(E) := \left(\frac{\sqrt{-1}}{2\pi}\right)^k \sigma_k(R)$.

   <u>Chern character</u>: $ch(E) = r + ch_1 + ch_2 + \cdots$

   $$ch_k = \frac{1}{k!} S_k\left(\frac{\sqrt{-1}}{2\pi} R\right).$$

2. $G = O(r)$

   k-th <u>Pontrjagin class</u> $p_k(X) = \left(\frac{1}{2\pi}\right)^{2k} \sigma_{2k}(R)$.

3. $G = SO(2m)$

   <u>Euler class</u> $e(E) := \left(\frac{1}{2\pi}\right)^m \det{}^{1/2}(R)$

   Pfaff $= 0$ for odd rank

   (Pfaffian defined for $2m\times 2m$ skew-sym. matrices. choice of root $\Leftrightarrow$ orientation. $(\mathrm{Pfaff})^2 = \det$.)

4. $G = U(r)$

   <u>Todd class</u>

   $$Td(E) = \det\left(\frac{R/2\pi\sqrt{-1}}{e^{R/2\pi\sqrt{-1}} - 1}\right)$$

   makes sense as a formal power series.

   $$= 1 + \frac{1}{2}c_1(E) + \frac{1}{12}\left(c_1(E)^2 + c_2(E)\right) + \cdots$$

5. <u>$G = SO(2m)$</u>

   <u>L-class</u>

   $$L(E) := \det{}^{1/2}\left(\frac{R/2\pi}{\tanh(R/2\pi)}\right) = 1 + \frac{1}{3}p_1(E) + \cdots$$

6. $G = SO(2m)$

$\hat{A}$ - class

$$\hat{A}(E) = \det{}^{1/2}\left(\frac{R/2\pi}{\sinh(R/2\pi)}\right)$$
$$= 1 - \frac{1}{24}p_1(E) + \cdots$$

Chern-Simons form

Pullback bundle
$$\begin{array}{ccc} \pi^*P & & P \\ \downarrow & & \downarrow \\ P & \xrightarrow[\pi]{} & X \end{array}$$
over P is trivial

$\Rightarrow$ $f(\Omega)$ is exact as a form on P.

can show

$\Rightarrow \exists\, \underset{(\text{form})}{w(Tf)}$ s.t. $dw(Tf) = f(\Omega)$

explicit recipe for constructing Tf.

$\Rightarrow$ • important in Chern-Simons theory (QFT on 3-mflds.)
(another noncommutative gauge theory)

• secondary characteristic class, which is nontrivial even when classical ones vanish.

§3. Yang-Mills theory on 4-manifolds.

Key feature in dimension 4 is the self-duality:

$$*: \Lambda^2 T^*X \to \Lambda^{4-2} T^*X = \Lambda^2 T^*X$$

satisfies $*^2 = (-1)^{2(4-2)} = +1$, i.e. $*$ is an automorphism of $\Lambda^2 T^*X$.

$\Rightarrow$ This induces an splitting
$$\Lambda^2 T^*X = \Lambda^2_+ T^*X \oplus \Lambda^2_- T^*X$$
into ± 1-eigenspaces

Write $R = R_+ \oplus R_-$ accordingly.

Crucially, curvature R is a 2-form.

Now, $R \wedge * R = (R_+ \wedge * R_+) + (R_- \wedge * R_-)$
$R \wedge R = (R_+ \wedge * R_+) - (R_- \wedge * R_-)$
by calculation.

Thus,

$$\mathcal{YM}(\omega) = \underbrace{\frac{1}{2}\int_X \langle R(\omega) \wedge R(\omega) \rangle}_{\displaystyle \overset{!!}{c(P,\omega)}} + \int_X \langle R_-(\omega) \wedge * R_-(\omega) \rangle$$

The first term is an Ad$_G$-invariant bilinear form in R
$\Rightarrow$ Chern-Weil theory tells that this is a closed form on X, and its cohomology class is independent of ω.
$\Rightarrow c(P,\omega) = c(P)$, characteristic number.

Example $G = SU(2)$ $\quad (\langle A, B \rangle = -\mathrm{tr}(AB))$.
We can show

$$c(P) = -8\pi^2 \int_X c_2(E) \qquad E = P \times_{\text{natural}} \mathbb{C}^2$$

$k := -\int_X c_2(E)$
is called topological charge (instanton number)

Assume $c(P) \geq 0$ (w.l.o.g.) $\quad$ ($c(P) \leq 0 \Rightarrow$ anti-self-dual or switch orientation.)

$$\mathcal{YM}(\omega) = \frac{1}{2} c(P) + \int_X \langle R_-(\omega) \wedge * R_-(\omega) \rangle$$

$$\geq \frac{1}{2} c(P) \geq 0$$

$\Rightarrow \mathcal{YM}$ is minimised iff $R_-(\omega) = 0 \Leftrightarrow *R(\omega) = R(\omega)$.

Def.
$\omega \in \mathcal{C}$ is self-dual if $*R(\omega) = R(\omega)$.
Self-dual connection is called an instanton (pseudoparticle solution in vacuum).

Note, if ω is self-dual,

$$D^{\omega *}R = *D^{\omega}\underset{\overset{\shortparallel}{R}}{\underline{*R}} = *D^{\omega}R = 0$$

by Bianchi identity.

<u>Remarks</u>. (significance of $c(P)$)

- $c(P) = 0 \Rightarrow$ only $R=0$ minimises $\mathcal{YM}$ (trivial solution)
- $c(P) > 0 \Rightarrow$ non-trivial minimum of $\mathcal{YM}$
  $\Rightarrow$ non-trivial solution of YM eq. forced by topology of P.

The moduli space of instantons is

$$\mathcal{M}^{+} := \{\text{instantons}\} / \mathcal{G}(P).$$

(Under some extra hypotheses (G semisimple, etc) this is a finite dimensional manifold with possible <u>cone</u> singularities in $\mathbb{CP}^2$.)

§4 Seiberg-Witten theory

$Spin^c(n)$ group is defined as

$$Spin^c(n) := Spin(n) \times U(1) \big/ (1,1) \sim (-1,-1).$$

$\Rightarrow$ double cover

$$0 \to \mathbb{Z}_2 \to Spin^c(n) \xrightarrow{\xi_0 := Ad \times id} SO(n) \times U(1) \to 1.$$

Spin^c-structure is a principal Spin^c-bundle over X which admits ~~an~~ Spin^c- equivariant bundle mapping

$$Prin(Spin^c(n)) \xrightarrow{\xi} P(TX, SO) \times Prin(U(1))$$

s.t.

$$\xi(pg) = \xi(p)\xi_0(g) \quad (\text{Spin}^c\text{-equivariance})$$

$$p \in Prin(Spin^c(n)),\ g \in Spin^c(n).$$

Def.

A spin^c-structure on $P(TX, SO(n))$ consists of a principal $U(1)$ bundle P_{U_1} and a principal $Spin^c(n)$ bundle P_{Spin^c} with a $Spin^c$-equivariant bundle map

$$P_{Spin^c} \longrightarrow P_{SO_n} \times P_{U_1}.$$

Def.

An oriented Riemannian mfd with a $Spin^c$-structure on TX is called a $Spin^c$-mfd.

Thm

Any 4-dim'l, compact oriented Riemannian mfds are $Spin^c$.

(Hirzebruch-Hopf etc.)

Get a connection A on P_{U_1}

$\Rightarrow$ $Spin^c(n) \longrightarrow SO(n) \times U(1)$ is a double cover

$\Rightarrow$ Levi-Civita on $P(TX, SO(n))$ and A on $U(1)$ lifts to a connection on the principal $Spin^c(n)$-bundle.

We can define the spinor bundle on X by noting

$$\Delta_{\mathbb{C}} \times z : Spin(n) \times U(1) \to U(N)$$
$$(g, z) \mapsto \Delta(g) z$$
$$\downarrow$$
$$Spin^c(n)$$

So we can define $\$_{\mathbb{C}}$

4-mfd $\Rightarrow$ $\$_{\mathbb{C}} = \$_{\mathbb{C}}^+ \oplus \$_{\mathbb{C}}^-$

SW eq.

$$D\!\!\!/_A \psi = 0 \quad \text{(Dirac eq.)}$$

$$F_A^+ = q(\psi) = \psi \otimes \psi^* - \frac{|\psi|^2}{2} id$$

(trace-free end)

under the identification $\Lambda_+^2 T^*X \cong$ trace-free $End(\$_{\mathbb{C}}^+)$

Gauge: $\mathcal{G}: \{\sigma \in \Gamma(P \times_{Ad} Spin^c) \mid \xi(\sigma) \in \Gamma(P \times_{Ad} SO(4)) \text{ is id.}\}$

$\cong \{\text{maps } M \to S^1\}$

~~X: simply connected to~~

So $\mathcal{G} \curvearrowright Sol$.

$\mathfrak{M} := Sol/\mathcal{G}$

By perturbing $F_A^+ = q(\psi)$ to

$F_A^+ + \delta = q(\psi) \qquad q \in \Gamma(\sqrt{-1}\Lambda_+^2 T^*X)$

and perturbing the metric,

we can set $\psi \neq 0$

$(\psi = 0 \Rightarrow \sigma \in S^1 \text{ fixes } (A, 0))$.

$\Rightarrow$ $\mathfrak{M}$ is a compact f.d. mfd.

X: compact oriented simply connected smooth 4-mfd, spin^c.

$b_+(M) > 0$ cobordism invariant.

$\Rightarrow \mathfrak{M}_\delta$ for generic δ is an oriented smooth mfd of dimension

$\dim \mathfrak{M}_\delta = 2\,\mathrm{ind}(\mathcal{D}_A^+) - b_+ - 1$

$= \int_M c_1(L)^2 - \frac{1}{2}\chi(X) - \frac{3}{4}\tau(X)$

pt $\in X$ fixed.

$\mathcal{G}_0 = \{\sigma \in \mathcal{G} \mid \sigma(pt) = 1 \in S^1\}$

$\Rightarrow Sol/\mathcal{G}_0 \to Sol/\mathcal{G}$ is an S^1-bundle Q_{S^1}.

$d = \dim \mathfrak{M}$ is even $\Rightarrow SW(L) = \int_{\mathfrak{M}_\delta} c_1^{d/2}(Q_{S^1})$