INTRODUCTION TO C^*-ALGEBRAS

TERUJI THOMAS

1. The rough idea: non-commutative geometry

As usual, in "the classical picture" there's some space S of 'states', and a commutative algebra of 'observables', functions on S. The idea of algebraic geometry is that, at least in good cases, one can study S purely in terms of the observables.

As usual, in "the quantum picture" there's some Hilbert spaces $\mathcal{H}$ of 'states', and a *non*-commutative algebra of 'observables', operators on $\mathcal{H}$. The idea of algebraic quantum field theory is that on should focus on the observables, and forget about $\mathcal{H}$. Can we define states in terms of observables? What kind of geometric structure do they have?

The theory of C^*-algebras gives a framework for understanding *both* the classical and quantum pictures.

2. Classical

We'll take S to be a compact, Hausdorff space. (The compactness assumption just makes things neater – it's not fundamental.) Elements of S are 'pure states.' In general, we define a state to be a probability measure μ on S. The usual explanation: "When we do an experiment, we can't specify the initial state precisely, only probabilistically." A pure state is a special kind of state: $s \in S$ corresponds to the delta-measure δ_s.

Let $C(S)$ be the space of continuous $\mathbb{C}$-valued functions on S. A probability measure μ is a special kind of functional $E_\mu\colon C(S) \to \mathbb{C}$

$$E_\mu(f) = \int_S f\, d\mu.$$

(E stands for 'expectation'; $E_\mu(f)$ is the expected value of f in the sense of probability theory).

The space of probability measures is *convex* and the extreme points are just the pure states. (Very roughly: A probability measure is a convex continuous sum of delta-measures.)

3. C^*-algebras

A [unital] C^*-algebra is

(a) A unital, associative $\mathbb{C}$-algebra $\mathcal{A}$ with
(b) A complete norm $\|\cdot\|$ with $\|1\| = 1, \|ab\| \leq \|a\|\|b\|$;
(c) An involution $*$ (an anti-linear anti-automorphism with $** = \mathrm{id}$); satisfying
(d) the C^*-condition: $\|aa^*\| = \|a\|^2$.

Some consequences: $*$ is automatically an isometry; any algebra homomorphism between C^* algebras, preserving $*$, is automatically continuous. The assumption that $\mathcal{A}$ is unital is again for simplification. It corresponds to the assumption that S is compact.

Example 3.1. $\mathcal{A} = \mathbb{C}$ with the usual norm and complex conjugation.

Example 3.2. $\mathcal{A} = C(S)$. The norm is $\|f\| = \sup_{x\in S} |f(x)|$, and $*$ is complex conjugation.

Theorem 3.1 (Gelfand)**.** *The association $S \to C(S)$ is an equivalence between compact Hausdorff spaces and commutative C^*-algebras. The inverse associates to each algebra $\mathcal{A}$ the set of maximal ideals in $\mathcal{A}$ (with the weak-$*$ topology).*

Example 3.3. $\mathcal{H}$ a Hilbert space; $\mathcal{A} = B(\mathcal{H})$, the bounded operators on $\mathcal{H}$. The involution is Hermitian conjugation; the norm is the operator norm $\|f\| := \sup |f(x)|/|x|$ (the condition that f is bounded just means that this sup exists).

Example 3.4. A $*$-closed subalgebra $\mathcal{A} \subset B(\mathcal{H})$ will be a C^*-algebra just in case it is closed in the uniform topology (i.e. the restriction of the norm to $\mathcal{A}$ is complete). Such C^*-algebras are called 'concrete' (much like a 'concrete group' is a subgroup of a permutation group).

A *representation* of $\mathcal{A}$ is a homomorphism $\mathcal{A} \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Theorem 3.2 (Gelfand-Naimark)**.** *Every $\mathcal{A}$ admits a faithful representation (so can be realised as a concrete C^*-algebra).*

Remark 3.1. $B(\mathcal{H})$ is in fact a "W^*" or "von Neumann" algebra – it means that it is the dual of a Banach space (here, the space of trace-class operators with the trace norm). A concrete C^*-algebra is a W^*-algebra just in case it is weak-$*$ closed. Any concrete C^*-algebra has an associated W^*-algebra – its weak-$*$ closure. As far as I understand, W^*-algebras have a nice classification and are generally a bit easier to handle.

4. States

For any C^*-algebra $\mathcal{A}$, we have a positive cone $\mathcal{A}^+ = \{aa^* : a \in \mathcal{A}\}$. For example, if $\mathcal{A} = \mathbb{C}$, then $\mathcal{A}^+ = \mathbb{R}^+$ the nonnegative real numbers.

Definition 4.1. A *state* is a linear map $E\colon \mathcal{A} \to \mathbb{C}$ that is positive ($E(\mathcal{A}^+) \geq 0$) and normalised ($\|E\| = 1$, where $\|E\| = \sup |E(x)|/|x|$).

Example 4.1. $\mathcal{A} = C(S)$. States are just probability measures on S.

Example 4.2. $\mathcal{A} = B(\mathcal{H})$ (or, more generally, $\mathcal{H}$ is a representation of $\mathcal{A}$). Given $v \in \mathcal{H}$, define $E_v(a) = (v, av)$. This is a state as long as v is normalised so that $(v, v) = 1$. These are called 'vector states' (relative to $\mathcal{H}$). These are the most traditional kinds of states in quantum mechanics.

Example 4.3. A bit more generally, if ρ is a trace-class operator on $\mathcal{H}$, then set $E_\rho(a) = \mathrm{tr}(\rho a)$. I think this is a state as long as (i) ρ is positive; and (ii) $\|\rho\|_{\mathrm{tr}} = 1$. Here the "trace norm" is $\|\rho\|_{\mathrm{tr}} = \mathrm{tr}\sqrt{aa^*}$ (in particular, if ρ is positive hence self-adjoint, it's just the trace). We call E_ρ a "normal state" (again relative to $\mathcal{H}$). A vector state E_v is a normal state: $E_v = E_\rho$ where $\rho(x) = (v, x)v$. Roughly speaking, a general state-defining ρ is an infinite convex sum of vector states – something like a probability distribution over vector states. This is what is physicists call a 'density matrix'.

Theorem 4.1 (Gelfand-Naimark-Segal)**.** *Every state E is a vector state with respect to some representation $\mathcal{H}_E$.*

Proof. Given E, define an Hermitian inner product on $\mathcal{A}$ by $(a, b) = E(a^*b)$. Let $I \subset \mathcal{A}$ be the set of norm-0 elements. It turns out to be a left ideal. Let $\mathcal{H}_E$ be the completion of $\mathcal{A}/I$. Let $\Omega \in \mathcal{H}_E$ be the image of $1 \in \mathcal{A}$. Then it is easy to check that $E_\Omega = E$. □

So it's not quite true that the states are naturally all vectors in a Hilbert space – they are vectors in *different* Hilbert spaces.

Example 4.4. $\mathcal{A} = C(S)$. If μ is a probability measure, then $\mathcal{H}_E = L^2(S, \mu)$.

5. Pure states and Folia

Theorem 5.1. *The set of states of $\mathcal{A}$ is convex and compact (in the weak-$*$ topology).*

(This follows pretty easily from the Banach-Alaoglu theorem.)

Definition 5.1. A *pure* state is an extreme point in the set of states.

Theorem 5.2. *$\mathcal{H}_E$ is an irreducible representation if and only if E is pure.*

A bit more precisely: if $E'' = pE + (1-p)E'$ then $\mathcal{H}_E, \mathcal{H}_{E'}$ are naturally quotient (hence sub) representations of $\mathcal{H}_{E''}$. This shows that, certainly, impure states can't give irreducible representations.

Example 5.1. $\mathcal{A} = C(S)$. Irreducible reps are 1-dimensional, and correspond exactly to maximal ideals, i.e. elements of S, i.e. pure states.

Given a representation $\mathcal{H}$, the *folium* of $\mathcal{H}$ is the set of normal states in $\mathcal{H}$. In particular, each state E determines a folium, the folium of $\mathcal{H}_E$. I *think* this is an equivalence relation on states, but I'm not sure [hmm, now it seems pretty unlikely].

Theorem 5.3. *The folium of a faithful representation is dense in the space of states.*

Example 5.2. $\mathcal{A} = C(S)$. As far as I can make out, each pure state is its own folium. I think that folia correspond to closed subsets of $\mathcal{A}$ – the folium of a probability measure is the support of the measure. If that's right, then the only dense folium consists of measures with support equal to S, so that $\mathcal{H}_E = L^2(S)$, which is certainly a faithful representation of $C(S)$.

5.4. **Summary.** In the classical picture, to a commutative C^*-algebra we associate a set of states. The extreme ones are *pure* states, and form a compact Hausdorff space; they correspond to maximal ideals. A state in general is a probability measure on the pure states. We've generalised this to a non-commutative C^*-algebra: we still get a compact, convex space of states, and the extreme ones parameterise irreducible representations. The states are also divided into folia, corresponding to the 'support' of a measure.

6. Example

Here's a non-commutative example. Let $\mathcal{H} = \mathbb{C}^2$, so $\mathcal{A} = B(\mathcal{H})$ is the algebra of 2×2 matrices. The invoution is just the conjugate-transpose. The map $(a, b) \mapsto \mathrm{tr}(ab)$ identifies $\mathcal{A}$ with its linear dual. So given $a \in \mathcal{A}$, we get a functional $E_a(b) = \mathrm{tr}(ab)$, and all functionals are of this form. Question: for which a is E_a a state?

First, it turns out that $\mathcal{A}^+$ is self-dual (i.e. E_a is positive if and only if $a \in \mathcal{A}^+$). The norm of E_a is just the trace-norm of a (recall that in general $B(\mathcal{H})$ is dual to the trace-class operators with the trace norm); for $a \in \mathcal{A}^+$, this is just the trace. So E_a is a state if and only if a is of the form $a = bb^*$ with $\mathrm{tr}(a) = 1$. A bit of calculation shows that this is equivalent to a being of the form

$$\begin{pmatrix} z & x+iy \\ x-iy & 1-z \end{pmatrix}$$

with $z \geq 0$ and $(z - \frac{1}{2})^2 + x^2 + y^2 \leq \frac{1}{4}$. So the states form a ball of radius 1/2 around $a = \mathrm{diag}(1/2, 1/2)$. The pure states form a sphere S^2.

If E is a pure state, then $\mathcal{H}_E$ is equivalent to the standard representation. If E is impure, then $\mathcal{H}_E$ is equivalent to the representation of $\mathcal{A}$ acting on itself by left-multiplication.

Remark 6.1. Note that proper left ideals in $\mathcal{A}$ correspond to lines in $\mathcal{C}^2$ (a line L gives the ideal I_L of matrices annihilating L). These lines form $\mathbb{P}^1 \cong S^2$. And $\mathcal{A}/I_L$ is an irreducible representation. This is surely essentially the same parametrisation of irreps by S^2, but I didn't work out the details.