QUANTIZATION OF ELEMENTARY DYNAMICAL SYSTEMS I (WILL DONOVAN)

1. Group actions and irreducibility

We've discussed the general problem of associating to a symplectic manifold (M, ω) a Hilbert space $\mathcal{H}$ with an irreducible representation of $C^\infty(M)$ (or, at least, a subalgebra). Now we consider the situation in which M carries an action of a Lie group G. Under some conditions (namely, when the action is 'Hamiltonian') we will get a representation of G on $\mathcal{H}$, and, at least naively, if G acts transitively on M, then the G-representation should be irreducible.

Remark 1.1. The 'states' in quantum theory are rays in $\mathcal{H}$, so what one really expects to get is a *projective* representation of G. An equivalent way to say this is that we may get a representation not of G but of some central extension (e.g. a covering group). This is what happened in the Heisenberg case, in two ways: (1) When $G = V$ acted on V by translations, we got a representation not of G but of the Heisenberg group, a central extension of V; (2) when $G = \mathrm{Sp}(V)$, we got a representation not of G but of $\mathrm{Mp}(V)$, a covering group.

Remark 1.2. There was some discussion, not yet conclusive, about the physical meaning of irreducibility (either under $C^\infty(M)$ or under G). Here are some comments that more or less came out of that discussion.

If we think about $C^\infty(M)$ as a Lie algebra acting on M, then the question of $C^\infty(M)$-irreducibility is formally the same kind of question as that of G-irreducibility. If M is connected, then $C^\infty(M)$ acts transitively (in the sense of Lie algebras). If M is not connected, then $\mathcal{H}$ probably won't be $C^\infty(M)$-irreducible. This kind of reducibility is probably what is called 'superselection'.

Normally for G to act 'by symmetries' means that it not only acts on (M, ω), but that it also preserves the Hamiltonian H. Now, (a) if G preserves H, then the decomposition of $\mathcal{H}$ into eigenspaces of the operator $\hat{H}$ is also a decomposition of $\mathcal{H}$ into G-representations, so $\mathcal{H}$ can't be G-irreducible, unless $\hat{H}$ is a scalar. But (b) if G acts transitively and preserves H, then H must be constant, so $\hat{H}$ is indeed a scalar. Note that (a) and (b) have nothing to do with the fact that H is the Hamiltonian – they work for any observable. Remember, also, from lecture 1, that it's not bad for the Hamiltonian to be constant – this is what happens in the formalism when we think of M as a space of trajectories rather than a space of states.

Example 1.1. A single particle moves in three dimensions. Then $M = T^*\mathbb{R}^3$ consists of pairs (position,momentum) of vectors. $G = \mathrm{SO}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ acts on M (that is: $\mathrm{SO}(3, \mathbb{R})$ acts by rotations, and $\mathbb{R}^3$ acts by translations). This action preserves the free Hamiltonian H (the length-squared of the momentum). But G doesn't act transitively. Rather, the orbits are parameterised by non-negative numbers $h \in \mathbb{R}^3$ (the values of H). These orbits are presymplectic manifolds – the integral curves of $\ker \omega$ are the (free) trajectories. The symplectic reduction of such an orbit (at least with $h \neq 0$) is a four-dimensional symplectic space, whose coordinates can be understood as choices of momentum and angular momentum (subject to the constraint $H = h$).

General (almost correct) idea of what we are doing: if G does not act transitively, consider one G-orbit at a time. These orbits are presymplectic spaces, and their reductions are certain nice symplectic manifolds with transitive G-actions: they are the 'coadjoint orbits'. The quantization of each coadjoint orbit, is an irreducible representation of G. The quantization of M is glued together (somehow) from these irreducible representations.

2. Coadjoint orbits

After this motivational story, we turn to some maths.

2.1. **Symplectic structure.** Given a Lie group G, the Lie algebra $\mathfrak{g} = T_1 G$ is a vector space with an action of G – the adjoint action. It is defined as follows: $g\colon h \mapsto ghg^{-1}$ defines an action of G on itself, preserving the identity; it therefore gives an action of G on the tangent space $\mathfrak{g}$.

We also get an action ('coadjoint') of G on the dual vector space $\mathfrak{g}^*$.

Theorem 2.2. *(Kostant-Kirillov-…) The 'coadjoint orbits' (the orbits of G in $\mathfrak{g}^*$) are naturally symplectic manifolds.*

Note that this has the otherwise mysterious consequence that the coadjoint orbits are even-dimensional.

The idea of the proof is as follows. Given $f \in \mathfrak{g}^*$ we can define a skew form ω_f on $\mathfrak{g}$: $\omega_f(x, y) = f([x, y])$. Thus we get a symplectic form on the quotient $\mathfrak{g}/\ker \omega_f$. The claim is that we can identify $\mathfrak{g}/\ker \omega_f$ with the tangent space $T_f\mathcal{O}_f$ to the orbit $\mathcal{O}_f$ at f. We have then defined a symplectic form on $T_f\mathcal{O}_f$, and as f varies we get a symplectic form on $\mathcal{O}_f$.

One way to do it: since G acts on $\mathcal{O}_f$, each $x \in \mathfrak{g}$ defines an 'infinitesimal transformation' of f, i.e. a tangent vector at f. The claim is that, as x varies, these span $T_f\mathcal{O}_f$, so that $T_f\mathcal{O}_f$ is a quotient of $\mathfrak{g}$. And in fact it is $\mathfrak{g}/\ker \omega_f$.

Equivalent (?) story: since $\mathcal{O}_f \subset \mathfrak{g}^*$, the cotangent space $T^*_f\mathcal{O}_f$ is naturally a quotient of $\mathfrak{g}$; in fact it is just $\mathfrak{g}/\ker \omega_f$. We therefore have a symplectic form on $T^*_f\mathcal{O}_f$, but this is the same thing as a symplectic form on $T_f\mathcal{O}_f$.

2.3. **Examples.** (1) For $G = \mathrm{SO}(3, \mathbb{R})$, we can realize $\mathfrak{g}^*$ as the set of skew-symmetric 3×3 matrices, with G acting by conjugation. Another way to describe $\mathfrak{g}$: it is a three-dimensional vector space generated by vectors i, j, k corresponding to infinitesimal rotations around the three axes. In this description, the coadjoint action is just the standard action of $\mathrm{SO}(3, \mathbb{R})$ on $\mathbb{R}^3$, and the orbits are spheres (including the origin as a sphere of zero radius).

(2) For $G = \mathrm{SL}(2, \mathbb{C})$, $\mathfrak{g}^*$ is the set of traceless 2×2 matrices, with G acting by conjugation; thus $\dim \mathfrak{g}^* = 3$. The orbits are as follows: for each real number $r \neq 0$, the set of $x \in \mathfrak{g}^*$ with $\det x = r$ (these are the 'semisimple' orbits); the set of $x \in \mathfrak{g}^*$ with rank 1; and $\{0\}$. The last two ('nilpotent') orbits form a quadratic cone in $\mathfrak{g}^* \cong \mathbb{C}^3$, called 'the nilpotent cone.'

(2) For $G = \mathrm{SL}(n, \mathbb{C})$, we can realize $\mathfrak{g}^*$ as the set of trace-free $n \times n$ matrices, again with G acting by conjugation. The different orbits correspond to different Jordan normal forms. So there is one for each nonincreasing list $(x_1, \ldots, x_n)$ of eigenvalues with $\sum x_i = 0$ – these are the 'semisimple' orbits, consisting of the diagonalizable matrices. The nilpotent orbits consist of nilpotent matrices. (Note some matrices are neither nilpotent nor diagonalizable.)

There is a certain relationship between the nilpotent orbits and *flag varieties*.