QUANTIZATION OF ELEMENTARY DYNAMICAL SYSTEMS II (WILL DONOVAN)

1. Some Lie Theory

Let $G = \mathrm{SL}(n, \mathbb{C})$. Much of what we say generalizes to arbitrary semisimple (or indeed reductive) Lie groups.

1.1. **Borels and Flags.** The Lie algebra $\mathfrak{g}$ is the space of trace-free $n \times n$ complex matrices (the Lie bracket is the usual commutator of matrices $AB - BA$). A *flag* F is a sequence of vector spaces

$$F = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{R}^n)$$

with $\dim F_d = d$. The stabilizer $\mathfrak{b}_F$ of F in $\mathfrak{g}$ is a Borel subalgebra (by definition, a Borel subalgebra is a *maximal solvable* Lie subalgebra of $\mathfrak{g}$). For example, with respect to the standard basis $e_1, \ldots, e_n$, let $F_d = \mathrm{span}(e_1, \ldots, e_d)$. Then $\mathfrak{b}_F$ is the space of traceless upper-triangular matrices.

A theorem of Engels says that $F \mapsto \mathfrak{b}_F$ is a G-equivariant bijection between the set $\mathcal{F}$ of flags and the set $\mathcal{B}$ of Borel subalgebras.

1.2. **The universal resolution.** The 'universal resolution' $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ is defined by

$$\tilde{\mathfrak{g}} = \{(f, \mathfrak{b}) : f \in \mathfrak{b}\} \subset \mathfrak{g} \times \mathcal{B}.$$

We have obvious maps $\mu\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$ and $\pi\colon \tilde{\mathfrak{g}} \to \mathcal{B}$. The fibres of μ are interesting.

First, $\mu^{-1}(0) = \mathcal{B}$.

Second, if $f \in \mathfrak{g}$ is regular semisimple (i.e. diagonalizable, with distinct eigenvalues), then $\mu^{-1}(f)$ has $n!$ elements and a natural action of the symmetric group S_n. That is, if $e_1, \ldots, e_n$ are the eigenvectors of f, then each $\sigma \in S_n$ defines a flag $F_d = \mathrm{span}(e_{\sigma(1)}, \ldots, e_{\sigma(d)})$. In other words, over the open set $\mathfrak{g}_{rs}$ of regular semisimple elements, μ is an principal S_n-bundle.

A third and most interesting case is when f is 'regular nilpotent': that is, it has a single block in Jordan normal form. (For $\mathrm{SL}_n(\mathbb{C})$, f is nilpotent if it is strictly upper triangular in some basis. In general, the condition is that $\mathrm{ad}(f)\colon \mathfrak{g} \to \mathfrak{g}$ is a nilpotent map. The nilpotent elements can be classified by $\dim \ker \mathrm{ad}(f)$; the 'regular' nilpotent elements are the ones where this number is as small as possible.)

1.3. **The nilpotent cone.** Let $\mathcal{N} \subset \mathfrak{g}$ be the set of all nilpotent elements. $\mathcal{N}$ is called the nilpotent cone; it *is* a cone, i.e. invariant under scaling. Let $\tilde{\mathcal{N}} = \mu^{-1}(\mathcal{N})$.

Proposition 1.4. *(1) If f is regular nilpotent, then there is a unique Borel $\mathfrak{b}$ containing f. (2) The set $\mathcal{N}_r$ of regular nilpotent elements is a dense G-orbit in $\mathcal{N}$.*

Thus the $n!$-fold cover of $\mathfrak{g}_{rs}$ ramifies to a 1-fold cover over $\mathcal{N}_r$. The proof of (1) is that f determines a flag $F_d = \ker f^d$. The fact that f is regular means that this is actually a flag in the sense we have defined. For the proof of (2) see for example Chriss/Ginzburg, Proposition 3.2.10.

Remark 1.1. There is a natural identification $\tilde{\mathcal{N}} = T^*\mathcal{B}$.

Example 1.1. For $\mathrm{SL}(2, \mathbb{C})$, $\mathfrak{g} = \mathbb{C}^3$, each non-zero semisimple orbit (automatically regular) is a hyperbola $x^2 - yz = c$, degenerating to a quadratic cone $x^2 - yz = 0$. The smooth part of the cone is the regular nilpotent orbit. The only other orbit is $\{0\}$.

1.5. **$\mathfrak{g}$ vs $\mathfrak{g}^*$.** We often make use of a G-equivariant isomorphism $\mathfrak{g} = \mathfrak{g}^*$. This is the same as choosing a G-invariant bilinear form on $\mathfrak{g}$. For $\mathrm{SL}(n, \mathbb{C})$, or for a semisimple group in general, such an isomorphism exists uniquely (up to scale) – the Killing form. For a reductive group, the Killing form is degenerate, but one can nonetheless define such an isomorphism. José says there is a classification of Lie algebras admitting invariant bilinear forms. They can all be constructed from one-dimensional Lie algebras and simple Lie algebras using a method called 'double extension.'

1.6. **Homogeneous spaces and line bundles.** One can show that $\mathcal{B} \cong G/B$ (where B is the subgroup of G corresponding to some standard Borel subalgebra $\mathfrak{b}$). The claim is just that B is its own normalizer, and that all Borel subgroups are conjugate.

This means that $\mathcal{B}$ has a natural principal B-bundle, $G \to \mathcal{B}$. (The fibres are the cosets of B, with B acting on the right of each coset.) For each character $\alpha\colon \mathcal{B} \to \mathbb{C}^\times$, we obtain a line bundle $\mathcal{L}_\alpha = G \otimes \alpha$ on $\mathcal{B}$.

Considering $\mathrm{SL}(n, \mathbb{C})$ once again, the characters of B factor through

$$B \to B/[B, B] \subset (\mathbb{C}^\times)^n$$

(projection to the diagonal) and are therefore given by sequences of integers $a = (a_1, \dots, a_n)$; the corresponding character is $\alpha_a(\mathrm{diag}(\lambda_1, \dots, \lambda_n)) = \prod \lambda_i^{a_i}$. If all the a_i are equal then $\alpha_a = 1$, so the space of characters is $\mathbb{Z}^n/\mathbb{Z}$. A character α_a is *dominant* if $a_1 \geq \cdots \geq a_n$. This turns out to be equivalent to the condition that the line bundle $\mathcal{L}_{\alpha_a}$ has non-zero holomorphic sections.

It is a well-known fact that finite-dimensional holomorphic representations of $\mathrm{SL}(n, \mathbb{C})$ are labelled by 'highest weights', which are nothing but dominant characters in the above sense. More precisely, one has the following theorem.

Theorem 1.7 (Borel-Weil). *If α is a dominant character, then the space $\Gamma(\mathcal{B}, \mathcal{L}_\alpha)$ of holomorphic sections of $\mathcal{L}_\alpha$ is a finite-dimensional irreducible holomorphic representation of G (with 'highest weight α'). All such representations arise in this way.*

(It should be clear that $\mathcal{L}_\alpha$ is G-equivariant, meaning that G *does* act on on $\Gamma(\mathcal{B}, \mathcal{L}_\alpha)$ in a geometrically natural way.)

Example 1.2. For $\mathrm{SL}(2, \mathbb{C})$, $\mathcal{B} = G/B = \mathbb{P}^1$. A dominant character is given by an integer $\alpha \geq 0$, and $\mathcal{L}_\alpha = \mathcal{O}_{\mathbb{P}^1}(\alpha)$ in the usual notation. Moreover, $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\alpha)) = \mathrm{Sym}^k(\mathbb{C}^2)^*$ (concretely, the space of homogeneous polynomials of degree k in two variables). This is the standard description of representations of $\mathrm{SL}(2, \mathbb{C})$.

2. Mystery

What does this have to do with geometric quantization? It *looks* like quantization in the following sense: we took $\mathcal{N}_r$, a transitive symplectic G-space. This is an open submanifold of $T^*\mathcal{B}$ (the projection $\mathcal{N}_r \to \mathcal{B}$ is the one that assigns to each $f \in \mathcal{N}_r$ the unique Borel containing it). We found irreducible representations of G by looking at sections of line bundles on the base $\mathcal{B}$. This looks just like the quantization story.

However, quantization should have given us *unitary* representations, and the representations we found are not unitary! ($\mathrm{SL}(n, \mathbb{C})$ has no non-trivial unitary finite-dimensional representations.)

It's remains a mystery to me what is going on. Here are three comments.

2.1. **Look at non-holomorphic sections.** The finite-dimensionality has to do with the fact that we considered holomorphic sections of $\mathcal{L}_\alpha$. In the geometric quantization prescription, we are supposed to (do something like) look at all L^2 sections. This makes sense at least when $\alpha = 0$. This is the picture: given a *unitary* character of $\mathcal{B}$, we can induce from $\mathcal{B}$ to $\mathcal{G}$ to get a unitary representation of $\mathcal{G}$. My vague memory is that the corresponding representations of $\mathcal{G}$ are irreducible (or almost so) and that *most* of the irreducible unitary representations arise in this way.

2.2. **Compact groups.** Instead of a complex group like $\mathrm{SL}(2,\mathbb{C})$, we can consider compact groups like $\mathrm{SU}(2)$. Compactness implies that one can make any finite-dimensional representation unitary. Now, the holomorphic representations of $\mathrm{SL}(2,\mathbb{C})$ restrict to unitary representations of $\mathrm{SU}(2)$, and this is a bijection. So we should be able to get the holomorphic representations of $\mathrm{SL}(2,\mathbb{C})$ by quantizing *coadjoint orbits in* $\mathrm{SU}(2)$.

For $\mathrm{SU}(2)$, the coadjoint orbits are spheres (remember that $\mathrm{SU}(2)$ and $\mathrm{SO}(2,\mathbb{R})$ have the same Lie algebra). These spheres are Kähler manifolds (think sphere=$\mathbb{P}^1$). Using the Kähler polarization, the Hilbert space resulting from geometric quantization is the space of holomorphic sections of a line bundle. So the construction of unitary representations of $\mathrm{SU}(2)$ looks exactly the same as the construction of holomorphic representations of $\mathrm{SL}(2,\mathbb{C})$ before: holomorphic sections of a line bundle on $\mathbb{P}^1$. But there is still a bit of a mystery in the sense that for $\mathrm{SU}(2)$ we have quantized semisimple orbits, while for $\mathrm{SL}(2,\mathbb{C})$ we dealt with the regular nilpotent orbit. It would be nice to understand the connection between these two stories (some kind of deformation?).

2.3. **Poincaré group.** The unitary representations of $\mathrm{SL}(2,\mathbb{C})$ are physically important in that $\mathrm{SL}(2,\mathbb{C})$ is the universal cover of the Lorentz group. But one is really interested in unitary representations of the Poincaré group, $P = \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$. By quantizing coadjoint orbits in P, one recovers the one-particle Hilbert spaces used in quantum field theory. The story here is explained in Woodhouse, and definitely involves the quantization of $\mathrm{SU}(2)$ explained above. It would be good to sort all this out.