TQFTs

- Introduced by Atiyah ('88)
- Goal: axiomatize various QFTs
- String-theoretic motivation
- Cat-theoretic approach

① Cobordisms

Say M, M' oriented $(n-1)$-mflds are <u>cobordant</u> if $\exists B$ n-mfld (<u>bordism</u>)

s.t. $\partial B = \overline{M} \amalg M'$

orientat^n reversed

eg

TQFT: "that thing with pairs of pants"!

<u>Obs</u> cobordism is equiv rel^n

- Forgets bordisms B
- Formally remember B's, as follows:

$\boxed{\text{Def}^{\underline{n}}}$ $\mathrm{Cob}(n)$ cobordism category

Obj: closed oriented $(n-1)$-mflds

Mor: ~~cobo~~ cobordisms (mod. diffeos)

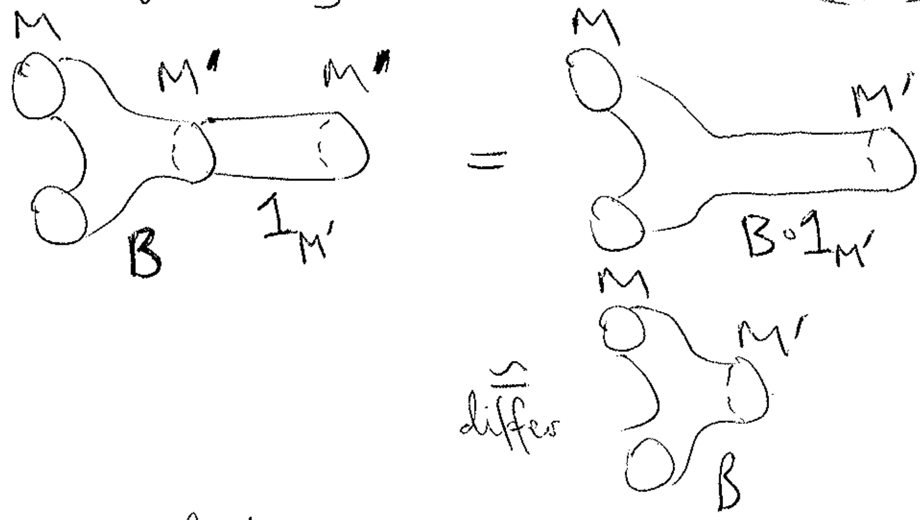
__Rmk__ : Say 'category' simply because have composition of morphisms

(eg) $n=1$

$$M \xrightarrow{B} M' \xrightarrow{B'} M'' \;=\; M \xrightarrow{B \amalg_{M'} B'} M''$$

& $\forall M$ have 'identity' morphism $\mathbb{1}_M$ corresponding to $B = M \times [0,1]$

(eg) $n=2$



(2) Symmetric monoidal structure

- Just terminology
- ~~Required~~ Used to define TQFT succinctly

(eg) $\mathrm{Vect}(k) :=$ category of v.s's / k field
(morphism are lin. maps)

$(\mathrm{Vect}(k), \underbrace{\otimes}_{\text{monoidal structure}}, \underbrace{k}_{\text{unit}})$ 1-d v.s

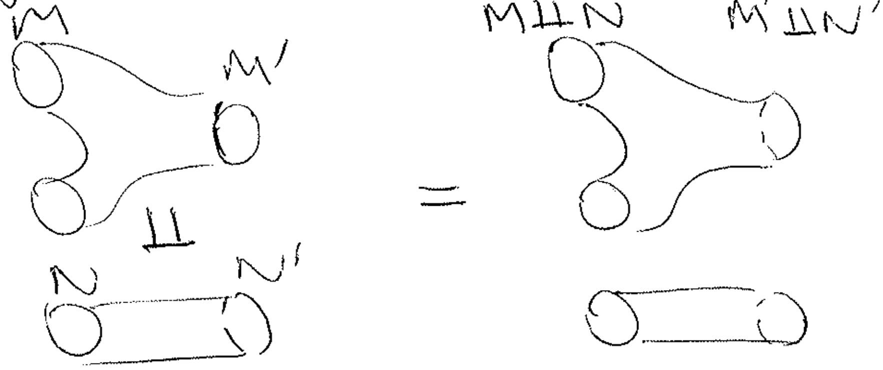
Symmetric : $V \otimes W \cong W \otimes V$

Unit : $V \otimes k \cong V$

Functoriality: $V \otimes W \xrightarrow{\phi \otimes \psi} V' \otimes W'$ ($V \xrightarrow{\phi} V'$, $W \xrightarrow{\psi} W'$)

eg $(\mathrm{Cob}(n), \amalg, \emptyset)$
monoidal structure ($\amalg$)

- Disjoint union of cobordisms:



- Symmetric: $M \amalg N \underset{\text{diffeo}}{\cong} N \amalg M$
- Unit: $M \amalg \emptyset = M$

3) TQFTs

Def$^{\underline{n}}$ (Atiyah) $\underline{\text{TQFT, dim}^{\underline{n}}\ n}$ is symm. monoidal f$^{\underline{n}}$tor $Z : \mathrm{Cob}(n) \longrightarrow \mathrm{Vect}(k)$
(k field)

Def$^{\underline{n}}$ (unpacked)

- v.s. $Z(M)$ for M closed oriented $(n-1)$-mfld.
- lin. $Z(B) : Z(M) \longrightarrow Z(N)$
  for bordism $B : M \longrightarrow N$
- $Z(\emptyset) \cong k$, $Z(M \amalg N) \cong Z(M) \otimes Z(N)$

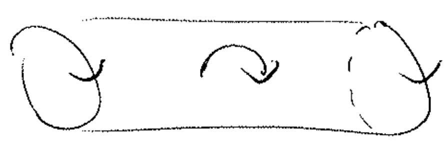
+ coherence cond$^{\underline{n}}$s.

Formal consequences

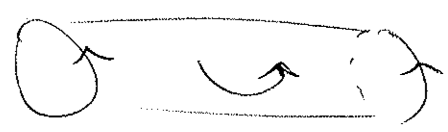
Take $B = M \times [0,1]$ (NB arbitrary M)

Can view as a bordism in various ways:

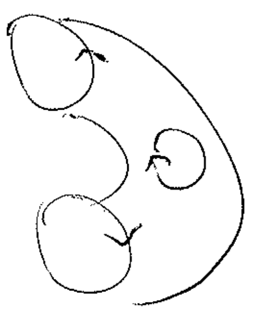
$M \xrightarrow{\mathbb{1}_M} M$



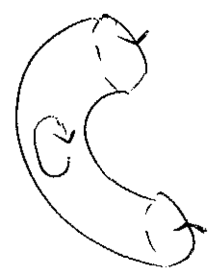
$\bar{M} \xrightarrow{\mathbb{1}_{\bar{M}}} \bar{M}$



$\bar{M} \sqcup M \xrightarrow{ev_M} \phi$



$\phi \xrightarrow{coev_M} M \sqcup \bar{M}$



Rmk: reasons for names will become clear.

Obs $\exists$ map

$$\begin{tikzcd} Z(\bar{M}) \otimes Z(M) \arrow[r, "\alpha"] \arrow[d, "\wr"] & k \\ Z(\bar{M} \sqcup M) \arrow[r, "Z(ev_M)"] & Z(\phi) \arrow[u, "\wr"'] \end{tikzcd}$$

monoidal structure

Can view as map $Z(\bar{M}) \longrightarrow Z(M)^\vee$

NB one diag  $\rightsquigarrow$ two maps

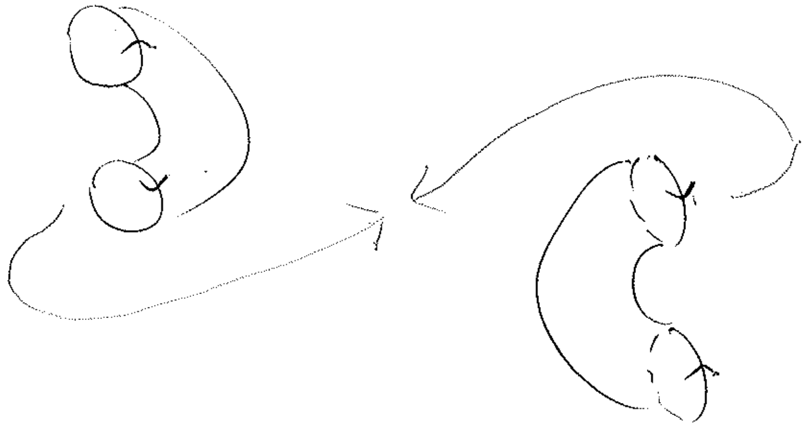
Prop $\alpha : Z(\bar{M}) \longrightarrow Z(M)^\vee$ iso

Proof Dually obtain

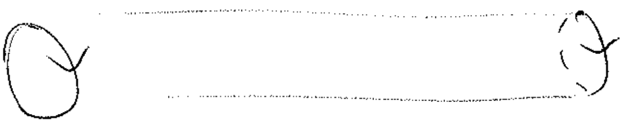
$\beta : Z(M)^\vee \longrightarrow Z(\bar{M})$

from coev_M!

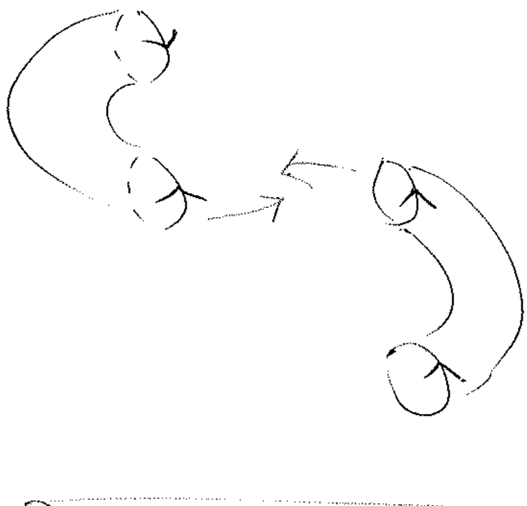
- Compose $Z(\bar{M}) \xrightarrow{\alpha} Z(M)^\vee \xrightarrow{\beta} Z(\bar{M})$



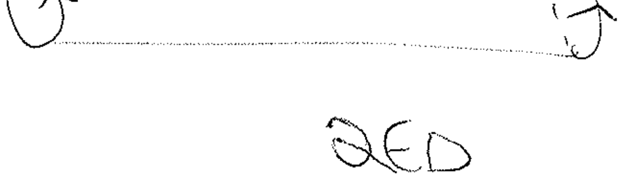
gives identity



- Compose $Z(M)^\vee \xrightarrow{\beta} Z(\bar{M}) \xrightarrow{\alpha} Z(M)^\vee$



also gives identity



QED

(4) 1-d TQFTs

Two orientations on point, $p^\pm$.

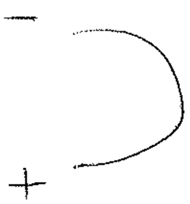
$\implies M = \coprod p^\pm$

Write $V := Z(p^+)$

deduce $V^\vee \simeq Z(p^-)$

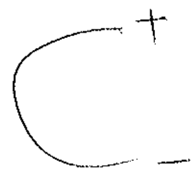
Obs nat. morphisms:

· $p^- \amalg p^+ \xrightarrow{ev_{p^+}} \emptyset$



$$\begin{tikzcd} V^\vee \otimes V \arrow[r, "ev"] & k \\ \mathrm{End}(V) \arrow[u, phantom, "\cong" rotate=90] \arrow[ur, "tr"'] & \end{tikzcd}$$

· $\emptyset \xrightarrow{coev_{p^+}} p^+ \amalg p^-$



$$\begin{tikzcd} k \arrow[r, "coev"] \arrow[dr, "\mathbb{1}_V"'] & V \otimes V^\vee \\ & \mathrm{End}(V) \arrow[u, phantom, "\cong" rotate=90] \end{tikzcd}$$

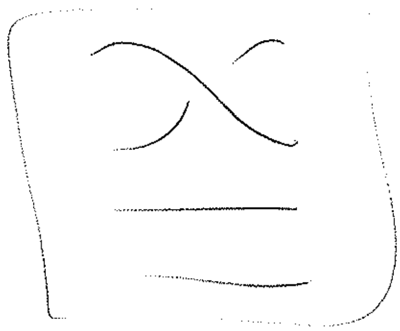
(eg) V = Borel-Weil Hilbert space

Construction

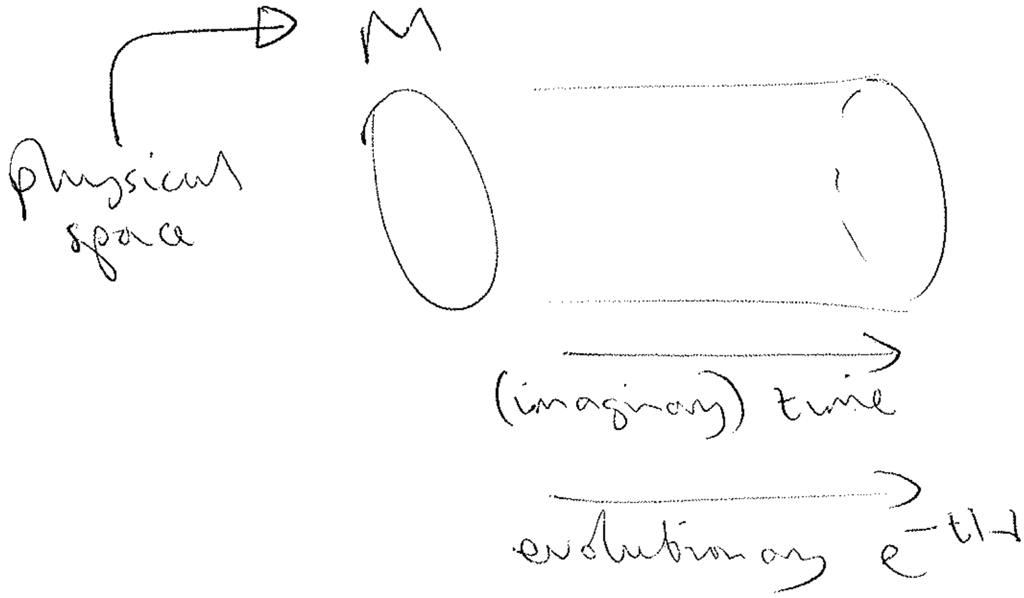
- cpct Lie grp G
- coadjoint orbit $\mathcal{O}$
- sympectic structure
- integral $\rightsquigarrow$ line bdl $\mathcal{L}$
- quantize $\rightsquigarrow$ G-irrep V

Rmks

- Lag = classical action = $\mathrm{Hol}(\mathcal{L})$
- Have action $S_n \longrightarrow V^{\otimes n}$ from permutation bordisms:



Physical picture



cf. e^{itH} as for ordinary time

topological $\iff H = 0$

eg Lagrangian with only first derivatives wrt. t.