GEOMETRIC QUANTIZATION I: SYMPLECTIC GEOMETRY AND MECHANICS

A useful reference is Simms and Woodhouse, *Lectures on Geometric Quantization*, available online.

1. Symplectic Geometry.

1.1. **Linear Algebra.** A *pre-symplectic* form ω on a (real) vector space V is a skew bilinear form $\omega\colon V\otimes V\to\mathbb{R}$. For any $W\subset V$ write $W^\perp=\{v\in V\,|\,\omega(v,w)=0\,\forall w\in W\}$. (V,ω) is *symplectic* if ω is non-degenerate: $\ker\omega:=V^\perp=0$.

Theorem 1.2. *If (V,ω) is symplectic, there is a "canonical" basis*

$$P_1,\dots,P_n,Q_1,\dots,Q_n$$

such that $\omega(P_i,P_j)=0=\omega(Q_i,Q_j)$ and $\omega(P_i,Q_j)=\delta_{ij}$.

P_i and Q_i are said to be *conjugate*. The theorem shows that all symplectic vector spaces of the same (always even) dimension are isomorphic.

1.3. **Geometry.** A symplectic *manifold* is one with a non-degenerate 2-form ω; this makes each tangent space T_mM into a symplectic vector space with form ω_m. We should also assume that ω is *closed*: $d\omega=0$.

We can also consider *pre-symplectic* manifolds, i.e. ones with a closed 2-form ω, which may be degenerate. In this case we also assume that $\dim\ker\omega_m$ is constant; this means that the various $\ker\omega_m$ fit tog ether into a *sub-bundle* $\ker\omega$ of TM.

Example 1.1. If V is a symplectic vector space, then each $T_mV=V$. Thus the symplectic form on V (as a vector space) determines a symplectic form on V (as a manifold). This is locally the only example:

Theorem 1.4. *Every symplectic manifold M is locally isomorphic to a symplectic vector space V. (In particular, for any $m\in M$ there are local 'canonical' coordinates $P_1,\dots Q_n$ corresponding to a canonical basis for V).*

Example 1.2. If M is any manifold, then T^*M is a symplectic manifold. In fact, there is a unique one-form θ on T^*M such that for any section $\alpha\colon M\to T^*M$ one has $\alpha^*(\theta)=\alpha$. Then $\omega:=d\theta$.

In general, if $\omega=d\theta$ then θ is called a *symplectic potential*. One always exists locally (because ω is closed) but not always globally.

Example 1.3. If M is a complex manifold with a Hermitian metric η then define $\omega_m(\xi_1,\xi_2)=\operatorname{Re}\eta_m(i\xi_1,\xi_2)$ for all $\xi_1,\xi_2\in T_mM$. Then ω is a non-degenerate 2-form; if it is symplectic (i.e. closed) then M is *Kähler*. For example: $M=\mathbb{P}^n(\mathbb{C})$ with η the unique $U_{n+1}(\mathbb{C})$-invariant metric. Note this is compact (unlike a cotangent bundle) and there is no symplectic potential.

1.5. **Hamiltonian Reduction.** Other examples are found using *Hamiltonian reduction*. Suppose first that W is a pre-symplectic vector space; then $W/\ker\omega$ is symplectic. In particular, if V is symplectic, and $W^\perp\subset W\subset V$ (i.e. W is 'coisotropic') then W is pre-symplectic and $W/W^\perp$ is symplectic. This construction globalises in the following way:

Suppose (N,ω) is pre-symplectic. Then $\ker\omega$ is (by assumption) a subbundle of TN. The fact that ω is closed means that $\ker\omega$ is *integrable*. This means there exists a ‘foliation’, i.e. a family N_α of submanifolds of N, such that $N = \sqcup_{\alpha \in I} N_\alpha$, and if $m \in N_\alpha$ then $T_m N_\alpha = \ker \omega_m$. In good cases, the set of ‘leaves’ $N/\ker\omega := I$ is a manifold, and then it is symplectic. Indeed, $T_\alpha(N/\ker\omega) = (T_m N)/\ker\omega_m$.

In particular, if (M, ω) is symplectic, we can choose some $N \subset M$ that is *coisotropic*: this means that $T_n N$ is a coisotropic subspace of $T_n M$, for any $n \in N$. In good cases, $(N, \omega|_N)$ will then be presymplectic, and $N/\ker\omega|_N$ will be symplectic.

Example 1.4. $V = \mathbb{R}^n$, so $M := TV \cong \mathbb{R}^{2n}$ is symplectic. Let $N \subset M$ be the unit sphere. Then the Hamiltonian reduction $N/\ker\omega|_N$ is naturally $\mathbb{P}^{n-1}(\mathbb{C})$. In this case the leaves of the foliation are circles on the sphere.

1.6. **Poisson Brackets.** The fact that M is symplectic endows $C^\infty(M)$ with the structure of a Lie algebra, under the *Poisson bracket*, defined as follows.

Given $f \in C^\infty(M)$, there exists a unique vector field X_f on M such that $\omega(X_f, -) = df$. Such a vector field is called *globally Hamiltonian.* For $f, g \in C^\infty(M)$, define $[f, g] = 2\omega(X_f, X_g)$. This makes $C^\infty(M)$ into a Lie algebra (the Jacobi identity is equivalent to the fact that ω is closed). This gives a short exact sequence of Lie algebras:

$$0 \to \mathbb{R} \to C^\infty(M) \to \{(\text{Glob. Ham. VFs})\} \to 0.$$

The Lie bracket on globally Hamiltonian vector fields is the usual bracket of vector fields.

1.7. **A Naive Idea About Quantization.** The idea of quantization is to associate to a symplectic manifold M a Hilbert space H, and to each ‘observable’ $f \in C^\infty(M)$, and operator O_f on H, such that the Poisson bracket on $C^\infty(M)$ becomes the commutator of operators. Naively, we can take $H = L^2(M)$ and $O_f = X_f$, acting by derivations. But life is more complicated.

2. Mechanics

The kind of situation we want to describe is that of a particle moving in space S. How can one describe the trajectories? We sketch three methods.

2.1. **Hamiltonian mechanics.** General setup: M a symplectic manifold; $H \in C^\infty(M)$ ‘the Hamiltonian.’ Points of M label instantaneous states; time evolution is given by flow along the vector field X_H. In other words, $\gamma\colon \mathbb{R} \to M$ is a trajectory if it satisfies the differential equation $d\gamma/dt = X_H$ (maybe with a minus sign).

Example 2.1. S space. Suppose that S has a Riemannian metric η (e.g. $S = \mathbb{R}^3$ with the usual inner product). We can use η to identify each tangent space $T_m S$ with the cotangent space $T^*_m S$, thus making $M = TS$ into a symplectic manifold. $H(s, \xi) = \frac{1}{2}\eta(\xi, \xi)$ for $s \in S$ and $\xi \in T_s S$. The trajectories $\mathbb{R} \to TS$ project to geodesics on S. (Note that any curve $\gamma\colon \mathbb{R} \to S$ lifts naturally to a curve $(\gamma, \gamma')\colon \mathbb{R} \to TS$.)

2.2. **Presymplectic mechanics.** Now N is pre-symplectic. Thus we have a foliation of N with tangent spaces $\ker\omega$. The trajectories are leaves of this foliation, i.e. unparameterized integral surfaces for $\ker\omega$. (The simplest case is when $\ker\omega_m$ is one-dimensional, so the leaves are cruves in N).

Example 2.2. With S as before, and H a function on TM, consider ‘spacetime’ $S \times \mathbb{R}$. Let $M = T^*(S \times \mathbb{R})$ and define $N \subset M$ to be those $(s, t; \alpha_s, \alpha_t) \in M$ such that $H(s, t) + \alpha_t = 0$. Then N is coisotropic and presymplectic, and the trajectories project to graphs of geodesics $\mathbb{R} \to S$.

This setup makes sense when H depends on time, and also in relativistic setting where there is no canonical decomposition of spacetime into space S and time $\mathbb{R}$.

Usually the situation is described in a slightly different way: one is given a 'relativistic Hamiltonian' $H_r \in C^\infty(M)$ (in the example, $H(s,t) + \alpha_t$), and the trajectories are again integral curves for X_{H_r}, but subject to the constraint $H_r = 0$.

2.3. **Nice Thing I Forgot to Say.** The Hamiltonian reduction $N/\ker\omega$ of the presymplectic manifold N is (by definition) the space of trajectories, which is therefore a symplectic manifold. In the example, we can, for each time $t \in \mathbb{R}$, identify $N/\ker\omega$ with the space T^*S of instantaneous states. In quantum theory, $N/\ker\omega$ is called the 'Heisenberg picture' (the 'states' are entire trajectories) whereas T^*S is called the 'Schrodinger picture' (the 'states' are instantaneous). As remarked before, the Schrodinger picture is not very natural in relativistic settings. In general, $N/\ker\omega$ might not even be isomorphic to a cotangent bundle.

2.4. **Lagrangian Mechanics.** Instead of characterising trajectories by differential equations, we use a 'variational principle' – like the one that says that geodesics minimize length. Lagrangian mechanics takes place in the tangent space TS. We are given a function $L \in C^\infty(TS)$. A trajectory $\gamma\colon \mathbb{R} \to S$ is one that extremizes the *action*

$$A(\gamma) = \int_{[0,1]} L(\gamma(t), \gamma'(t))\, dt.$$

The relation to Hamiltonian mechanics is roughly as follows: one can use L to define a (non usually linear) map $T_m S \to T_m^* S$, called the *Legendre transfrom.* In good cases, the Legendre transform is a local diffeomorphism, making TS into a symplectic manifold. Flows are determined by an appropriate Hamiltonian.

Example 2.3. Take $L(s,\xi) = \frac{1}{2}\eta(\xi,\xi)$. Then the action is the length[1] of γ, and the trajectories are geodesics. More generally, if we had a Hamiltonian $H(s,\xi) = \frac{1}{2}\eta(\xi,\xi) + U(s)$, then the corresponding Lagrangian is $L(s,\xi) = \frac{1}{2}\eta(\xi,\xi) - U(s)$.

The Lagrangian formalism is ubiquitous in physics, but it's not yet clear to me how important it will be for our immediate aims.

[1]Actually not quite the length, which would be the integral of the *square-root* of the Lagrangian; but it turns out not to matter.