GEOMETRIC QUANTIZATION II: PREQUANTIZATION AND THE HEISENBERG GROUP

1. Prospectus

1.1. **Representation.** In the first lecture we defined a Lie algebra structure on $C^\infty(M)$, where M is any symplectic manifold. The general aim is to define a representation of this Lie algebra on some Hilbert space $\mathcal{H}$ associated to M. It's not always clear what the rules are, but here are some pointers:

(a) For each $f \in C^\infty(M)$, the corresponding operator on $\mathcal{H}$ should be skew-Hermitian.
(b) Any constant function f should act by multiplication-by-if.
(c) We may only be able to represent some subalgebra of $C^\infty(M)$.
(d) $\mathcal{H}$ should not be 'too big'; in particular, it should be an irreducible representation.

There is some kind of tradeoff between the size of $\mathcal{H}$ and the size of the subalgebra that we can represent. In the end, what we do will be judged by how well we can handle important examples.

1.2. **Hermiticity.** Given a Lie group G, it is usual to consider *unitary* representations of G. This corresponds to Lie-algebra representations in which Lie(G) acts by skew-Hermitian opearators. For various reasons, physicists prefer to work with Hermitian operators. If X is Hermitian, iX is anti-Hermitian. So from this point of view, we want for each $f \in C^\infty(M)$, a *Hermitian* operator $\hat{f}$, such that

$$[\hat{f}, \hat{g}] = -i\widehat{[f,g]}. \tag{1}$$

On the right, $[,]$ is the Poisson bracket. The passage between Hermitian and skew-Hermitian operators is the source of most factors of i throughout.

2. Prequantization

2.1. **Hilbert Space.** Suppose we have a manifold M with a measure μ. Then we can form the Hilbert space $L^2(M, \mu)$ of complex-valued L^2 functions on M.

We need a more general construction. Suppose given a complex line bundle $\mathcal{L}$ on M with a Hermitian form $\eta\colon \overline{\mathcal{L}} \otimes_{\mathbb{C}} \mathcal{L} \to \mathbb{C}_M$. (Here $\mathbb{C}_M$ is the trivial complex line bundle on M.) Then the space of sections $C^\infty(M, \mathcal{L})$ has a natural Hermitian pairing

$$(f, g) = \int_M \eta(f, g)\, \mu$$

and we again obtain a Hilbert space $L^2(M, \mathcal{L}, \mu)$. Here $\eta(f, g)$ is a function on M, so $\eta(f, g)\, \mu$ is a measure.

Remark 2.1. A slightly more canonical construction is to forget about the measure μ, and assume given a Hermitian pairing $\eta\colon \overline{\mathcal{L}} \otimes_{\mathbb{C}} \mathcal{L} \to \Delta_1(M)$, where now $\Delta_1(M)$ is the complex line bundle whose sections are smooth (complex) measures on M. Then $\eta(f, g)$ is a measure, and can be integrated, so we get a Hilbert space $L^2(M, \mathcal{L})$. In this context, I talked a bit about spaces of densities and forms; we ended up not needing this yet, so I will come back to this subject later in the notes.

If M is a symplectic manifold then there is already a canonical (up to scale) measure μ on M, corresponding to the volume form $\omega \wedge \cdots \wedge \omega$. We will use this canonical measure in what follows.

2.2. **Operators.** Recall that for each $f \in C^\infty(M)$ we have a 'globally Hamiltonian' vector field X_f. This vector field X_f acts on $L^2(M, \mu)$ by derivations. More generally, we must assume that our line bundle $\mathcal{L}$ has a connection, which allows us differentiate sections of $\mathcal{L}$, and in particular gives us an operator ∇_{X_f} on $L^2(M, \mathcal{L}, \mu)$. If we define

$$\hat{f}(s) = -i\nabla_{X_f} s + fs$$

then $\hat{f}$ is a Hermitian operator, and if f is constant then $\hat{f}$ is multiplication-by-f. However, to fulfill our mission, we must also require (1).

Theorem 2.3. *(a) Equation* (1) *holds if and only if*

$$\omega(\xi_1, \xi_2)s = \frac{i}{2}([\nabla_{\xi_1}, \nabla_{\xi_2}] - \nabla_{[\xi_1,\xi_2]})s$$

for all sections s of $\mathcal{L}$ (i.e. ω is the curvature of ∇).

(b) There exists a triple $(\mathcal{L}, \eta, \nabla)$ with curvature ω if and only if ω is integral.

(c) If ω is integral, then the triples $(\mathcal{L}, \eta, \nabla)$ with curvature ω are parameterised by $H^1(M, \mathbb{R}/\mathbb{Z})$.

The condition in (b) that ω is integral means that its class in $H^2(M, \mathbb{R})$ comes from $H^2(M, 2\pi\mathbb{Z})$, or, more geometrically, that the integral of ω over every closed, oriented 2-surface in M is an integer times 2π. Suppose that there exists some $(\mathcal{L}, \eta, \nabla)$ such that ∇ has curvature ω. Then η is determined uniquely by $(\mathcal{L}, \nabla)$; but given $(\mathcal{L}, \eta)$, the choices of ∇ are parameterized by $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.

The representation $f \mapsto \hat{f}$ of $C^\infty(M)$ on $\mathcal{H} = L^2(M, \mathcal{L}, \mu)$ is called a *prequantization.*

Example 2.1. Suppose that $M = T^*N$ is a cotangent bundle. Then ω is exact, so there exists a prequantization. If M is simply connected, then there exists at most one prequantization. In the simplest case, if $M = V$ is a symplectic vector space, then there is a unique prequantization.

2.4. **Construction.** I sketch how to construct the unique $(\mathcal{L}, \eta, \nabla)$ when M is simply connected.

First define $\mathcal{L}_0$ to be the set of triples (m, z, γ) where $m \in M$, $z \in \mathbb{C}$, and γ is a path from some basepoint m_0 to m. Introduce an equivalence relation on $\mathcal{L}_0$ such that $(m, z, \gamma) \sim (m', z', \gamma')$ if and only if $m = m'$ and

$$\frac{z}{z'} = \exp(i \int_S \omega)$$

for any oriented 2-surface whose boundary is the concatenation of paths $\gamma^{-1} * \gamma'$. Such a surface exists for any γ, γ', because we assumed M simply connected; the right-hand side $\exp(i \int_S \omega)$ is independent of S by the integrality of ω. It is thus easy to see that $\mathcal{L} = \mathcal{L}_0/\sim$ is a line-bundle on M. We write $\mathcal{L}_m$ for the fibre at m, and continue to label elements of $\mathcal{L}_m$ by equivalence classes of pairs (z, γ).

The Hermitian structure on $\mathcal{L}$ is given by $\eta(z, \gamma; z', \gamma) = \bar{z}z'$ where, note, the paths γ are the same. The connection ∇ is defined by *parallel transport* as follows. Suppose given a path α from m to m'. Given $(z, \gamma) \in \mathcal{L}_m$ there should be a unique way to lift $\alpha\colon [0, 1] \to M$ to $\tilde{\alpha}\colon [0, 1] \to \mathcal{L}$ so that $\nabla_{\alpha'(t)}\tilde{\alpha}(t) = 0$ (i.e. the section $\tilde{\alpha}$ of $\mathcal{L}$ over α is constant). The association $(z, \gamma) = \tilde{\alpha}(0) \mapsto \tilde{\alpha}(1)$ is 'parallel transport' from m to m', and these parallel transport maps determine ∇. The definition is that parallel transport is given by $(z, \gamma) \mapsto (z, \alpha * \gamma)$.

3. Example

The problem with prequantization is that the Hilbert space is 'too big.' The fundamental example is as follows. Suppose that $S = \mathbb{R}$ (with coordinate q) and $M = T^*S = \mathbb{R}\oplus\mathbb{R}$ (with coordinates (q, p)). Quantum mechanics gives the following prescription: let $\mathcal{H} = L^2(S, dq)$, and define $\hat{q}s(q) = qs(q)$ and $\hat{p}s(q) = -i\partial_q s(q)$, for every $s \in L^2(S, dq)$. These operators satisfy the expected 'canonical' commutation relation (cf. (1))

$$[\hat{p}, \hat{q}] = -i.$$

There are two lessons to be learnt:

(a) We get a representation of the coordinate functions p and q (and some others), but not all of $C^\infty(M)$.
(b) The Hilbert space consists of functions on S, not anything like functions on M (as we would have expected from prequantisation). However, it is an irreducible representation of the Lie algebra generated by p and q.

To preview the general case, the specified quantization depends on the additional datum of the projection $T^*S \to S$, which is called a *polarization*. If $M = T^*N$ is a cotangent bundle, we can always use the projection $M \to N$, but in general there may be many possible reasonable choices of polarization.

4. The Heisenberg Group

We can abstract the above example in the following way. We take V to be a symplectic vector space. For each $v \in V$ we have the linear function $f_v := \omega(v, -) \in C^\infty(V)$. The Poisson bracket is

$$[f_v, f_w] = \omega(v, w).$$

Thus these linear functions generate a Lie algebra $\mathfrak{h}(V)$ (the *Heisenberg Lie algebra*): $\mathfrak{h}(V) = V \times \mathbb{R}$ as a set, $\mathbb{R}$ is central, and $[v, w] = (0, \omega(v, w))$ for $v, w \in V$.

A quantization of V should yield (at least) a representation of $\mathfrak{h}(V)$, in which constants $a \in \mathbb{R}$ act by ia. There are reasons to require this to be an *irreducible* representation. (The rough reason is that $\mathfrak{h}(V)$ acts transitively on V, so ought to act 'transitively', i.e. irreducibly, on $\mathcal{H}$.)

The Heisenberg Lie *group* associated to $\mathfrak{h}(V)$ is $H(V) = V \times U(1)$ as a set; multiplication is defined so that the circle $U(1)$ is central, and the product of $(v, 0), (w, 0) \in V$ is

$$vw = (v + w, \exp(\tfrac{i}{2}\omega(v, w))).$$

So, to rephrase: *a quantization of V is an irreducible unitary represenation of $H(V)$ in which $U(1)$ acts by scalar multiplication.*

Remark 4.1. Of course there are many different Lie groups with the same Lie algebra $\mathfrak{h}(V)$. We choose this one because the representations of $\mathfrak{h}(V)$ we are interested in correspond to representations of our $H(V)$.

Theorem 4.1 (Stone-von Neuman)**.** *There exists a unique irreducible reprsentation of $H(V)$ such that $U(1)$ acts by scalar multiplication.*

'Unique' as usual means unique-up-to-isomorphism. However, the usual construction of the representation depends on a choice of a *Lagrangian subspace* of V – this is essentially the choice of a *polarization* mentioned before.

José says that in the infinite dimensional case, the choice of the Lagrangian is a serious matter, resulting in 'different physics' (I think this means: inequivalent representations). But even in our finite-dimensional case, where the representations are isomorphic, there is some subtle structure to the choices, related to the metaplectic group.