GEOMETRIC QUANTIZATION II: PREQUANTIZATION AND
THE HEISENBERG GROUP

1. Prospectus

1.1. Representation. In the first lecture we defined a Lie algebra structure on
$C^\infty(M)$, where $M$ is any symplectic manifold. The general aim is to define a
representation of this Lie algebra on some Hilbert space $\mathcal{H}$ associated to $M$. It’s
not always clear what the rules are, but here are some pointers:

(a) For each $f \in C^\infty(M)$, the corresponding operator on $\mathcal{H}$ should be skew-
Hermitian.
(b) Any constant function $f$ should act by multiplication-by-$if$.
(c) We may only be able to represent some subalgebra of $C^\infty(M)$.
(d) $\mathcal{H}$ should not be ‘too big’; in particular, it should be an irreducible repre-
sentation.

There is some kind of tradeoff between the size of $\mathcal{H}$ and the size of the subalgebra
that we can represent. In the end, what we do will be judged by how well we can
handle important examples.

1.2. Hermiticity. Given a Lie group $G$, it is usual to consider unitary represen-
tations of $G$. This corresponds to Lie-algebra representations in which $\text{Lie}(G)$ acts
by skew-Hermitian operators. For various reasons, physicists prefer to work with
Hermitian operators. If $X$ is Hermitian, $iX$ is anti-Hermitian. So from this point
of view, we want for each $f \in C^\infty(M)$, a Hermitian operator $\hat{f}$, such that

\[ [\hat{f}, \hat{g}] = -i[\hat{f}, \hat{g}] \]

On the right, $[,]$ is the Poisson bracket. The passage between Hermitian and skew-
Hermitian operators is the source of most factors of $i$ throughout.

2. Prequantization

2.1. Hilbert Space. Suppose we have a manifold $M$ with a measure $\mu$. Then we
can form the Hilbert space $L^2(M, \mu)$ of complex-valued $L^2$ functions on $M$.

We need a more general construction. Suppose given a complex line bundle $\mathcal{L}$
on $M$ with a Hermitian form $\eta: \mathcal{L} \otimes \mathbb{C} \to \mathbb{C}_M$. (Here $\mathbb{C}_M$ is the trivial complex
line bundle on $M$.) Then the space of sections $C^\infty(M, \mathcal{L})$ has a natural Hermitian
pairing

\[ (f, g) = \int_M \eta(f, g) \mu \]

and we again obtain a Hilbert space $L^2(M, \mathcal{L}, \mu)$. Here $\eta(f, g)$ is a function on $M$,
so $\eta(f, g) \mu$ is a measure.

Remark 2.1. A slightly more canonical construction is to forget about the measure
$\mu$, and assume given a Hermitian pairing $\eta: \mathcal{L} \otimes \mathbb{C} \to \Delta_1(M)$, where now $\Delta_1(M)$
is the complex line bundle whose sections are smooth (complex) measures on $M$. Then
$\eta(f, g)$ is a measure, and can be integrated, so we get a Hilbert space $L^2(M, \mathcal{L})$. In
this context, I talked a bit about spaces of densities and forms; we ended up not
needing this yet, so I will come back to this subject later in the notes.
If $M$ is a symplectic manifold then there is already a canonical (up to scale) measure $\mu$ on $M$, corresponding to the volume form $\omega \wedge \cdots \wedge \omega$. We will use this canonical measure in what follows.

2.2. Operators. Recall that for each $f \in C^\infty(M)$ we have a ‘globally Hamiltonian’ vector field $X_f$. This vector field $X_f$ acts on $L^2(M, \mu)$ by derivations. More generally, we must assume that our line bundle $\mathcal{L}$ has a connection, which allows us to differentiate sections of $\mathcal{L}$, and in particular gives us an operator $\nabla_{X_f}$ on $L^2(M, \mathcal{L}, \mu)$. If we define

$$\hat{f}(s) = -i\nabla_{X_f}s + fs$$

then $\hat{f}$ is a Hermitian operator, and if $f$ is constant then $\hat{f}$ is multiplication-by-$f$. However, to fulfill our mission, we must also require (1).

Theorem 2.3. (a) Equation (1) holds if and only if

$$\omega(\xi_1, \xi_2)s = \frac{i}{2}([\nabla_{\xi_1}, \nabla_{\xi_2}] - \nabla_{[\xi_1, \xi_2]})s$$

for all sections $s$ of $\mathcal{L}$ (i.e. $\omega$ is the curvature of $\nabla$).

(b) There exists a triple $(\mathcal{L}, \eta, \nabla)$ with curvature $\omega$ if and only if $\omega$ is integral.

(c) If $\omega$ is integral, then the triples $(\mathcal{L}, \eta, \nabla)$ with curvature $\omega$ are parameterised by $H^1(M, \mathbb{R}/\mathbb{Z})$.

The condition in (b) that $\omega$ is integral means that its class in $H^2(M, \mathbb{R})$ comes from $H^2(M, 2\pi \mathbb{Z})$, or, more geometrically, that the integral of $\omega$ over every closed, oriented 2-surface in $M$ is an integer times $2\pi$. Suppose that there exists some $(\mathcal{L}, \eta, \nabla)$ such that $\nabla$ has curvature $\omega$. Then $\eta$ is determined uniquely by $(\mathcal{L}, \nabla)$; but given $(\mathcal{L}, \eta)$, the choices of $\nabla$ are parameterized by $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.

The representation $f \mapsto \hat{f}$ of $C^\infty(M)$ on $\mathcal{H} = L^2(M, \mathcal{L}, \mu)$ is called a prequantization.

Example 2.1. Suppose that $M = T^*N$ is a cotangent bundle. Then $\omega$ is exact, so there exists a prequantization. If $M$ is simply connected, then there exists at most one prequantization. In the simplest case, if $M = V$ is a symplectic vector space, then there is a unique prequantization.

2.4. Construction. I sketch how to construct the unique $(\mathcal{L}, \eta, \nabla)$ when $M$ is simply connected.

First define $\mathcal{L}_0$ to be the set of triples $(m, z, \gamma)$ where $m \in M$, $z \in \mathbb{C}$, and $\gamma$ is a path from some basepoint $m_0$ to $m$. Introduce an equivalence relation on $\mathcal{L}_0$ such that $(m, z, \gamma) \sim (m', z', \gamma')$ if and only if $m = m'$ and

$$\frac{z}{z'} = \exp(i \int_S \omega)$$

for any oriented 2-surface whose boundary is the concatenation of paths $\gamma^{-1} \ast \gamma'$. Such a surface exists for any $\gamma, \gamma'$, because we assumed $M$ simply connected; the right-hand side $\exp(i \int_S \omega)$ is independent of $S$ by the integrality of $\omega$. It is thus easy to see that $\mathcal{L} = \mathcal{L}_0/\sim$ is a line-bundle on $M$. We write $\mathcal{L}_m$ for the fibre at $m$, and continue to label elements of $\mathcal{L}_m$ by equivalence classes of pairs $(z, \gamma)$.

The Hermitian structure on $\mathcal{L}$ is given by $\eta(z, \gamma; z', \gamma') = zz'$ where, note, the paths $\gamma$ are the same. The connection $\nabla$ is defined by parallel transport as follows. Suppose given a path $\alpha$ from $m$ to $m'$. Given $(z, \gamma) \in \mathcal{L}_m$ there should be a unique way to lift $\alpha: [0, 1] \rightarrow M$ to $\tilde{\alpha}: [0, 1] \rightarrow \mathcal{L}$ so that $\nabla_{\alpha'(t)} \tilde{\alpha}(t) = 0$ (i.e. the section $\tilde{\alpha}$ of $\mathcal{L}$ over $\alpha$ is constant). The association $(z, \gamma) \mapsto \tilde{\alpha}(1)$ is ‘parallel transport’ from $m$ to $m'$, and these parallel transport maps determine $\nabla$. The definition is that parallel transport is given by $(z, \gamma) \mapsto (z, \alpha \ast \gamma)$. 

3. Example

The problem with prequantization is that the Hilbert space is ‘too big.’ The fundamental example is as follows. Suppose that \( S = \mathbb{R} \) (with coordinate \( q \)) and \( M = T^*S = \mathbb{R} \oplus \mathbb{R} \) (with coordinates \((q, p)\)). Quantum mechanics gives the following prescription: let \( H = L^2(S, dq) \), and define \( \hat{qs}(q) = qs(q) \) and \( \hat{ps}(q) = -i\partial_{q}s(q) \), for every \( s \in L^2(S, dq) \). These operators satisfy the expected ‘canonical’ commutation relation (cf. (1))

\[ [\hat{p}, \hat{q}] = -i. \]

There are two lessons to be learnt:

(a) We get a representation of the coordinate functions \( p \) and \( q \) (and some others), but not all of \( C^\infty(M) \).

(b) The Hilbert space consists of functions on \( S \), not anything like functions on \( M \) (as we would have expected from prequantisation). However, it is an irreducible representation of the Lie algebra generated by \( p \) and \( q \).

To preview the general case, the specified quantization depends on the additional datum of the projection \( T^*S \to S \), which is called a polarization. If \( M = T^*N \) is a cotangent bundle, we can always use the projection \( M \to N \), but in general there may be many possible reasonable choices of polarization.

4. The Heisenberg Group

We can abstract the above example in the following way. We take \( V \) to be a symplectic vector space. For each \( v \in V \) we have the linear function \( f_v := \omega(v, -) \in C^\infty(V) \). The Poisson bracket is

\[ [f_v, f_w] = \omega(v, w). \]

Thus these linear functions generate a Lie algebra \( \mathfrak{h}(V) \) (the Heisenberg Lie algebra): \( \mathfrak{h}(V) = V \times \mathbb{R} \) as a set, \( \mathbb{R} \) is central, and \( [v, w] = (0, \omega(v, w)) \) for \( v, w \in V \).

A quantization of \( V \) should yield (at least) a representation of \( \mathfrak{h}(V) \), in which constants \( a \in \mathbb{R} \) act by \( ia \). There are reasons to require this to be an irreducible representation. (The rough reason is that \( \mathfrak{h}(V) \) acts transitively on \( V \), so ought to act ‘transitively’, i.e. irreducibly, on \( \mathcal{H} \).)

The Heisenberg Lie group associated to \( \mathfrak{h}(V) \) is \( H(V) = V \times U(1) \) as a set; multiplication is defined so that the circle \( U(1) \) is central, and the product of \((v, 0), (w, 0) \in V \) is

\[ vw = (v + w, \exp(\frac{i}{2}\omega(v, w))). \]

So, to rephrase: a quantization of \( V \) is an irreducible unitary representation of \( H(V) \) in which \( U(1) \) acts by scalar multiplication.

Remark 4.1. Of course there are many different Lie groups with the same Lie algebra \( \mathfrak{h}(V) \). We choose this one because the representations of \( \mathfrak{h}(V) \) we are interested in correspond to representations of our \( H(V) \).

Theorem 4.1 (Stone-von Neuman). There exists a unique irreducible representation of \( H(V) \) such that \( U(1) \) acts by scalar multiplication.

‘Unique’ as usual means unique-up-to-isomorphism. However, the usual construction of the representation depends on a choice of a Lagrangian subspace of \( V \) – this is essentially the choice of a polarization mentioned before.

José says that in the infinite dimensional case, the choice of the Lagrangian is a serious matter, resulting in ‘different physics’ (I think this means: inequivalent representations). But even in our finite-dimensional case, where the representations are isomorphic, there is some subtle structure to the choices, related to the metaplectic group.