GEOMETRIC QUANTIZATION III: METAPLECTIC GROUP

1. The Heisenberg Group, Continued

1.1. **Construction of Representations.** The unique-up-to-isomorphism irreducible representation of the Heisenberg group is constructed in the following way. First one chooses a *Lagrangian* subspace $L \subset V$: this means that $L^\perp = L$. Then $L \oplus U(1) \subset H(V)$ is a maximal abelian subgroup, and it has a one-dimensional representation, the projection $\psi\colon L \oplus U(1) \to U(1)$. Finally, $(\mathcal{H}_L, \rho_L)$ is the induced representation $\mathrm{Ind}_{L\oplus U(1)}^{H(V)}\psi$.

More concretely, this amounts to the following. The space $\delta_{1/2}(V/L)$ of *half-densities* is a one-dimensional complex vector space with an isomorphism

$$\delta_{1/2}(V/L) \otimes \delta_{1/2}(V/L) = \delta_1(V/L),$$

where $\delta_1(V/L)$ is the space of translation-invariant measures on V/L.[1] The Hilbert space $\mathcal{H}_L$ is the completion of the space of smooth functions $\phi\colon V \to \delta_{1/2}(V/L)$ satisfying the condition

$$\phi(x+a) = \phi(x)e^{i\omega(x,a)/2}$$

for all $x \in V, a \in L$, and finite with respect to the norm

$$\|\phi\|^2 := \int_{V/L} \bar{\phi}\phi.$$

(Here note that $\bar{\phi}\phi\colon V \to \delta_1(V/L)$ is constant along L, so it defines a function $V/L \to \delta_1(V/L)$, i.e. a measure on V/L.) The action of $(v,t) \in H(V)$ is given by

$$\rho_L(v,t)\phi(x) = \phi(x-v)e^{i\omega(v,x)/2}t.$$

1.2. **Change of Lagrangian.** The main thing to take away is that although the isomorphism class of $(\mathcal{H}_L, \rho_L)$ is independent of L, there is no canonical isomorphism $\mathcal{H}_L \to \mathcal{H}_{L'}$ for different Lagrangians L, L'. (Of course such isomorphisms exist, and, because the representations are irreducible, they are unique *up to phase*.)

The dependence on the choice of Lagrangian is neatly summarised by considering the action of the symplectic group $\mathrm{Sp}(V) = \mathrm{Aut}(V,\omega)$. $\mathrm{Sp}(V)$ acts on $H(V)$ by group automorphisms $g \cdot (v,t) = (gv,t)$. It's then natural to ask whether the representation $(\mathcal{H}_L, \rho_L)$ extends to a representation of the semi-direct product $\mathrm{Sp}(V) \ltimes H(V)$.

The answer is *no*. However, there is a natural representation of $\mathrm{Mp}(V) \ltimes H(V)$ on $\mathcal{H}_L$, where $\mathrm{Mp}(V)$ is the unique non-trivial double cover of $\mathrm{Sp}(V)$.

More precisely: Let Λ be the manifold of all Lagrangian subspaces of V. $\mathrm{Sp}(V)$ acts transitively on Λ, and each $g \in \mathrm{Sp}(V)$ naturally determines an isomorphism $\mathcal{H}_L \to \mathcal{H}_{gL}$ *of Hilbert spaces*. As mentioned above, there exists an isomorphism *of representations* $\mathcal{H}_{gL} \to \mathcal{H}_L$. The composition of these two maps defines an operator $\rho_L(g)$ on $\mathcal{H}_L$. The question is whether one can choose these operators in such a way that $\rho_L(g)\rho_L(g') = \rho_L(gg')$. It is automatic that this will work, if not for $g, g' \in \mathrm{Sp}(V)$, then for a covering group of $\mathrm{Sp}(V)$. The claim is that the correct covering group is $\mathrm{Mp}(V)$.

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[1] I always consider complex-valued (so, not necessarily positive) measures. Thus $\delta_1(V/L)$ is a one-dimensional complex vector space. A smooth function $V/L \to \delta_1(V/L)$ is the same thing as a smooth measure on V/L.

1.3. **Weyl Quantization.** So far we found a representation of the Lie subalgebra of $C^\infty(V)$ generated by linear functions on V. Can one use this construction to represent other functions as well?

There is a very natural way to try, but it *does not* satisfy our original (somewhat arbitrary) criterion for quantization. For a sufficiently nice (e.g. Schwartz) function f on V, define the *Weyl transform* $W(f)$ to be the operator on $\mathcal{H}_L$ defined by

$$W(f)\phi(x) = \int_{v\in V} f(v)\rho_L(v,0)\phi(x)\,dv.$$

It turns out that W defines an isomorphism of vector spaces from $L^2(V,dv)$ to the algebra of 'Hilbert-Schmidt' operators on $\mathcal{H}_L$ (i.e. those represented by L^2-integral kernels). We can pull back the multiplication of operators to define a product $\star$ on $L^2(V,dv)$ making W into an isomorphism of algebras. This $\star$ is the 'star' or 'Moyal' product.

So we have associated operators $W(f)$ on $\mathcal{H}_L$ to a large class of functions f on V. But for this to count as quantization, we must have $[W(f_1),W(f_2)] = iW([f_1,f_2])$, or, in terms of $\star$,

$$f_1 \star f_2 - f_2 \star f_1 = i[f_1,f_2] \quad \text{(Poisson bracket)}.$$

This does not hold. However, something else, possibly just as good, is true.

Instead of a single symplectic form ω, introduce a real parameter $\hbar \in \mathbb{R}$ and for each $\hbar$ a symplectic form $\omega_\hbar = \hbar\omega$. Let $\star_\hbar$ be the corresponding star product on $L^2(V,dv)$. As $\hbar \to 0$, $\star_\hbar$ becomes the standard, commutative, multiplication of functions on V – this is 'the classical limit'. Moreover,

$$f_1 \star_\hbar f_2 - f_2 \star_\hbar f_1 = i\hbar[f_1,f_2] + o(\hbar^2).$$

So we do have a quantization to first order in $\hbar$. It is not immediately clear whether the higher-order terms are problematic – maybe our initial criterion for 'quantization' was too naive.

Remark 1.1. Iain points out that the star product generalises to arbitrary symplectic manifolds, which mathematically, at least, suggests it is a good object to consider. Also, $\star$ is independent of the choice of Lagrangian L. (One can see this by first making $L^2(H(V))$ into an algebra, the group algebra of $H(V)$. Then $L^2(V,dv)$ is just the largest quotient of $L^2(H(V))$ on which $U(1)$ acts by scalars.)

2. Half-forms and the metaplectic gerbe

The idea of a gerbe is not used in the references I have given, but it is a beautiful and (in some sense) elementary thing, so let's plunge in.

2.1. **Gerbes.** Recall that any group G can understood as a category BG with one object whose automorphism group is G. A 'connected G-groupoid' is any category equalvalent to BG. An analogous idea: a G-torsor is any set with a simply transitive action of G. (In what follows, I am thinking of G as a discrete group, but there are analogues for Lie groups.)

One can with reasonable clarity say that a covering space of a manifold M is a locally trivial family of sets parameterized by M. Similarly, a vector bundle is a locally trivial family of vector spaces. Similarly, a principal G-bundle is a locally trivial family of G-torsors.

A G-gerbe is nothing but a locally trivial family of connected G-groupoids. Note that covering spaces, vector bundles, principal bundles, etc., are families of sets, and are classified by first cohomology groups $H^1(M,-)$. A gerbe on the other hand is a family of categories, and we will see that G-gerbes are classified by $H^2(M,G)$.

Let us flesh out the definition a bit more. Fix a cover $M = \{U_i\}$ of M by open sets. A G-gerbe (with fixed local trivialisations) should amount to the following data. To each U_i we associate the trivial G-groupoid BG. On each intersection $U_i \cap U_j$ we associate a *gluing map*, i.e. an equivalence of categories $\alpha_{ij} \colon BG \to BG$. On triple intersections we should have natural isomorphisms

$$\beta_{ijk} \colon \alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ki} \to \mathrm{id}.$$

(For families of sets, we would here require an *equality.* But the general rule is that instead of *requiring an equality* between functors, we must *give a natural isomorphism* between them.)

A bit of thought shows that a natural isomorphism β_{ijk} is nothing but an element of G. The association of $\beta_{ijk} \in G$ to the triple intersection $U_i \cap U_j \cap U_k$ defines a class in $H^2(M, G)$, and these classes determine the gerbe up to isomorphism.

A principal G-bundle over a manifold M is again a manifold. In contrast, a G-gerbe over M is not a manifold, but some kind of categorified manifold (a stack). However, it has the right structure to do a lot of geometry, and one can often get away with thinking of a gerbe as some kind of covering space.

2.2. **Square-roots of line bundles.** Suppose we have a complex line bundle $\mathcal{B}$ over a manifold M. We would like to find a line bundle $\mathcal{A}$ and an isomorphism $\mathcal{A} \otimes \mathcal{A} \to \mathcal{B}$. In general, $\mathcal{A}$ does not exist. That is: it does not exist globally. The situation is very similar to the familiar one in which the square-root of a given function cannot be defined globally on M, but *is* defined on a double-cover. In our case, $\mathcal{A}$ is defined not on M but on a μ_2-gerbe $\tilde{M}$ over M, where $\mu_2 = \{\pm 1\}$. For each $m \in M$, the fibre, the connected μ_2-groupoid $\tilde{M}|_m$, is the category of pairs $(\mathcal{A}_m, \gamma_m)$, where $\mathcal{A}_m$ is a one-dimensional vector space and γ_m is an isomorphism $\gamma_m \colon \mathcal{A}_m \otimes \mathcal{A}_m \to \mathcal{B}_m$.

Thus a point of $\tilde{M}$ is a triple $(m, \mathcal{A}_m, \gamma_m)$. For each such triple, we have a line $\mathcal{A}_m$, and these form a line bundle $\mathcal{A}$ on $\tilde{M}$. By construction, $\mathcal{A} \otimes \mathcal{A}$ is the pullback of $\mathcal{B}$ to $\tilde{M}$.

Of course, it may happen that there *is* such an $\mathcal{A}$ on M itself. This happens when $\tilde{M}$ is the *trivial* μ_2-gerbe, meaning that there is a section $M \to \tilde{M}$. For example, on $\mathbb{P}^1$, the canonical bundle $O(-2)$ has a square-root $O(-1)$.

2.3. **Half forms.** In discussing representations of $H(V)$, we used the space $\delta_{1/2}(V/L)$ of *half-densities* to define the Hilbert space. In general, it works out better to use a space $\Delta_{1/2}(V/L)$ of *half-forms.* This is a one-dimensional complex vector space with an isomorphism

$$\Delta_{1/2}(V/L) \otimes \Delta_{1/2}(V/L) = \Delta_1(V/L)$$

where now $\Delta_1(V/L) = \wedge^{\dim V/L} \mathrm{Hom}_{\mathbb{R}}(V/L, \mathbb{C})$ is the space of (complex) translation-invariant volume forms on V/L. The difficulty is that while $\delta_{1/2}(V/L)$ can be defined canonically in terms of L, $\Delta_{1/2}(V/L)$ cannot. Rather, following the above gerbey discussion, one has the following situation.

There is a line bundle $\mathcal{B} = \Delta_1$ on Λ with fibre $\Delta_1(V/L)$ over L. There is a μ_2-gerbe $\tilde{\Lambda}$ over Λ with a line-bundle $\mathcal{A} = \Delta_{1/2}$ such that $\Delta_{1/2} \otimes \Delta_{1/2}$ is the pull-back of Δ_1. By definition, 'the' space $\Delta_{1/2}(V/L)$ of half-forms is the fibre $\mathcal{A}_{\tilde{\Lambda}}$, for some choice of $\tilde{L} \in \tilde{\Lambda}$ over $L \in \Lambda$.

2.4. **Metaplectic again.** The symplectic group $\mathrm{Sp}(V)$ acts on Λ and indeed on the total space of the line-bundle Δ_1. But it does not naturally act on $\tilde{\Lambda}$ or on $\Delta_{1/2}$. Rather, the metaplectic group $\mathrm{Mp}(V)$ acts. In fact, this gives a kind of construction of $\mathrm{Mp}(V)$. We'll come back to this next time when we talk about half-forms on manifolds.