GEOMETRIC QUANTIZATION IV: POLARIZATIONS AND QUANTIZATION

1. Polarizations

We have seen that for a vector space (V, ω), quantizations correspond to choices of Lagrangians $L \subset V$. For a manifold (M, ω), the analogous object is a *polarization*: a subbundle $P \subset TM$ whose fibres are Lagrangian subspaces, and which is involutive – i.e. there exists a foliation by integral surfaces

$$M = \bigsqcup_{\alpha \in M/P} M_\alpha.$$

(In other words, $m \in M_\alpha \implies T_m M_\alpha = P_m$.) We also assume that the set M/P of integral surfaces is a Hausdorff manifold.

1.1. **Examples.** (1) If $M = V$ is a symplectic vector space, then $T_m V = V$, and the choice of a Lagrangian $L \subset V$ defines a polarization with $P_m = L$.

(2) If $M = T^*N$, then there is a 'vertical' polarization such that the integral surfaces M_α are just the fibres of $T^*N \to N$.

Remark 1.1. This is the definition of a 'real' polarization. More generally, one can consider complex Lagrangian subbundles of the complex symplectic bundle $(TM)^{\mathbb{C}}$. For example, if M is a Kähler manifold, then a complex polarization is given, at each $m \in M$, by the span of the tangent vectors $\partial/\partial\bar{z}$. In general, a complex polarization is almost the same thing as a Kähler structure on a Hamiltonian reduction of M (see Woodhouse, ch. 3, for a precise statement). The real case is well suited to cotangent bundles, and obviously the complex case is well suited for Kähler manifolds (José says there are some pseudo-physical applications, but maybe a more important example is when one interprets Borel-Weil in terms of quantization – a future talk).

2. Half-forms

In the case of (V, ω), quantization used the space $\delta_{1/2}(V/L)$ of half-densities on V/L. In the general picture, it is preferable to use half-*forms*. Here are the definitions.

2.1. **General nonsense about principal bundles.** Given a group G, a G-torsor (over a point) is just a set S with a simply transitive action of G. A principal G-bundle over a manifold M (or again 'a G-torsor over M') is a locally trivial bundle of G-torsors. Given a G-torsor S over a point, and a representation (α, W) of G, then one can form $S \otimes_G \alpha := \{(s, w)\}/(sg, w) \sim (s, gw)$, a vector space non-canonicaly isomorphic to W. If S is a torsor over M, then $S \otimes_G \alpha$ is a vector bundle over M (the 'associated vector bundle').

The main example is as follows. If $X \to M$ is a vector bundle of rank n, the *frame bundle* $\mathrm{Fr}(X)$ is defined so that a section of X is an isomorphism of vector bundles $\mathbb{R}^n \times M \to X$. In other words, the fibre $\mathrm{Fr}(X)_m$ is the set of ordered bases for X_m. This is a principal $\mathrm{GL}_n(\mathbb{R})$ bundle. (In the lecture I gave a more complicated explanation, ah well.)

In particular, take $X = TM$, and $\alpha = \det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}$. Sections of the associated bundle $\Delta_1(M) := TM \otimes \det$ (a complex line-bundle) are volume forms on M.

Remark 2.1. Of course, α is real-valued, so $\Delta_1(M)$ is the complexification of a real line-bundle (real volume forms on M).

2.2. **Half-forms.** We want a line-bundle $\Delta_{1/2}(M)$ with an isomorphism $\Delta_{1/2}(M)\otimes\Delta_{1/2}(M) = \Delta_1(M)$.

First let $\widetilde{\mathrm{GL}}_n(\mathbb{C})$ be the double-cover of $\mathrm{GL}_n(\mathbb{C})$ on which the function $\sqrt{\det}$ is well-defined. Let $\widetilde{\mathrm{GL}}_n(\mathbb{R})$ be the preimage of $\mathrm{GL}_n(\mathbb{R})$ in $\widetilde{\mathrm{GL}}_n(\mathbb{C})$. A *metalinear structure* on $\mathrm{Fr}(TM)$ is a $\widetilde{\mathrm{GL}}_n(\mathbb{R})$-torsor $\widetilde{\mathrm{Fr}}(TM)$ over M with a compatible double-covering map $\widetilde{\mathrm{Fr}}(TM) \to \mathrm{Fr}(TM)$. Given a metalinear structure (which may or may not exist), we can define $\Delta_{1/2}(M) = \widetilde{\mathrm{Fr}}(TM) \otimes \sqrt{\det}$.

Remark 2.2. $\widetilde{\mathrm{GL}}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ is topologically a trivial double cover. But not group-theoretically, i.e. there is no group-homomorphic section. In the books they use complex frames and so on, so their 'frame bundle' is a principal $\mathrm{GL}_n(\mathbb{C})$ bundle. This would be important if we were considering complex polarizations, but I think what I have said here is nonetheless correct.

Remark 2.3. The existence of the metalinear structure, which is equivalent to the existence of an appropriate line bundle $\Delta_{1/2}(M)$, depends on the vanishing of a certain class in $H^2(M, \mu_2)$. In the language used last time, this is equivalent to the triviality of a certain μ_2-gerbe on M. Paul says that this is the same as the obstruction to the existence of a spin structure, and that one can define a spin structure just to be a metalinear structure on $\mathrm{Fr}(TM)$, without having to introduce a metric.

There is a natural way to differential sections of $\Delta_{1/2}(M)$ along vector fields, as follows. If $\xi \in \Gamma(TM)$ is a vector field on M, there is a natural way to push-forward tangent vectors, hence frames, along ξ. This means that ξ lifts to a vector field ξ' on $\mathrm{Fr}(TM)$. Then again, $\widetilde{\mathrm{Fr}}(TM) \to \mathrm{Fr}(TM)$ is a double cover, so ξ' lifts naturally to $\tilde{\xi}$ on $\widetilde{\mathrm{Fr}}(TM)$. Now, on the other hand, we can reinterpret the definition of $\Delta_{1/2}(M)$ to see that a section f of it is the same as a function $\tilde{f}$ on $\widetilde{\mathrm{Fr}}(TM)$ such that $\tilde{f}(bg) = \tilde{f}(b)\sqrt{\det g}$ for each $b \in \widetilde{\mathrm{Fr}}(TM)$ and $g \in \widetilde{\mathrm{GL}}_n(\mathbb{R})$. Thus we can define the derivative $\mathcal{L}_\xi$ of f along ξ by

$$\widetilde{\mathcal{L}_\xi f} = \tilde{\xi}\tilde{f}$$

as functions on $\widetilde{\mathrm{Fr}}(TM)$.

2.3. ***P*-half-forms.** Instead of considering the tangent bundle TM, we use the quotient TM/P for some polarization P. A metalinear structure $\widetilde{\mathrm{Fr}}(TM/P) \to \mathrm{Fr}(TM/P)$ gives us a line-bundle $\Delta_{1/2}(TM/P) = \widetilde{\mathrm{Fr}}(TM/P) \otimes \det$ of 'P-half-forms'. The considerations above show that we can differentiate sections of this $\Delta_{1/2}(TM/P)$ along vector fields ξ that preserve P, i.e. such that $[\xi, \Gamma(P)] \subset \Gamma(P)$.

3. Quantization

We start with a symplectic manifold (M, ω). The quantization procedure involves various choices, each of which may be problematic. First we must find a line bundle $\mathcal{L}$ with a Hermitian structure η and a connection ∇ with curvature ω. Next we must find a polarization P and a metalinear structure $\widetilde{\mathrm{Fr}}(TM/P) \to \mathrm{Fr}(TM/P)$.

3.1. **Construction of the Hilbert space.** The Hilbert space will be constructed out of sections of $\mathcal{A} := \mathcal{L} \otimes \Delta_{1/2}(TM/P)$. Using the connection on $\mathcal{L}$, the Lie derivative on $\Delta_{1/2}(TM/P)$, and the Leibniz rule, we can differentiate sections of $\mathcal{A}$ along vector fields ξ that perserve P. In particular, this is possible when ξ is a section of P. Let $\mathcal{H}_0$ be the space of sections of $\mathcal{A}$ that are *constant* along P.

Remark 3.1. It seems that $\mathcal{H}_0$ could be zero, depending on the topology of the integral surfaces of P. Mathematically, surely one would expect to allow for higher cohomology here?

The Hermitian form on $\mathcal{L}$ and the construction of $\Delta_{1/2}(TM/P)$ lead naturally to a Hermitian pairing

$$\bar{\mathcal{H}}_0 \otimes \mathcal{H}_0 \to \Gamma(M/P, \Delta_1(M/P) \xrightarrow{\int} \mathbb{C}.$$

Let $\mathcal{H}$ be the Hilbert space defined by completing the finite-norm elements of $\mathcal{H}_0$.

3.2. **Representation of Operators.** If $f \in C^\infty(M)$ is a function such that the vector field X_f preserves P, then we can define an operator $\hat{f}$ on $\mathcal{H}$ by

$$\hat{f}s = fs - i\nabla_{X_f}s.$$

The condition that X_f preserves P seems to mean that f is at most linear along P. Thus we don't get representations of all of $C^\infty(M)$, but still quite a large subalgebra (compare to the Heisenberg case, where we only explicitly quantized functions linear on V). However, this isn't really good enough. It standard physical applications, with $M = T^*N$, we should at least be able to represent the Hamiltonian, which is at least quadratic along the vertical polarization.

3.3. **Metaplectic and Metalinear.** Understanding how to quantize more general functions f seems to be a tricky question without good answers, in general. The first idea is that, since X_f changes the polarization, we must understand how the quantization depends on P.

The only thing I will explain here is how a *metaplectic* structure can be used to relate *metalinear* structures on different polarizations.

Let $\mathbb{R}^{2n}$ be the standard symplectic vector space with 'canonical' symplectic basis $\{P_1, \dots, P_n, Q_1, \dots, Q_n\}$. The symplectic frame bundle $\mathrm{Fr}_s(TM)$ is such that a section is an isomorphism of symplectic bundles $\mathbb{R}^{2n} \times M \to TM$. It is a principal $\mathrm{Sp}(\mathbb{R}^{2n})$-bundle on M. A *metaplectic structure* on M is a principal $\mathrm{Mp}(\mathbb{R}^{2n})$-bundle $\widetilde{\mathrm{Fr}}_s(TM)$ with a compatible map $\widetilde{\mathrm{Fr}}_s(TM) \to \mathrm{Fr}(TM)$.

I will explain how a metaplectic structure on M determines a metalinear structure $\widetilde{\mathrm{Fr}}(TM/P) \to \mathrm{Fr}(TM/P)$ for any polarization P. Let $G \subset \mathrm{Sp}(\mathbb{R}^{2n})$ be the subgroup preserving the Lagrangian $\mathbb{R}^n = \langle P_1, \dots, P_n \rangle \subset \mathbb{R}^{2n}$. Then G has a double cover fitting into the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{GL}}(\mathbb{R}^n) \arrow[d] & \tilde{G} \arrow[l, "\pi"'] \arrow[r] \arrow[d] & \mathrm{Mp}(\mathbb{R}^{2n}) \arrow[d] \\
\widetilde{\mathrm{GL}}(\mathbb{R}^n) & G \arrow[l] \arrow[r] & \mathrm{Sp}(\mathbb{R}^{2n})
\end{tikzcd}.$$

Let $\mathrm{Fr}_G(TM)$ be the principal G-bundle such that a section is an isomorphism of symplectic bundles $\mathbb{R}^{2n} \times M \to TM$ restricting to $\mathbb{R}^n \to P$. Define

$$\widetilde{\mathrm{Fr}}_G(TM) \times_{\mathrm{Fr}_s(TM)} \widetilde{\mathrm{Fr}}_s(TM).$$

This is a principal $\tilde{G}$-bundle. Now push out to get a principal $\widetilde{\mathrm{GL}}(\mathbb{R}^n)$-bundle, i.e. define

$$\widetilde{\mathrm{Fr}}(TM/P) = \widetilde{\mathrm{Fr}}_G(TM) \otimes_{\tilde{G}} \widetilde{\mathrm{GL}}(\mathbb{R}^n).$$

(Thus, fibrewise, $\widetilde{\mathrm{Fr}}(TM/P)_m$ is $\widetilde{\mathrm{Fr}}_G(TM)_m \times \widetilde{\mathrm{GL}}(\mathbb{R}^n)$ modulo the relation $(bg, h) \sim (b, \pi(g)h)$ for all $g \in \tilde{G}$.) Because $\mathrm{Fr}_G(TM) \otimes_G \mathrm{GL}(\mathbb{R}^n)$ is nothing but $\mathrm{Fr}(TM/P)$, there is a projection $\widetilde{\mathrm{Fr}}(TM/P) \to \mathrm{Fr}(TM/P)$, and we have defined a metalinear structure.

This is a global version of the story we had last time, with $\mathrm{Mp}(V)$ acting on $\tilde{\Lambda}$.