THE COBORDISM HYPOTHESIS

1. References

This talk refers to the paper "Higher-Dimensional Algtebra and Topological Quantum Field Theory" by Baez and Dolan, which formulates the hypothesis, and "On the classification of Topological Field Theories" by Jacob Lurie, which proves a version of it. Both are wonderfully clear papers with lots of big ideas, so I only hope to give some of the flavour.

2. Formulation

Recall that for each $n \geq 1$ we have a cobordism category, now to be called nCob, in which the objects are $(n-1)$-dimensional closed, oriented manifolds, and the morphisms are n-dimensional oriented manifolds with boundary. If $\mathcal{C}$ is a symmetric monoidal category, e.g. $\mathcal{C} = \mathrm{Vect}$ the category of vector spaces over some field, then an n-dimensional TQFT is just a symmetric monoidal functor $Z\colon n\mathrm{Cob} \to \mathcal{C}$.

2.1. **Dimension 1.** We've seen that for $n = 1$, a TQFT with values in Vect is just a finite-dimensional vector space $Z(+)$ (where $\pm$ denote the two orientations of the one-point manifold). Why finite-dimensional? Recall that $Z(-) = Z(+)^*$, and $Z(+) = Z(-)^*$. This can only be true if $Z(+)$ is finite-dimensional. The generalisation is that a 1-dimensional TQFT with values in $\mathcal{C}$ is an object $V = Z(+)$ of $\mathcal{C}$ with a well-behaved dual; this means that there is an object V^* and morphisms

$$\mathrm{tr}\colon V \otimes V^* \to 1 \qquad u\colon 1 \to V^* \otimes V$$

such that the compositions $(\mathrm{tr}\otimes 1)\circ(\mathrm{id}\otimes u)\colon V \to V\otimes(V^*\otimes V) \cong (V\otimes V^*)\otimes V \to V$ and $(1\otimes \mathrm{tr})\circ(u\otimes \mathrm{id})V^* \to (V^*\otimes V)\otimes V^* \cong V^*\otimes(V\otimes V^*) \to V^*$ are identity maps. It follows that duals are unique when they exist (in a symmetric monoidal category).

The picture can be reformulated (definitionally) in the following way: 1Cob *is the free symmetric monoidal category with duals, generated by one object.*

(Compare: $\mathbb{Z}$ is the free group generated by one object; to give a homomorphism $\mathbb{Z} \to M$, M a monoid, is just to give an invertible element of M.)

2.2. **Dimension 2.** We've seen that a 2-dimensional TQFT with values in Vect is a commutative Frobenius algebra. The definition of 'commutative Frobenius algebra' makes sense in any symmetric monoidal category, and the same argument shows that a 2-dimensional TQFT with values in $\mathcal{C}$ is just a 'Frobenius algebra in $\mathcal{C}$'. The way Baez and Dolan formulate this is to say that 2Cob *is the "free rigid symmetric monoidal category on one commutative monoid object with nondegenerate trace."* Again, these words just mean: to give a functor $2\mathrm{Cob} \to \mathcal{C}$ is to give a Frobenius algebra in $\mathcal{C}$.

2.3. **The Cobordism Hypothesis.** The Cobordism Hypothesis is a higher-dimensional genreralisation (strictly speaking, it generalises the first example, not the second). First, instead of nCob, it deals with something called Cob_n, which is a '(∞, n)-category'. The hypothesis is that Cob_n *is the free symmetric monoidal (∞, n)-cateogry with duals, generated by one object.*

A version of this hypothesis has been proved by Lurie. As far as I understand, the hypothesis should be understood as a guide to defining what (∞, n)-categories, rather than as a conjecture about a pre-existing notion.

3. Some explanation

The rest of the talk was an attempt to clarify what nCob is. The discussion in the references is probably much clearer!

3.1. **Infinity categories.** A category is something with objects (also called '0'-morphisms) and morphisms (also called '1-morphisms'). An n-category is something with k-morphisms for all $0 \leq k < n+1$ (this includes possibly $n = \infty$). How exactly to organize all these morphisms (and their compositions) is one of the difficulties in this subject. Roughly, though, if X_k is the collection of all k-morphisms, there should be 'source' and 'target' maps $X_k \to X_{k-1}$, and if f, g are k-morphisms such that the source of f is the target of g, then there should be a composition $f \circ g \in X_k$. But there will be other data as well: e.g. if x, y are 0-morphisms, then the 1-morphisms $x \to y$ come organized into an $(n-1)$-category, and we should require that the composition of 1-morphisms is appropriately functorial.

3.2. **First example: categories of categories.** The collection of all categories forms a 2-category. The objects are categories, the 1-morphisms are functors, the 2-morphisms are natural transformations. More generally, the collection of all n-categories should form an $(n+1)$-category. An important thing to note: one should never ask whether two functors are equal, only whether or not they are isomorphic. For example, to say that a functor F is an isomorphism (in usual language, 'an equivalence of categories') is to say that there is a functor G in the opposite direction such that $F \circ G$ and $G \circ F$ are themselves isomorphic to the identity functors. This is a general theme: never ask whether two k-morphisms are equal, only whether or not they are isomorphic (via a $(k+1)$-morphism). In particular, the composition of morphisms is not required to be associative, but only associative up to higher isomorphism. The specification of these 'associativity' isomorphisms must itself form part of the data defining the n-category...

3.3. **Second example: fundamental groupoids.** Let X be a topological space. Then there should be an ∞-category ΠX such that $X_0 = X$, X_1 is the set of paths in X, X_2 is the set of homotopies between paths (fixing their endpoints), X_3 is the set of homotopies between homotopies... A little more precisely, X_k is the space of maps $B_k \to X$ (B_k being the k-dimensional unit ball). If we divide the boundary $S^{k-1} = \partial B_k$ into two hemispheres (i.e. two copies of B_{k-1}), then we get two restriction maps $X_k \to X_{k-1}$. There are standard ways to compose paths and homotopies.

Some comments about this example. First, the composition of paths is not associative: $(\gamma_1 \circ \gamma_2) \circ \gamma_3 \neq \gamma_1 \circ (\gamma_2 \circ \gamma_3)$. However, these two paths are homotopic, i.e. related by a 2-morphism. Moreover, this 2-morphism – which should be part of the data defining ΠX – is an isomorphism (again: in the sense that it has an inverse-up-to-3-morphism). In fact, every morphism in ΠX is an isomorphism.

An (∞, n)-category is an ∞-category in which all N-morphisms are isomorphisms, for $N > n$. So ΠX is an $(\infty, 0)$-category.

Even in this example, it is not clear how to organize all the possible compositions of morphisms, all the isomorphisms giving associativity, and so on. However, it is a guiding principal (going back to Grothendieck, I think) that however ∞-categories are defined, **every $(\infty, 0)$-category should be isomorphic to ΠX for some topological space X**. Moreover, ΠX (up to isomorphism) should determine X (up to homotopy equivalence).

In a sense, we can turn this into a definition, and say that an $(\infty, 0)$-category just *is* a homotopy type. This shortcut gives us some hope of giving a rigorous definition of (∞, n)-categories in general: for if f, g are two n-morphisms with the same source and target, the morphisms between them should form an $(\infty, 0)$-category. So we only have to organize 'combinatorially' the morphisms up to dimension n; after that we have homotopy theory.

3.4. **Informal Definition of Cob_n.** Cob_n is supposed to be an (∞, n)-category. The objects are 0-manifolds. The 1-morphisms are 1-manifolds with boundary. The 2-morphisms are '2-manifolds with corners'. (Again, several ways to make this precise. But the basic idea is this: suppose given two 0-manifolds x, y, and two cobordisms f, g between them. Then by gluing f and g along $\{x, y\}$, we obtain a closed 1-manifold. A '2-manifold with corners' is a 2-manifold M whose boundary ∂M is presented in this way; thus ∂M can be understood as a cobordism between some 1-morphisms f and g). In general, a k-morphism is a k-dimensional manifold whose boundary has some additional 'corner' structure. This continues for k up to n. Given two n-morphism X, Y, we then define the $(\infty, 0)$-category of maps between X and Y to be the topological space of diffeomorphisms betwen X and Y (thus we use the equivalence between $(\infty, 0)$-categories and homotopy types).

Strictly speaking, Cob_n is defined using *framed* manifolds – ones with trivialised tangent bundles. But there are elaborations of the hypothesis using oriented manifolds (and more generally).

3.5. **Consequences of the cobordism hypothesis.** As explained in Lurie's paper, the cobordism hypothesis determines the homotopy type of the geometric realisation of Cob_n. This was calculated more directly by Galatius-Madsen-Tillmann-Weiss.

3.6. **Why It Should Be True.** When we proved the result about Frobenius algebras, the key observations were that (1) there was essentially only one object, the circle; (2) every 2-manifold can be carved up into pairs of pants and discs. These pairs of pants and discs in some sense generated 2Cob, and the question was to find the relations. But in higher dimensions, it's much harder to figure out such generators and relations.

The idea of the cobordism hypothesis is to allow us to carve up manifolds into much simpler pieces – essentially to *triangulate* them. Now, an n-simplex is (if we get the definitions right!) a manifold with corners, an n-morphism in Cob_n. So the fact that any manifold can be triangulated means that Cob_n is generated by simplices. But the reasoning we deployed in the 1-dimensional cases shows that these simplices must essentially be identity morphisms. This hand-waving argument suggests that Cob_n is really very simple: it has an object (the point); it has a bunch of identity morphisms; and everything else is determined by disjoint unions and duals.