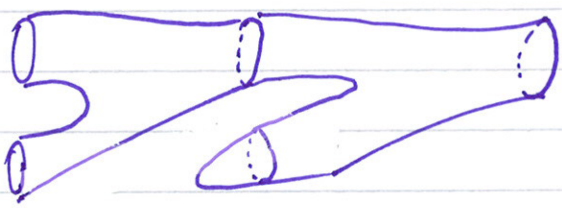
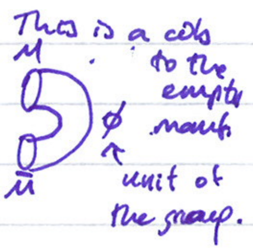
Main motivation : Classification of manifolds. ~~and of~~ Not just manifolds themselves but also the diffeomorphisms
I will be talking about compact, smooth manifolds, and they might be oriented, might have a spin structure, they may be framed, almost complex and so on.
I am going to concetrate for the presentation in oriented manifolds.

Definition : Two closed orient $(d-1)$-dimensional manifolds M_0 and M_1 are cobordant if there exists a compact oriented d-dimensional manifold W with boundary $\partial W = \bar{M}_0 \sqcup W_1$
↳ M_0 has here the reversed orientation.



$$M_0 \to W \leftarrow M_1 \longrightarrow W' \longleftarrow M_2$$

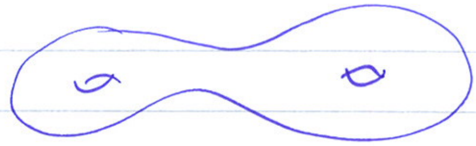
It is a very simple exercise to see that actually this defines an equivalence relation – go through exercise.
So the equivalence classes I can form, and denote by Ω^+_{d-1}
And there is more structure, group structure with product given by $\sqcup$ and inverse $M^{-1} = \bar{M}$



Graded ring $\bigoplus_{d \geq 0} \Omega^+_{d-1}$ with multiplication $\times$.

Examples :

$\Omega_0^+ = \mathbb{Z}$ $\qquad$ $\Omega_1^+ = \{0\}$ $\qquad$ $\Omega_2^+ = \{0\}$



S^1 is always the boundary of a surface, so the cob group is 0.

The group is 0 also in this case, because if we have an oriented surface, we can embed it in $\mathbb{R}^3$. So it bounds a handlebody $\Rightarrow$ it is the boundary of a 3-mfld.

Theorem (Thom) : $\Omega_{d-1}^+ = \pi_{d-1} \Omega^\infty MSO$

we can interpret these cobordism group as the homotopy group of a "rather large" space.

What is $\Omega^\infty MSO$? It is the infinite loop space of a spectrum. Writing it out explicitly, we have

$$\Omega^\infty MSO := \lim_{n\to\infty} \lim_{k\to\infty} \operatorname{map}\left(S^n, (U_{n,k})^c\right)$$

and $U_{n,k} \to Gr^+(n,k)$ is the universal n-dimensional bundle over the Grassmann manifold of oriented n-planes in $\mathbb{R}^{n+k}$

So given

$$\Omega^+_{d+1} = \Pi_{d-1}\, \Omega^\infty MSO$$

You might ask yourself: the elements in Ω^+_{d-1} are represented by mnflds, what kind of element does that give me in $\Pi_{d-1}\, \Omega^\infty MSO$. It has to be some kind of map of spheres. So here is the construction:

You take M and every compact manifold can be embedded in some large Euclidean space $\mathbb{R}^{d-1+n}$

$$M \subset \text{Tubular neighbourhood } N(M) \subset \mathbb{R}^{d-1+n}.$$

~~an~~ ~~Once~~ Once we choose a Tub Neigh $N(M)$ we can define an element in the group $\Pi_{d-1}\, \Omega^\infty MSO$.

An element in $\Pi_{d-1}\, \Omega^\infty MSO$ is given by a map from an S^{n+d-1}

$$f_M : \underbrace{S^{d-1+n} = (\mathbb{R}^{d-1+n})^c}_{\text{identify}} \xrightarrow[\text{everything outside the neighbourhood I map to a point}]{\text{Thom collapse}} N(M)^c \xrightarrow{\phi_{N(M)}} (U_{n,K})^c$$

identify this sphere as the compactification of my euclidean space

everything outside the neighbourhood I map to a point

can identify this compactification of $N(M)$ with the compactification of the normal bundle so I can push it forward to a map to the bundle $(U_{n,R})^c$

Thom computed the oriented case, and showed that it is a rational polynomial ring with generator in every dimension $4i$, and they can be represented by complex projective planes $\mathbb{C}P^2$, $\mathbb{C}P^4, \ldots$

$$\Omega_n^+ \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \ldots]$$

Proof: This calculation can be found in Milnor-Stasheff

$$\Pi_n(\Omega^\infty MSO) \otimes \mathbb{Q} = \Pi_{*+n}\left((U_{n,k})^c\right) \otimes \mathbb{Q}$$

$$= H_{*+n}\left((U_{n,k})^c\right) \otimes \mathbb{Q} \quad \text{by Serre}$$

$$= H_n\left(Gr^+(n,k)\right) \otimes \mathbb{Q} \quad \text{by Thom}$$

Wall computed the 2-torsion.

Milnor showed there is no odd torsion.

This is the background which was done in the 50's '60s and then in the late '80s Atiyah and Segal where introducing Topological Field Theories.

TFT

So here Cobordism comes in a quite different role.

Cobd is a cobordism category with

objects : compact, closed, oriented $(d-1)$ dimensional manifolds

Morphisms: d-dim cobordisms W with $\partial W = \bar{M}_0 \sqcup M_1$
Note that another cobordism W' with $\partial W' = \bar{M}_0 \sqcup M_1 = \partial W$ defines the same morphisms if there is a diffeomorphism relative to the boundary taking W' to W.

Def: A d-dim TQFT is a symmetric monoidal functor

$$\mathcal{F} : \underset{\text{cobodism category}}{\mathrm{Cob}_d} \longrightarrow \underset{\text{vector space}}{V}$$

that takes disjoint union of manifolds to tensor product of vector spaces.

Motivation : d-dimensional TQFTs define topological invariants for d-dimensional closed manifolds : If $\partial W = \emptyset$ then

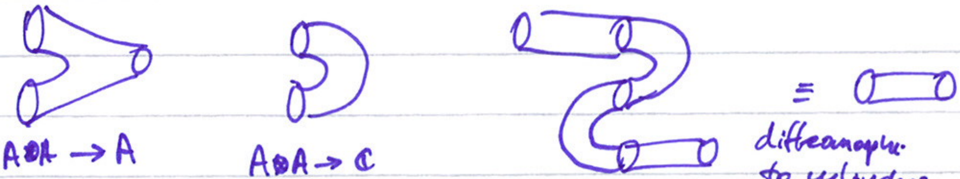
$$\mathcal{F}(W) : \underset{\substack{\shortparallel \\ \mathbb{C}}}{\overset{\text{source mfld}}{\mathcal{F}(\emptyset)}} \longrightarrow \underset{\substack{\shortparallel \\ \mathbb{C}}}{\overset{\text{target manifld}}{\mathcal{F}(\emptyset)}}$$

So $\mathcal{F}(W)$ assigns a number to W depending only on its topology.

In this case the unit for the tensor product is $\mathbb{C}$. A closed mfld gives just a linear map $\mathbb{C} \to \mathbb{C}$, just a number

<u>Theorem</u> 2-dim TQFTs are in one-one relation to Frobenius algebra:

Let $F(S^1) = \underset{\text{vector space}}{A}$ The functor associates to a circle S^1, a vector space A. Then to the disjoint union of circles it associates a tensor product.



$A \otimes A \to A$ product

$A \otimes A \to \mathbb{C}$

$=$ diffeomorphic to cylinder.

bilinear form is non degenerate

Physical inspiration

Quantum field theories are local:

one should be able to compute what happens in the system if we know what happens in the first interval of time and then in the second interval, and so on.

There is a ~~the~~ time evolution from

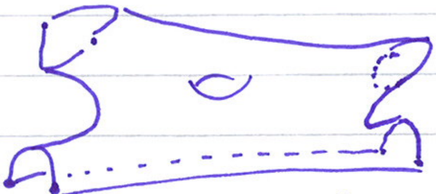
to

another one

But you can take your system even more locally, because we can also try to cut down our input manifolds.
So if you believe that the system should be so local that every should (like in ~~to~~ a homology theory) should be determined by a point. Then this leaves us to the categorification.

So you start off actually with a 0-dim manifold and look at a ~~cobord~~. between them. And then you look at cobordism between cobordisms.

This is categorification



Replace the cobordism category Cob_d by the d-fold category extended category $exCob_d$, where you start with the 0-manifld, ~~then~~ and also consider a d-fold symmetric monoidal category V_d.

Baez and Dolan in the mid '90s had the following hypothesis or conjecture. Namely that extended TQFT (those that start from a point) should be determined by their value on a point:
So the Field Theory should be local in that sense.

Here we have an example in dimension 1

So

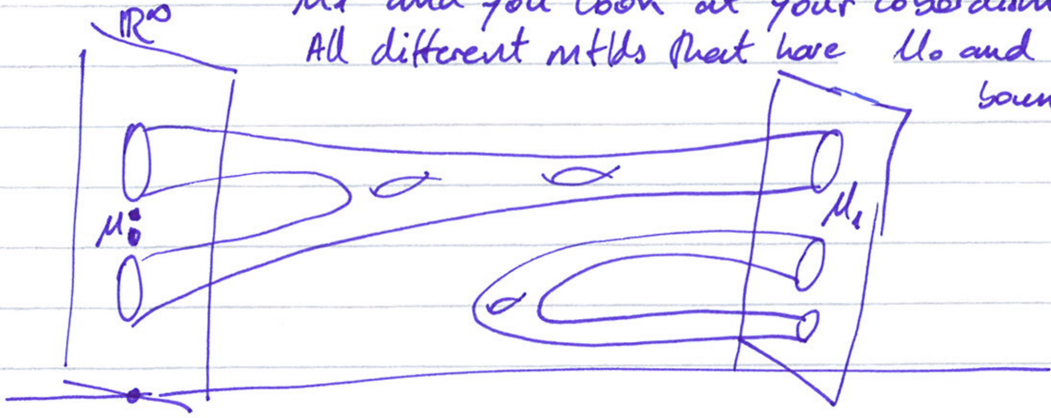
Enriched TQFT's

The idea of extended category is probably too local and not so interesting from the point of view of a topologist as other interpretations of TQFT.

The idea is now to enrich those cobordisms. So instead of considering $(d-1)$-dimensional manifolds and bordisms between them, we also put in the diffeomorphisms.

You don't just identify two cobordisms ~~that~~ when they are diffeomorphic, but you remember the diffeomorphism.

One way to get a very nice topological category is as follows:
So you look at your objects as being embedded in $\mathbb{R}^\infty$ across that. So you take any embedded mfld M_0 and M_1 and you look at your cobordism
All different mflds that have M_0 and M_1 as boundaries.



So that's a Topological category. We can clearly compose such a picture, if we had another one ... and they agree on M_1 ...
So the homotopy type of this space is a very interesting one.

Fix M_0 and M_1. The diffeomorphism space is actually the disjoint union of the classifying space diffeomorphism of the different cobordisms W that we can have in $\mathbb{R}^\infty$

So in particular this is an interesting space to study if we are interested in diffeomorphisms of manifolds.

What is this group $\coprod_W B\,\mathrm{Diff}(W, \partial)$?

Embeddings of W in large Euclidean space, say $\mathbb{R}^\infty$ is actually contractible by the Whitney embedding theorem, and then since we are looking at embedded mnifolds, rather than the embeddings themselves we mod out by the diffeomorphism group, so we get a BDiff here.

We have the following theorem:

Theorem (Galatius, Madsen, Tillmann, Weiss)

$$\Omega B(\mathrm{Cob}_d) \simeq \Omega^\infty MTSO(d) = \lim_{n\to\infty} \Omega^{d+n}\left((U_{d,n}^{\perp})^c\right)$$

This is the one where we had the morphisms being the diffeomorphisms of the d-dimensional manifolds.

This topological category $\Omega B(\mathrm{Cob}_d)$ was computed like this (as in the statement of the theorem).
And it looks very much like the one that appeared in Thom's Theorem.
So let us see what are the differences:

$U_{d,n}^{\perp}$ is the orthogonal complement of the universal bundle $U_{d,n} \to Gr^+(d,n)$

So we are now taking the Thom space of the orthogonal complement.

Also taking ~~the~~ loops is similar as before, but notice that the Thom class here $\left(\lim_{n\to\infty} \Omega^{d+n}(\underline{U_{d,n}^{\perp}})^c\right)$ is in dimension n,
and I am taking loops of $d+n$, so really the Thom class in this spectrum $MTSO(d)$ is in dimension $-d$.
(have shifted things down)

The characteristic map $\overset{\alpha}{\vee}$ from the empty set to $\emptyset$ is an embedded mnfld

$$\mathrm{Morph}_{Cob_d}(\emptyset,\emptyset) \ni W \underset{\substack{\text{I can} \\ \text{take again} \\ \text{a Tubular neighbourhood} \\ \text{of this mnfld}}}{\subset} N(W) \subset \mathbb{R}^{d+n}$$

and try to understand what it gives me here

and it is exactly like before:

$$\alpha(W) : S^{d+n} = \underbrace{(\mathbb{R}^{d+n})^c}_{\substack{\text{we take this Euclidean} \\ \text{space and think of it as} \\ \text{a sphere}}} \xrightarrow[\substack{\text{apply the} \\ \text{Thom} \\ \text{collapse} \\ \text{map}}]{\text{collapse}} N(W)^c \xrightarrow{\phi_{T(W)}} (U^{\perp}_{d,n})^c$$

and this time we push forward by the classification map of the tangent bundle of W

so

$$N(W)^c \xrightarrow{\phi_{T(W)}} (U^{\perp}_{d,n})^c$$

$$(\underset{\substack{\uparrow \\ W}}{x}, \underset{\substack{\text{is} \\ \text{a normal} \\ \text{vector to} \\ x}}{v}) \xmapsto[\text{map to}]{\text{This is a}} (\underbrace{T_x W}_{\substack{\text{the tangent} \\ \text{space at } x}}, v)$$

To compare it with Thom's Theory, there we used the classification map for the normal bundle

$$(x,v) \longrightarrow (N_x W, v) \in (U_{n,d})^c$$

But note that $(U_{n,d})^c \underset{\text{homeomorphic}}{\simeq} (U^{\perp}_{d,n})^c$

So the calculation given of the rational homotopy groups of the cobordism ring earlier on, goes through exactly here as well, only that we have now a dimension shift, the $-d$ comes up. But otherwise it is exactly the same thing.

$$H^*(\Omega_0^\infty MTSO(d), \mathbb{Q}) \simeq \Lambda^*(H^{*>0}(BSO(d);\mathbb{Q})[-d])$$

What is the characteristic map in different cases

$$\boxed{d=2} : \quad B\,\mathrm{Diff}(F_g) \xrightarrow{\alpha} \underline{\Omega^\infty MTSO(2)}$$

In this case the cobordism is an oriented surface of genus g, F_g, and the map α is now again a homology isomorphism in degrees $* \leq (2g-2)/3$
$\Rightarrow$ Mumford's conjecture

The homology of this can be easily computed as a polynomial algebra on even dimensional classes.