TQFT 2: FROBENIUS ALGEBRAS AND FINITE GAUGE THEORY

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I will largely follow the lecture notes by Freed.

1. 0+1 Dimensions

Recall the main idea of a TQFT. We have two symmetric monoidal categories. First, the category Cob_{n+1} of $(n+1)$-cobordisms, whose objects are oriented, closed n-dimensional manifolds. Second, the category of vector spaces over some field k. An $(n+1)$-dimensional TQFT is a symmetric monoidal functor Z between them (i.e. Z takes disjoint unions to tensor products).

The picture can be dressed up in many ways; to begin with, we could consider Riemannian manifolds, or we could consider graded vector spaces.

As Will explained, a $(0+1)$-dimensional TQFT is completely determined by the vector space $V := Z(+)$, where $+$ is the 'positively oriented' point. Roughly speaking, V is supposed to be the Hilbert space of quantum mechanics, representing the set of physically possible states at a given time. If I is an interval (a cobordism $+ \to +$), we should think of it as an interval of time; the linear map $Z(I)\colon V \to V$ is supposed to describe the evolution of the physical state. We have seen that for a TQFT we must have $Z(I) = \mathrm{id}$. If we considered I as a Riemannian manifold, then it would have an associated length t, and $Z(I)$ would be of the form e^{itH} for some 'Hamiltonian' $H\colon V \to V$. Thus the 'topological' case can be thought of as the case when $H = 0$, making the time evolution independent of the metric on I.

2. $1+1$ Dimensions

For $n = 1$ every object in Cob_2 is a disjoint union of circles S^1. Note also that S^1 has a unique orientation (up to diffeomorphism). So the first piece of data for a 2-dimensional TQFT is

(a) A vector space $V := Z(S^1)$.

For any other object M, $Z(M)$ is determined to be a tensor-power of V.

Next, we must have for every cobordism B from M to M' a linear map $Z(B)$ from $Z(M)$ to $Z(M')$. In general, any oriented 2-manifold B with boundary can be cut up into *cylinders* C, *disks* D and *pairs of pants* P. As usual, $Z(C) = \mathrm{id}$. Considering a disk D as a cobordism $\emptyset \to S^1$ or dually $S^1 \to \emptyset$, and P as a cobordism $S^1 \sqcup S^1 \to S^1$ or dually $S^1 \to S^1 \sqcup S^1$, we see that the additional data will be

(b) A map $u = Z(D)\colon k \to V$ and dually $u^*\colon V \to k$.

(c) A map $m = Z(P)\colon V \otimes V \to V$ and dually $m^*\colon V \to V \otimes V$.

In general $Z(B)$ will be determined as a composition of such maps. The fact that this composition is independent of how B is cut up gives certain relations between the data. For one thing, $u^* \circ m\colon V \otimes V \to k$ will be a non-degenerate, symmetric inner product. This identifies V with V^*, and with respect to this identification, u^* and m^* are literally the duals of u and m. Altogether, the relations say that that V is a *commutative Frobenius algebra*: it is a vector space with a unital, commutative, associative multiplication, and a 'trace' $u^*\colon V \to k$ such that $u^* \circ m\colon V \otimes V \to k$ is a nondegenerate inner product.

Theorem 2.1. *There is an equivalence of categories between* $(1+1)$*-dimensional TQFTs and commutative Frobenius algebras.*

The proof is not much more than what we have said.

2.2. **An example of a Frobenius algebra.** Semi-simple commutative Frobenius algebras V over $\mathbb{C}$ are all of the form $V = \mathbb{C}^n$. In other words, there is an orthogonal basis $e_1, \ldots, e_n$ for V such that $e_i e_j = \delta_{ij}$. The trace u^* must be given by the dot-product with some vector $(d_1, \ldots, d_n)$. These numbers $d_1, \ldots, d_n$ determine V up to isomorphism.

Here is an example from group theory. Let G be a finite group. The group algebra $\mathbb{C}[G]$ is a Frobenius algebra, but not commutative (unless G is). However, the center $V = \mathbb{C}[G]^G$ (which is also just the subalgebra fixed by G-conjugation) is a commutative Frobenius algebra. If we think of $\mathbb{C}[G]$ as the space of functions $G \to \mathbb{C}$, then the trace u^* is evaluation at the identity element (normalised by $1/\#G$). Multiplication is given by the usual convolution formula

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

The inner product is thus

$$u^*(f_1 * f_2) = \frac{1}{\#G} \sum_{g \in G} f_1(g) f_2(g^{-1}).$$

Now, V can also be understood as the space of *class functions*, i.e. functions on the conjugacy classes of G. For each irreducible representation ρ of G, the character $\operatorname{Tr}\rho$ is a class function. These characters form a basis for V of the type described above.

According to the theorem, taking $V = \mathbb{C}[G]^G$ determines a TQFT. However, this defines the TQFT in terms of 'generators and relations': to find $Z(B)$ for an arbitrary cobordism B, we must choose a way to cut up B into pairs of pants. It would be nicer to have a more canonical, geometric explanation of how this V arises from a TQFT. That is the subject of the rest of this talk.

Remark 2.1. The inner product here is slightly different from the Hermitian inner product usually used in representation theory. To connect the two, it is useful to note that the representations of G are unitary and that therefore $\operatorname{Tr}\rho(g^{-1}) = \overline{\operatorname{Tr}\rho(g)}$.

Remark 2.2. As far as I understand, this is how representation theory was originally invented, by trying to understand the structure of $\mathbb{C}[G]^G$ as a Frobenius algebra. Characters came before representations.

3. The Path Integral Formalism

Physicists typically think about QFTs in terms of path integrals. The picture (adapted to TQFTs) is basically as follows.

3.1. **Classical dynamics.** Suppose we have the following data:

(a) For each object $M \in \mathrm{Cob}_{n+1}$, a 'configuration space' X_M.
(b) Similarly, for each cobordism $B\colon M \to M'$, a 'path space' X_B.
(c) 'Boundary' maps $\pi_1\colon X_B \to X_M$ and $\pi_2\colon X_B \to X_{M'}$.

For a simple example, we can just take $X_M = M$; if B is a bordism $M \to M'$, then take X_B to be the space of paths $\gamma\colon [0,1] \to B$ with $\gamma(0) \in M$ and $\gamma(1) \in M'$. The boundary maps send γ to $\gamma(0)$ and $\gamma(1)$.

(d) An 'action' function $S\colon X_B \to \mathbb{R}$ for each B.

To continue the simple example, suppose that B were actually a Riemannian manifold. Then we could take S to be the length of γ. Physically, this example describes the following situation. We think of a particle moving in space. The laws of physics say that if it moves from $a \in M$ to $b \in M'$, then it does so along a geodesic, i.e. it moves on a path that minimizes S.

More generally, while M represents physical space, X_M is a space of configurations. If we had two particles, we would take $X_M = M \times M$ corresponding to their two positions. If instead of particles we were to think about fields (like the electromagnetic field), then X_M would be the set of field configurations (roughly, a space of functions on M). The 'laws of physics' are of the following form: for each $a \in X_M$ and $b \in X_{M'}$, there is a unique physically possible path $\gamma \in X_B$ from a to b: it is the one that minimizes $S(\gamma)$.

3.2. **Quantum Dynamics.** The quantum picture uses the same data in a different way. Given a, b, the 'laws of physics' determine a *probability* that the system evolves from a to b. More precisely, they determine a 'transition amplitude' $\langle a, b\rangle_B \in \mathbb{C}$ whose modulus-squared is a probability. Instead of there being one privileged path from a to b, *every* such path contributes $\exp(iS(\gamma))$ to the amplitude; in other words, $\langle a, b\rangle_B$ is given by a sum or integral

$$\langle a, b\rangle_B = \int_\gamma e^{iS(\gamma)}\, D\gamma$$

over all $\gamma \in X_B$ with boundary values a, b. Here $D\gamma$ is an appropriate measure. In this lecture X_B will be a finite set, and the integral a finite sum. In more serious situations, X_B is huge and there is no simple way to make sense of the integral.

A different way of getting at the same idea is to define $Z(M) = L^2(X_M)$ (just a finite-dimensional vector space, if X_M is a finite set). This has a basis of delta-functions δ_a corresponding to points $a \in X_M$. The various transition amplitudes $\langle a, b\rangle_M$ are the matrix coefficients of some linear map $Z(B)\colon Z(M) \to Z(M')$.

This looks just like a TQFT. The only problem is that to get functoriality we have to say something about gluing together path spaces. I'll leave this story to one side.

4. Finite Gauge Theory

Finally, I briefly explain how the Frobenius algebra $\mathbb{C}[G]^G$ arises from a natural TQFT. The short answer is that X_M is the set of isomorphism classes of G-bundles over M. Similarly, X_B is the set of isomorphism classes of G-bundles over B. A G-bundle over B can be restricted to a G-bundles on M and M', whence the boundary maps. Finally, the action is $S = 0$.

To see how this gives us the group algebra, one can show that, in general, for M connected, the set of G-bundles is $\mathrm{Hom}(\pi_1(M), G)/G$ (where G acts on itself by conjugation). For $M = S^1$, this is just the set of conjugacy classes. Therefore $Z(S^1) = L^2(X_{S^1})$ is the space of class functions, as promised.

It is a nice exercise to use the path integral formalism to rederive the product on $Z(S^1)$. Since $S = 0$, we have $\exp(iS(\gamma)) = 1$, and the path integral just counts the number of G-bundles with the specified restriction to the boundary.

Remark 4.1. In more general 'gauge theory' G would be a compact group and X_M the set (really stack) of G-bundles-with-connection over M. (When G is finite, any G-bundle has a unique connection.) The path integral is hard to define, but the basic dynamical data in 3.1 make perfect sense.