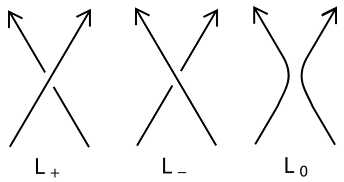
CHERN-SIMONS THEORY

This material is based on Witten's 'QFT and the Jones Polynomial', as well as the notes by Himpel on 'Lie Groups and Chern-Simons Theory.' A number of works by Freed are also of interest.

1. Jones Polynomial

A knot is an embedding of a circle in (most classically) S^3. A link is an embedding of several circles, i.e. a union of disjoint knots. The Jones polynomial V_L is an element of $\mathbb{Z}(q^{1/2})$ – i.e. a Laurent polynomial in a variable $q^{1/2}$ associated to each link L, and invariant under continuous deformations. The q here is 'the' q that appears everywhere in mathematics. The basic definition of the Jones polynomial is by a sort of induction: $V_L = 1$ if L is the unknot (e.g. a geodesic circle in S^3); and then there is a 'skein relation' that explains how V_L changes when you move strands through each other: Suppose that links L_+, L_-, L_0 are the same outside some small ball, where they look as follows (picture from Wikipedia):



Then the skein relation is that $(q^{1/2} - q^{1/2})V_{L_0} = q^{-1}V_{L_+} - qV_{L_-}$. A curious feature of this definition is that it is 'two dimensional': the pictures only make sense if we consider a projection of the links into the plane, and so one has to show that V_L is ultimately independent of the projection. Witten frames his paper as giving a genuinely three-dimensional definition of V_L (and of some big generalisations) using a $2+1$ dimensional TQFT.

2. Lagrangian Field Theory

Let's recall the basic picture of Lagrangian field theory – in a bit more generality than we've had before. For every $(n+1)$-manifold X with boundary, we have M_X a 'space of fields'; for every closed n-manifold ∂X we similarly have $M_{\partial X}$, and a 'restriction' map $\pi\colon M_X \to M_{\partial X}$. We assume also that $M_{\partial X}$ has over it a Hermitian line-bundle $L_{\partial X}$, and that on M_X we are given a section S of the pulled-back line bundle $\pi^* L_{\partial X}$. This S is the (exponentiated) *action*.

All this data should satisfy some properties, which we will skip. The main point of these properties is that we should be able to get a TQFT as follows.

The Hilbert space $Z(\partial X)$ is the space of L^2-sections of $L_{\partial X}$,

$$Z(\partial X) = L^2(M_{\partial X}, L_{\partial X}).$$

Thinking, for simplicity, of X as a cobordism $\emptyset \to \partial X$, the value $Z(X)$ should be an element of $Z(\partial X)$, defined (at least heuristically) by a 'path integral'

$$Z(X) = \pi_* S.$$

(That is, to get $Z(X)$, we are supposed to integrate S along the fibres of π. Remember S is a section of $\pi^* L_{\partial X}$, so we would expect $\pi_* S$ to be a section of $L_{\partial X}$, hence an element of $Z(\partial X)$.) The basic problem with this definition is that it is generally hard to make sense of the path integral π_*.

Example 2.1. Remember we saw an example of Freed in $1+1$ dimensions: fix a finite group G, and let M_X be the groupoid of G-bundles over X. Take the action $S = 1$, and $L_{\partial X}$ the trivial bundle. In this case the path integral is well defined as a sum over finite sets (more carefully, as the sum over a finite groupoid). The Frobenius algebra corresponding to this TQFT was the algebra of class functions on G.

3. Chern-Weil-Simons

Largely recollections from Yoshi's talk.

3.1. **Characteristic classes.** We take G to be a Lie group (most importantly $SU(2)$) and let X be an $(n+1)$-manifold (most importantly $n = 2$) with boundary. There is a space BG, the moduli space of G-bundles. What this means is that giving a map $f\colon X \to BG$ is the same as giving a G-bundle on X. (This defines BG as a stack, but one could consider it in other categories, e.g. as a homotopy type.) If $\alpha \in H^*(BG)$ is a cohomology class, then $f^*\alpha \in H^*(X)$ is an invariant of P, called a 'characteristic class'.

Example 3.1. If $G = U(1)$, so that a G bundle is a Hermitial line bundle. then we can take $BG \cong \mathbb{CP}^\infty$, $H^*(BG, \mathbb{Z}) = \mathbb{Z}[x^2]$ (i.e. polynomials in a variable of degree 2). The characteristic class $f^*(x^2) \in H^2(X, \mathbb{Z})$ is the first Chern class of the line bundle.

Example 3.2. If $G = SU(2)$ then, analogously, $BG \cong \mathbb{HP}^\infty$ (infinite quaternionic projective space), $H^*(BG, \mathbb{Z}) = \mathbb{Z}[x^4]$. This lecture is in fact basically all about the characteristic class $f^*(x^4)$.

3.2. **Chern-Weil.** Chern-Weil theory gives a deRham model of characteristic classes. You take a connection A on P, with curvature form F_A: this is a 2-form on X with values in $P \times_G \mathfrak{g}$. Then, given $f \in (Sym^m \mathfrak{g}^*)^G$, we get a $2m$-form $f(F_A)$ on X (with values in $\mathbb{R}$). The basic theorem is that $f(F_A)$ is closed and that its cohomology class depends only on P, not on A.

Example 3.3. If $G = U(1)$, we can take $s\colon \mathfrak{g} \cong \mathbb{R}$, so $m = 1$. Thus we get a 2-form on X representing the first Chern class. If $G = SU(2)$ we can take f to be the killing form, so $m = 2$. Then we get a four-form on X.

3.3. **Chern-Simons.** We can sharpen the previous discussion as follows. If A, A' are two connections on P, then $f(F_A) - f(F_{A'})$ must be an exact $2m$-form, and it turns out there is a canonical $(2m-1)$ form $c(A, A')$ with this coboundary. This and variations on it are called Chern-Simons forms.

Now we assume that every G bundle on X is trivial – e.g. this is true when $\dim X = 3$ and $G = SU(2)$. We fix a trivialisation s of P, and take A' to be the connection such that s is flat. Then we can define $c_k(A) = kc(A, A')$. (In this definition $k \in \mathbb{Z}$ is just a convenient additional parameter, so we will get slightly different theories for different values of k. See remarks at the end for some explanation.)

4. Gauge Theory

G again is a Lie group. Gauge theory is an example of Lagrangian field theory in which M_X (similarly $M_{\partial X}$) is the moduli space of G-bundles-with-connection-on-X (henceforth, just 'connections'). Thus a point of M_X is a G-bundle with connection over X. These form a groupoid (i.e. different connections can be isomorphic or not). Given a bundle P, the *gauge group* $G_X = C^\infty(X, G)$ acts on it, two connections

(P, A) and (P, A') on the same bundle P are isomorphic if and only if exists $g \in G_X$ such that $(P, A') = g \cdot (P, A) := (P, g^*A)$.

To give a function on M_X is to give an appropriately smooth functor $M_X \to \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers; in other words, for each connection (P, A) we get a number $f(A)$, and this number is gauge-invariant: $f(A) = f(g^*A)$. To give a line bundle on M_X is to give a functor $M_X \to \mathrm{Pic}_{\mathbb{C}}$, where $\mathrm{Pic}_{\mathbb{C}}$ is the category of one-dimensional complex lines; so for each (P, A) we should have a complex line $f(A)$ which transformas as a one-dimensional representation of G_X.

4.1. **Chern-Simons.**

Theorem 4.2. *(For $\dim X = 3$, $G = SU(2)$, but more general.) If X is closed, then $\int_X c_k(A) \in \mathbb{R}$ is gauge-independent modulo $\mathbb{Z}$.*

Thus, for X closed, we can define the action $S(A) = \exp(2\pi i \int_X c_k(A)) \in U(1) \subset \mathbb{C}$ and hope to define a path integral $Z(X) = \pi_* S \in Z(\emptyset) = \mathbb{C}$ which is then a topological invariant of the three-manifold X. These invariants of *closed* 3-manifolds are already interesting, but, more importantly for our purposes, what happens if X has a boundary? We still define $S(A) = \exp(2\pi i \int_X c_k(A)) \in \mathbb{C}$. Now, this is no longer a gauge-*invariant* function of A, but it is gauge-*equivariant*, which means we can consider S as a section of a line bundle on M_X. Now, what we have to argue is that this line bundle is the pullback of a line bundle $L_{\partial X}$ on $M_{\partial X}$. And what this amounts to is the statement (generalising the above theorem) that $S(g^*A)/S(A)$ depends only on the restriction of $A \in A_X$ and $g \in G_X$ to the boundary ∂X. (This explanation is obviously rather sketchy.)

Remark 4.1. The above sketch defines a Lagrangian field theory. My understanding is that Witten presents some consistency/plausibility checks to show that the corresponding TQFT (essentially: the path integral) can be defined, but the rigorous construction of it was found by Reshitikhin-Turaev.

5. Jones Polynomial again

How is the Jones polynomial supposed to emerge from this picture? For that we have to understand what the observables of a gauge theory are. An observable, in general, is a function $\mathcal{O}$ on M_X. The main thing one can calculate is its *expectation* $\langle \mathcal{O} \rangle$, which is given (again heuristically) by the path integral

$$\langle \mathcal{O} \rangle = \pi_*(\mathcal{O} \cdot S) \in Z(\partial X)$$

(so if $\partial X = \emptyset$ then $\langle \mathcal{O} \rangle$ is a number). Now suppose that γ is knot (i.e. an embedded circle) in X. We can calculate the holonomy $H(\gamma, A) \in G$ for any connection A. But this is not gauge invariant. But now fix a representation ρ of G. Then $\mathcal{O}_\gamma(A) = \mathrm{Tr}\, \rho(H(\gamma, A))$ is an observable (i.e. it is a function of A and it is gauge-invariant). More generally, we get an observable $\mathcal{O}_\gamma$ for any link γ in X, as long as each knot in the link is labelled by a representation of G.

I believe Witten's basic theorem is as follows. Take $G = SU(2)$, $X = S^3$, ρ always the standard representation of G. Define $q = \exp(-2\pi i/(2 + k))$. Then $\langle \mathcal{O}_\gamma \rangle = V_\gamma(q^{1/2})$ (the Jones polynomial). Here k can be considered as a variable, so we reconstruct the whole Jones polynomial, not just one value.

By allowing different Lie groups and various representations, we get a whole lot of link invariants generalising the Jones polynomial.

5.1. **Remarks.**
Some tid-bits that I only vaguely understand.

The basic idea of the proof of Witten's theorem is as follows. First, we explained how to define Chern-Simons theory as a functor from the $2 + 1$-dimensional cobordism category. But we should really consider 'enriched cobordisms' in which an object is a closed manifold with marked points and a cobordism is a manifold with boundary and an embeddded tangle. (A tangle is like a link, but it also can have non-closed strands that have end-points on the boundary. I am still skipping over further complications, e.g. we should consider ribbon tangles.) If you do this, then the skein relation is exactly the sort of thing you would expect to get when you cut and paste manifolds with tangles.

The use of tangles is very closely related to considering TQFTs with corners (i.e. extending the TQFT to lower dimensions, as in the cobordism hypothesis). A lot of interesting ideas by way of categorification were worked out by Freed.

The cobordism hypothesis, applied to 3-dimensional TQFTs, says roughly that a TQFT is the same thing as a modular tensor category. As far as I understand, the modular tensor category for Chern-Simons theory is the category of representations of the quantum group $U_q(sl_2)$ (the same q as before).

The classical solutions to Chern-Simons theory are *flat* connections. In other words, the QFT we constructed (for some fixed cobordism X) is supposed to be a quantization of the moduli space of flat connection on X. Remember we have to choose the symplectic form on this phase space to be integral; the various choices of symplectic form are parameterised by the integer k.