An Asymptotic Analysis of an Optimal Hedging Model for Option Pricing with Transaction Costs

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Abstract

Davis, Panas & Zariphopoulou (1993) and Hodges & Neuberger (1987) have presented a very appealing model for pricing European options in the presence

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of rehedging transaction costs. In their papers the `maximization of utility' leads to a hedging strategy and an option value. The latter is different from the Black-Scholes fair value and is given by the solution of a three-dimensional free boundary problem. This problem is computationally very time-consuming. In this paper we analyse this problem in the realistic case of small transaction costs, applying simple ideas of asymptotic analysis. The problem is then reduced to an inhomogeneous diffusion equation in only two independent variables, the asset price and time. The advantages of this approach are to increase the speed at which the optimal hedging strategy is calculated and to add insight generally. Indeed, we find a very simple analytical expression for the hedging strategy involving the option's gamma.

Keywords: Option pricing, transaction costs, asymptotic analysis, nonlinear diffusion

1 Introduction

Option pricing in the presence of transaction costs has recently become a very popular subject for research. There are two main approaches to this work in the literature: local in time and global in time. The former was started by Leland (1985) and extended by Boyle & Vorst (1992), Hoggard, Whalley & Wilmott (1993) and Whalley & Wilmott (1993). The first three of these assume hedging takes place at given discrete intervals (Boyle & Vorst is actually a binomial model) and the last assumes flexible trading periods. In all cases the decision whether or not to rehedge is based upon minimizing the current level of risk as measured by the variance of the hedged portfolio. Such models are often used in practice and are invariably quick to compute. They typically result in two-dimensional nonlinear or inhomogeneous diffusion equations for the value of an option. The global-in-time models can be illustrated by the model of Hodges & Neuberger (1987) and Davis, Panas & Zariphopoulou (1993). Such models achieve an element of `optimality', since they are based on the approach of utility maximization. The appeal of optimality is obvious, but, on the other hand, such models do have a number of disadvantages. Two of these disadvantages are speed of computation and the necessity of prescribing the investor's utility function. The models are slow to compute since they usually result in three- or four-dimensional free boundary problems. There is great practitioner resistance to the idea of utility theory.

In this paper we perform a simple asymptotic analysis of the Davis, Panas & Zariphopoulou (1993) model. We show how, in the limit of small transaction costs, their three-dimensional free boundary problem reduces to a much simpler two-dimensional inhomogeneous diffusion equation of the form found in the local-in-time models. We thus bring together the competing philosophies behind modelling transaction costs. The asymptotic formulae for the hedging strategy we present here have been tested empirically by Mohamed (1994), and found to be the best strategy he tested.

Perturbation analysis is a very powerful tool of applied mathematics. It is used to great effect in areas such as fluid mechanics (Hinch, 1991), because it reveals the salient features of the problem whilst remaining a good approximation to the full but more complicated model. As yet the technique has, to our knowledge, rarely been used in finance. For this reason, we shall at times walk the reader very slowly through the calculations. For comparison, for an asymptotic analysis of the Morton & Pliska (1993) portfolio management problem with transaction costs see Atkinson & Wilmott (1993).

In section 2 we very briefly describe the model of Davis, Panas & Zariphopoulou (1993), the interested reader should read that paper carefully in conjunction with this. In section 3 we consider the asymptotic limit of small transaction costs. This results in an inhomogeneous diffusion equation for the price of an option. In section 4 we compare the model with others and draw conclusions.

Recall that in the absence of transaction costs the Black-Scholes equation for the value of an option is

$$W_t + rSW_S + \frac{\sigma^2 S^2}{2} W_{SS} - rW = 0. \tag{1}$$

Here S is the underlying asset price, t is time, r the interest rate, assumed deterministic, σ the volatility of the underlying and $W(S,t)$ is the value of an option. This equation must be solved for $t < T$ and $0 \leq S < \infty$. On $t = T$ we must impose a final condition, amounting to the payoff function for the option in question. For example, for a call option with strike price E we have

$$W(S,T) = \max(S - E, 0).$$

This is the problem to be solved in the absence of costs. In the presence of costs, we shall find an equation similar to the Black-Scholes equation but with additional small terms which allow for the cost of rehedging and which are nonlinear in the option's gamma. In common with the Davis, Panas & Zariphopoulou paper, we are initially considering the valuation of a short European call option. We shall continue to use W to denote the Black-Scholes value of a European option.

2 The model of Davis, Panas & Zariphopoulou

In the model of Davis et al the writing price of a European option is defined in terms of a utility maximization problem. Simply put, the option price (to the writer) is obtained by a comparison of the maximum utilities of trading with and without the obligation of fulfilling the option contract at expiry. When there are no costs this results in the Black-Scholes value for the option (Black-Scholes, 1973).

The asset price S is assumed to follow the random walk

$$dS = \mu S\,dt + \sigma S\,dX,$$

where μ and σ are constant and X is a Brownian motion.

When the utility function takes the special form $U(x) = 1 - \exp(-\gamma x)$ (so that γ is the index of risk aversion) Davis et al find that the option price $V(S, t)$ is given by

$$V(S, t) = \frac{\delta(T, t)}{\gamma} \log \left(\frac{Q_w(S, 0, t)}{Q_1(S, 0, t)} \right), \tag{2}$$

where T is the expiry date, $\delta(T, t) = e^{-r(T-t)}$ and $Q_1(S, y, t)$ and $Q_w(S, y, t)$ both satisfy the following equation

$$\min \left\{ \frac{\partial Q}{\partial y} + \frac{\gamma(1+\epsilon)SQ}{\delta}, \; -\frac{\partial Q}{\partial y} - \frac{\gamma(1-\epsilon)SQ}{\delta}, \; \frac{\partial Q}{\partial t} + \mu S \frac{\partial Q}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 Q}{\partial S^2} \right\} = 0.$$

Here ϵ measures the transaction costs: a trade of N shares will result in a loss of ϵNS. This cost structure represents bid-offer spread,[1] or more generally commissions and costs which are proportional to the value of the assets traded. The independent variable y measures the number of shares held in the optimally hedged portfolio. The two functions Q_1 and Q_w must satisfy certain final conditions, analogous to the payoff profile of the option; for example, for a call option

$$Q_1(S, y, T) = \exp(-\gamma c(S, y)) \tag{3}$$

and

$$Q_w(S, y, T) = \begin{cases} \exp(-\gamma c(S, y)) & S \le E \\ \exp(-\gamma (c(S, y) + E - S)) & S > E \end{cases} \tag{4}$$

where

$$c(S, y) = \begin{cases} (1+\epsilon) y S & y < 0 \\ (1-\epsilon) y S & y \ge 0. \end{cases}$$

So the final condition for the second problem (with subscript w) is equal to that of the first problem (with subscript 1) modified by the effects of the potential liability at expiry of the European call (after transaction costs). Note we are assuming here that the option is settled in cash. For options with delivery of the asset on exercise the analysis below remains the same; the final conditions merely alter.

[1] Davis et al consider the slightly more general case in which there are different levels of cost for buying and selling.

Finally, to fully pose the problem we must specify that for $t < T$, Q, $\partial Q/\partial S$ and $\partial^2 Q/\partial S^2$ must all be continuous.

This is a free boundary problem. It is explained by Davis et al how the $(S; y)$ space divides into three regions, shown schematically in Figure 1. The writer of the option must always maintain his portfolio in the region of the $(S; y)$ space bounded by the two outer curves. Whilst inside this region he does not transact. Should a movement of the asset price take the writer to the edge of this no-transaction region he must trade so as to just stay inside. If he hits the top boundary he must sell shares, if he hits the bottom boundary he must buy shares. The middle line in Figure 1 is the curve along which the investor must move in the absence of transaction costs, this curve is denoted by

$$y = y^*(S; t).$$

Both y^* and the position of the upper and lower boundaries are to be found. We shall find simple analytical expressions for all three of these curves.

In the buy region we have

$$\frac{\partial Q}{\partial y} + \frac{\gamma(1 + \epsilon)SQ}{\delta} = 0. \tag{5}$$

In the sell region we have

$$\frac{\partial Q}{\partial y} + \frac{\gamma(1 - \epsilon)SQ}{\delta} = 0. \tag{6}$$

In the no-transaction region we have

$$\frac{\partial Q}{\partial t} + \mu S \frac{\partial Q}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 Q}{\partial S^2} = 0. \tag{7}$$

This is the free boundary problem we shall shortly solve asymptotically. Since the two problems for Q_1 and Q_w are identical except for the final data we need only perform the analysis for one of them. When we come to apply the final data we will distinguish between Q_1 and Q_w as necessary. As mentioned above, across the two free boundaries (the outer curves in Figure 1) Q, $\partial Q/\partial S$ and $\partial^2 Q/\partial S^2$ must all be continuous.

3 Asymptotic analysis for small levels of transaction costs

Equation (5) is very easy to solve explicitly. The solution for $Q(S,y,t)$ in the buy region is found to be

$$Q = \exp\left(-\frac{\gamma S y}{\delta} - \frac{\gamma S \epsilon y}{\delta} + H^-(S,t;\epsilon)\right) \tag{8}$$

where H^- is, as yet, an arbitrary function of S and t that comes from solving the ordinary differential equation (5): in this equation S and t are effectively parameters.

In the sell region we can similarly solve (6) to get

$$Q = \exp\left(-\frac{\gamma S y}{\delta} + \frac{\gamma S \epsilon y}{\delta} + H^+(S,t;\epsilon)\right). \tag{9}$$

This contains another arbitrary function H^+. The two expressions (8) and (9) are the exact, general solutions of (5) and (6).

The solution in the no-transaction region is much harder to find. Indeed, we shall not find the general solution, rather we shall find the asymptotic solution valid for small ϵ. The first stage in determining this solution is to expand Q in an asymptotic series in powers of ϵ.

We write the solution in the no-transaction region as

$$\begin{aligned} Q = \exp\Big(&-\frac{\gamma S y^*}{\delta} + H_0(S,t) - \epsilon^{1/3}\frac{\gamma S Y}{\delta} + \epsilon^{1/3}H_1(S,t) + \epsilon^{2/3}H_2(S,t) \\ &+ \epsilon H_3(S,t) + \epsilon^{4/3}H_4(S,Y,t) + \epsilon^{5/3}H_5(S,Y,t) + \cdots\Big). \end{aligned} \tag{10}$$

There are two very important things to note about this expression. First, we have chosen to expand in powers of $\epsilon^{1/3}$. This is not an arbitrary choice. We shall see as we perform our analysis, how such a choice is the natural one. (Shreve (1994) has results which suggest a similar asymptotic scale for the width of the no-transaction interval for an optimal investment and consumption model with transaction costs under a different utility function, and notes that Fleming, Grossman, Vila and Zariphopoulou

(1990) have also obtained this scale.) Second, we have translated the y coordinate according to

$$y = y^*(S,t) + \epsilon^{1/3} Y. \tag{11}$$

Thus Y is a rescaled variable, see Figure 1. It is a measure of the difference between the number of shares actually held in the portfolio and the ideal number we would hold in the absence of transction costs, y^*. We shall find an explicit expression for y^* as a function of S and t. Y turns out to be a more natural variable to use than y. The factor of $\epsilon^{1/3}$ represents the scale of the asymptotic width of the no-transaction region for this type of transaction costs (proportional to value traded).

Observe how, in (10), there is Y dependence at $O(\epsilon^{1/3})$ and $O(\epsilon^{4/3})$. The former is forced by the leading terms in (8) and (9) and continuity of slope at the boundary of the no-transaction region. The reason for the latter is similar and the details will become apparent. It is such continuity requirements that actually force on us the special choice of $\epsilon^{1/3}$.

As yet (10) does not satisfy the equation in the no-transaction region. We must now find the functions H_i such that this equation and all relevant boundary and smoothness conditions are satisfied. We shall see, in performing this analysis, that the choice of a series expansion in powers of $\epsilon^{1/3}$ is inevitable.

Since the derivatives in (7) are with respect to t and S keeping y fixed, then

$$\frac{\partial}{\partial y} \to \epsilon^{-1/3}\frac{\partial}{\partial Y};$$

$$\frac{\partial}{\partial S} \to \frac{\partial}{\partial S} - \epsilon^{-1/3} y^*_S \frac{\partial}{\partial Y};$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \epsilon^{-1/3} y^*_t \frac{\partial}{\partial Y}.$$

Thus we readily find from (10) and (11) that

$$\frac{\partial Q}{\partial t} \equiv Q_t = \left(-\frac{\gamma S y^*_t}{\delta} + H_{0_t}(S,t) + \frac{r\gamma S y^*}{\delta} + \epsilon^{1/3}\frac{r\gamma S Y}{\delta} \right.$$

$$\left. + \epsilon^{1/3} H_{1_t}(S,t) + \epsilon^{2/3} H_{2_t}(S,t) + \epsilon H_{3_t}(S,t) \right.$$

$$+\epsilon^{4/3}H_{4_t}(S,Y,t)+y_t^*\left(\frac{\gamma S}{\delta}-\epsilon H_{4_Y}\right)+\cdots\Bigg)Q;$$

$$\frac{\partial Q}{\partial S}\equiv Q_S=\Bigg(-\frac{\gamma S y_S^*}{\delta}+H_{0_S}(S,t)-\frac{\gamma y^*}{\delta}-\epsilon^{1/3}\frac{\gamma Y}{\delta}$$
$$+\epsilon^{1/3}H_{1_S}(S,t)+\epsilon^{2/3}H_{2_S}(S,t)+\epsilon H_{3_S}(S,t)$$
$$+\epsilon^{4/3}H_{4_S}(S,Y,t)+y_S^*\left(\frac{\gamma S}{\delta}-\epsilon H_{4_Y}\right)+\cdots\Bigg)Q;$$

and

$$\frac{\partial^2 Q}{\partial S^2}\equiv Q_{SS}=\Bigg(\bigg(-\frac{\gamma S y_S^*}{\delta}+H_{0_S}(S,t)-\frac{\gamma y^*}{\delta}-\epsilon^{1/3}\frac{\gamma Y}{\delta}$$
$$+\epsilon^{1/3}H_{1_S}(S,t)+\epsilon^{2/3}H_{2_S}(S,t)+\epsilon H_{3_S}(S,t)+y_S^*\left(\frac{\gamma S}{\delta}-\epsilon H_{4_Y}\right)+\cdots\bigg)^2$$
$$+\Big(H_{0_{SS}}(S,t)+\epsilon^{1/3}H_{1_{SS}}(S,t)+\epsilon^{2/3}H_{2_{SS}}(S,t)$$
$$+\epsilon H_{3_{SS}}(S,t)-\epsilon y_{SS}^*H_{4_Y}+\epsilon^{2/3}y_S^{*2}H_{4_{YY}}-2\epsilon y_S^*H_{4_{YS}}+\epsilon y_S^{*2}H_{5_{YY}}+\cdots\Big)\Bigg)Q:$$

It will be observed that each of the above can be slightly simplified. We have retained them in this form to help the reader perform his own calculations.

The advantage of asymptotic analysis will now become clear when we perform the next step, to substitute these expressions into (7) and equate powers of $\epsilon^{1/3}$.

3.1 The O(1) equation

To leading order ($O(1)$) we find that

$$Q_t=\left(H_{0_t}+\frac{r\gamma S y^*}{\delta}\right)Q;$$
$$Q_S=\left(H_{0_S}-\frac{\gamma y^*}{\delta}\right)Q;$$
$$Q_{SS}=\left(H_{0_{SS}}+\left(H_{0_S}-\frac{\gamma y^*}{\delta}\right)^2\right)Q:$$

Thus to leading order equation (7) becomes

$$H_{0_t}+\frac{r\gamma S y^*}{\delta}+\mu S\left(H_{0_S}-\frac{\gamma y^*}{\delta}\right)+\frac{\sigma^2S^2}{2}\left(H_{0_S}-\frac{\gamma y^*}{\delta}\right)^2+\frac{\sigma^2S^2}{2}H_{0_{SS}}=0: \quad (12)$$

3.2 The $O(\epsilon^{1/3})$ equation

We can take this procedure to the next order, equating powers of $\epsilon^{1/3}$. We find that

$$\frac{r\gamma SY}{\delta} + H_{1t} + \mu S\left(-\frac{\gamma Y}{\delta} + H_{1S}\right) + \sigma^2 S^2\left(-\frac{\gamma Y}{\delta} + H_{1S}\right)\left(H_{0S} - \frac{\gamma y^*}{\delta}\right) + \frac{\sigma^2 S^2}{2}H_{1SS} = 0.$$

This equation contains a term proportional to Y and one independent of Y. Since all the other terms in the equation are independent of Y, these terms must separately be zero. From the first of these we find that

$$y^*(S,t) = \frac{\delta}{\gamma}H_{0S} + \frac{\delta(\mu - r)}{\gamma S\sigma^2}. \tag{13}$$

Thus, if we can find H_0 then we have found the leading order expression for y^*.

Equation (13) determines the hedging strategy in the absence of transaction costs, y^*, in terms of the leading order 'option value' H_0. If we substitute this back into (12) we find that H_0 satisfies

$$H_{0t} + \frac{\sigma^2 S^2}{2}H_{0SS} + rSH_{0S} = \frac{(\mu - r)^2}{2\sigma^2}. \tag{14}$$

If we write

$$H_0(S,t) = \frac{\gamma}{\delta}V_0(S,t),$$

we have

$$y^*(S,t) = V_{0S} + \frac{\delta(\mu - r)}{\gamma S\sigma^2}. \tag{15}$$

as given by Davis et al, and equation (14) becomes

$$V_{0t} + \frac{\sigma^2 S^2}{2}V_{0SS} + rSV_{0S} - rV_0 = \frac{\delta(\mu - r)^2}{2\gamma\sigma^2}.$$

The particular solution of this with zero final data is

$$-\frac{\delta(\mu - r)^2(T - t)}{2\gamma\sigma^2}.$$

The general solution is thus any solution satisfying the Black-Scholes equation plus this particular solution.

We then retrace our steps to get from V_0 to V, the option price, using (10) and (2) (for both Q_w and Q_1). We find that the leading order final data in the portfolio without the option liability, (Q_1), is $V_0(S,T) = 0$, whereas in the portfolio with the call option liability, (Q_w), it has the usual payoff functional form $V_0(S,T) = -\max(S-E,0)$. So from the linearity of (3.2) we see that, to leading order, (or in the absence of any costs) the option value is simply the Black-Scholes value. Similarly the extra number of shares required in the portfolio with the additional option liability is, to leading order, the Black-Scholes delta value.

We now consider the terms independent of Y, which give an equation for H_1

$$H_{1t} + \mu S H_{1S} + \sigma^2 S^2 H_{1S}\left(H_{0S} - \frac{\gamma y^*}{\delta}\right) + \frac{\sigma^2 S^2}{2} H_{1SS} = 0.$$

If we substitute for $H_{0S} - \frac{\gamma y^*}{\delta}$ using (15), and set $V_1 = \delta H_1/\gamma$ as above we find that V_1 satisfies the Black-Scholes equation. The final condition for this equation for both Q_w and Q_1 is $V_1(S,t) = 0$. (This is found by expanding the final conditions in powers of $\epsilon^{1/3}$ and considering the terms of $O(\epsilon^{1/3})$.)

Thus V_1 is identically zero for all S and $t < T$, and so the leading order correction to the Black-Scholes value occurs at the $O(\epsilon^{2/3})$ level.

3.3 The $O(\epsilon^{2/3})$ equation

We now take the analysis to higher order to find the correction to the Black-Scholes value due to transaction costs. If we examine the $O(\epsilon^{2/3})$ terms in (7), we find that

$$H_{2t} + rSH_{2S} + \frac{\sigma^2 S^2}{2} H_{2SS} + \frac{\sigma^2 S^2}{2} y_S^{*2} H_{4YY} + \frac{\gamma^2 \sigma^2 S^2 Y^2}{2\delta^2} = 0.$$

This is an ordinary differential equation[2] for H_4 which is easily integrated to give

$$H_4(S,Y,t) = -\frac{Y^2}{\sigma^2 S^2 y_S^{*2}}\left(H_{2t} + rSH_{2S} + \frac{\sigma^2 S^2}{2} H_{2SS}\right) - \frac{\gamma^2 Y^4}{12\delta^2 y_S^{*2}} + aY + b.$$

We now have to join this solution in the no-transaction cost region with the solutions (8) and (9) in the buy and sell regions respectively.

[2] The inhomogeneous term, proportional to Y^2, at this order has forced Y dependence in H_4.

Let us use the notation $Y^+(S;t)$ and $-Y^-(S;t)$ to denote the Y-coordinates of the boundaries of the no-transaction region. These are, of course, unknown and must be determined as part of the solution by imposing suitable smoothness conditions. As stated above we require Q and its first two derivatives with respect to Y to be continuous at $Y = Y^+$ and $Y = -Y^-$. From (8) and (9) we can see that continuity of the gradient of Q at $Y = Y^+$ and $Y = -Y^-$ is ensured by

$$H_{4_Y} = \frac{\gamma S}{\delta} \quad \text{on} \quad Y = Y^+$$

and

$$H_{4_Y} = -\frac{\gamma S}{\delta} \quad \text{on} \quad Y = -Y^-:$$

Thus

$$-\frac{2Y^+}{\sigma^2 S^2 y_S^{*2}}\left(H_{2_t} + rSH_{2_S} + \frac{\sigma^2 S^2}{2}H_{2_{SS}}\right) - \frac{\gamma^2 Y^{+3}}{3\delta^2 y_S^{*2}} + a = \frac{\gamma S}{\delta}$$

and

$$\frac{2Y^-}{\sigma^2 S^2 y_S^{*2}}\left(H_{2_t} + rSH_{2_S} + \frac{\sigma^2 S^2}{2}H_{2_{SS}}\right) + \frac{\gamma^2 Y^{-3}}{3\delta^2 y_S^{*2}} + a = -\frac{\gamma S}{\delta}:$$

The second derivative of Q with respect to Y must also be continuous, that is, zero, at $Y = Y^+$ and $Y = -Y^-$. If this were not the case then there could be no finite value for the option price.[3] Thus

$$\frac{2}{\sigma^2 S^2 y_S^{*2}}\left(H_{2_t} + rSH_{2_S} + \frac{\sigma^2 S^2}{2}H_{2_{SS}}\right) = -\frac{\gamma^2 Y^{+,-2}}{\delta^2 y_S^{*2}}$$

and $a = 0$. We conclude from this that the no-transaction region is to leading order symmetric about the Black-Scholes hedging strategy, i.e. $Y^- = Y^+$. Eliminating Y^+ and Y^- from these equations we arrive at

$$H_{2_t} + rSH_{2_S} + \frac{\sigma^2 S^2}{2}H_{2_{SS}} = -\frac{1}{2}\left(\frac{3\gamma^2 S^4 \sigma^3 y_S^{*2}}{2\delta^2}\right)^{2/3}: \qquad (16)$$

[3] Recall that the number of the underlying asset held contains a term V_{0_S}, as in Black-Scholes, to leading order. The infinite number of trades in a finite time required at the boundary of the no-transaction region would lead to an infinite cost unless the gamma of the option is zero at the boundary.

We also find that the edges of the 'hedging bandwidth', $Y = Y^+$ and $Y = -Y^-$, are given by

$$Y^+ = Y^- = \left(\frac{3S\delta y_S^{*2}}{2\gamma} \right)^{1/3}; \qquad (17)$$

to leading order.

We cannot stress the importance of this last result enough. As far as implementation of the optimal hedging is concerned, we need to know the boundaries of the no-transaction region. These are given by very simple analytic expressions in terms of y_S^*, via equation (17), which in turn is simply related to the option's gamma by equation (13). We shall see this more clearly in the final section of this paper.

Equation (16) is to be solved subject to the final condition

$$H_2(S;T) = 0:$$

By letting

$$H_2 = \frac{\gamma}{\delta} V_2(S;t)$$

we can write (16) as

$$V_{2_t} + rSV_{2_S} + \frac{\sigma^2 S^2}{2} V_{2_{SS}} - rV_2 = -\frac{\delta}{2\gamma} \left(\frac{3\gamma^2 S^4 \sigma^3}{2\delta^2} \right)^{2/3} \left(\left| V_{0_{SS}} - \frac{\delta(\mu - r)}{\gamma S^2 \sigma^2} \right| \right)^{4/3} \qquad (18)$$

It is now important to distinguish between the two problems for Q_w, the problem including the option liability, and Q_1, the problem without the option. The V_2 component of Q_1 satisfies (18) with $V_{0_{SS}} = 0$ i.e.

$$V_{2_t} + rSV_{2_S} + \frac{\sigma^2 S^2}{2} V_{2_{SS}} - rV_2 = -\frac{1}{2} \left(\frac{3}{2\sigma} \right)^{2/3} \frac{\delta(\mu - r)^{4/3}}{\gamma}:$$

This has solution with zero final data

$$V_2 = \frac{1}{2} \left(\frac{3}{2\sigma} \right)^{2/3} \frac{\delta(\mu - r)^{4/3}(T - t)}{\gamma}:$$

Using $W(S;t)$ to denote the Black-Scholes option value we see that the V_2 component of Q_w satisfies (18) with $V_{0_{SS}}$ being the Black-Scholes value for the gamma, i.e. W_{SS}. Thus we see that the option value correct to $O(\epsilon^{2/3})$ is simply

$$V(S,t) = W(S,t) + \epsilon^{2/3}\left(V_2(S,t) - \frac{1}{2}\left(\frac{3}{2\sigma}\right)^{2/3}\frac{\delta(\mu - r)^{4/3}(T-t)}{\gamma}\right) + \cdots;$$

where V_2 satisfies (18) with $V_{0_{SS}} = W_{SS}$.

3.4 The $O(\epsilon)$ equation

It is remarkable that the algebraic complexity of the problen is still manageable at the $O(\epsilon)$ level. We can thus take the asymptotic analysis even further.

The $O(\epsilon)$ terms in expression (7) give

$$H_{3_t} - y^*_t H_{4_Y} + \mu S(H_{3_S} - y^*_S H_{4_Y}) + \frac{\sigma^2 S^2}{2}\Big(H_{3_{SS}} - y^*_{SS} H_{4_Y} - 2y^*_S H_{4_{YS}}$$
$$+ y^{*2}_S H_{5_{YY}} + 2\left(H_{0_S} - \frac{\gamma y^*}{\delta}\right)(H_{3_S} - y^*_S H_{4_Y}) - \frac{2\gamma Y}{\delta} H_{2_S}\Big) = 0. \qquad (19)$$

This may be interpreted first as an ordinary differential equation for H_5 and then, given sufficient boundary conditions, as a partial differential equation for H_3. (Just as in the H_2, H_4 problem of Section 3.3.) To determine the correct boundary conditions recall that we must have continuity of first and second derivatives with respect to Y at all orders of ϵ. Thus

$$H_{4_Y} + \epsilon^{1/3} H_{5_Y} = \pm\frac{\gamma S}{\delta} \qquad (20)$$

on the top and bottom free boundaries. By going to higher order we must also expand the position of the free boundaries as power series in $\epsilon^{1/3}$. Transferring the boundary condition (20) onto the known leading order boundaries $y = Y^+$ and $y = -Y^-$, we find that

$$H_{5_Y} = 0 \quad \text{on} \quad y = Y^+ \text{ and } y = -Y^-;$$

since $H_{4_{YY}} = 0$ on $y = Y^+$ and $y = -Y^-$.

Now integrate (19) from $y = -Y^-$ to $y = Y^+$. We find that

$$H_{3_t} + rSH_{3_S} + \frac{\sigma^2 S^2}{2} H_{3_{SS}} = 0 \qquad (21)$$

(since $\int_{-Y^-}^{Y^+} H_{4_Y}\,dY = 0$). With $H_3 = \gamma V_3/\delta$ we can now see that V_3 satisfies the Black-Scholes equation.

The final data for this equation is, for both the 1 and the w problems,

$$H_3(S,T) = \frac{(\mu - r)}{\sigma^2}.$$

This is found by expanding (3) and (4) in powers of $\epsilon^{1/3}$. The solution of (21) with this final data is simply

$$H_3(S,t) = \frac{(\mu - r)}{\sigma^2}.$$

The only remaining step in calculating the option value to $O(\epsilon)$ is to apply continuity between the no-transaction region and the buy region. We have

$$H^- = H_0 + \epsilon^{2/3} H_2 + \epsilon \left(H_3 + \frac{\gamma S}{\delta} y^* \right).$$

Finally, since the option value depends on $Q(S,0,t)$, we need the result

$$Q(S,0,t) = \exp(H^-).$$

From (2) we now have

$$V(S,t) = W(S,t) + \epsilon^{2/3} \left(V_2(S,t) - \frac{1}{2} \left(\frac{3}{2\sigma} \right)^{2/3} \frac{\delta(\mu - r)^{4/3}(T - t)}{\gamma} \right) + \epsilon S W_S + O(\epsilon^{4/3});$$

where V_2 satisfies (18) with $V_{0_{SS}} = W_{SS}$. Observe that the $O(\epsilon)$ correction to our earlier result is simply the cost of changing the number of shares in the portfolio in order to set up the initial hedge! Recall that it is assumed that the option obligation will be held until maturity, and that the final condition incorporates any transaction costs payable at maturity in order to unwind the hedge.

4 Results and conclusions

In this section we give the results of our asymptotic limit of the Davis et al model and make comparisons with their numerical results. To be specific we have concentrated on examples given in Davis et al.

First, we consider a European call option with exercise price $E = 0.5$ and time to expiry 0.3. Other parameters are $r = 0.07$, $\sigma = 0.2$, $\mu = 0.1$ and $\gamma = 1.0$. The level of transaction costs is such that $\epsilon = 0.002$.

In Figure 2 we plot the solution for y^* and the hedging boundaries against S for the first problem (denoted by subscript 1) which does not have the option liability at expiry. The solution in the absence of costs, y^*, is the middle curve. The outer, bold curves are the boundaries of the no-transaction region when there are non-zero transaction costs as detailed above. Recalling our expressions for y^*, equation (13), and Y^+ and Y^-, equations (17), these three curves are given by

$$y = y^*(S, t) = \frac{\delta(\mu - r)}{\gamma S \sigma^2}$$

and

$$\begin{aligned} y &= \frac{\delta(\mu - r)}{\gamma S \sigma^2} \pm \epsilon^{1/3} \left(\frac{3S\delta y_S^{*2}}{2\gamma} \right)^{1/3} \\ &= \frac{\delta(\mu - r)}{\gamma S \sigma^2} \pm \epsilon^{1/3} \left(\frac{3\delta^3(\mu - r)^2}{2\gamma^3 \sigma^4 S^3} \right)^{1/3}. \end{aligned}$$

In Figure 3 we plot the equivalent solutions for the second problem (denoted by the subscript w) which includes the option liability at expiry. Again y^*, the solution in the absence of transaction costs, is the middle curve and two bold curves are the boundaries of the no-transaction region. These three curves are given by

$$y = y^*(S, t) = W_S + \frac{\delta(\mu - r)}{\gamma S \sigma^2}$$

and

$$\begin{aligned} y &= W_S + \frac{\delta(\mu - r)}{\gamma S \sigma^2} \pm \epsilon^{1/3} \left(\frac{3S\delta y_S^{*2}}{2\gamma} \right)^{1/3} \\ &= W_S + \frac{\delta(\mu - r)}{\gamma S \sigma^2} \pm \epsilon^{1/3} \left(\frac{3S\delta}{2\gamma} \left[W_{SS} - \frac{\delta(\mu - r)}{\gamma S^2 \sigma^2} \right]^2 \right)^{1/3} \end{aligned}$$

where W is the Black-Scholes call value.

This plot is of particular interest. Because y^* has turning points, the width of the no-transaction region (which is proportional to $(y_S^*)^{2/3}$) goes to zero. This gives the 'string of sausages' shape shown in Figure 3. This result has an obvious financial interpretation. At the two turning points of y^*, a relatively large change in the share price can be tolerated before rehedging is necessary. In stochastic terms, to leading order, we have

$$dy^* = y^*_S dS + \cdots.$$

Away from turning points dy^* is of the same order as dS. However, at the two turning points dy^* becomes deterministic and of higher order. Thus it is possible to impose tighter bounds on the no-transaction region and this is exactly what is seen.

In deriving these plots we have not had to solve any differential equation since the functions y^* and Y^+ depend only on W, the Black-Scholes call value.

We now move on to another example. The parameters in this case are $\epsilon = 0.002$, $\gamma = 1.0$, $\sigma = 0.05$, $r = 0.085$ and $\mu = 0.1$. We consider a European call with exercise price 20 and with up to three years until expiry.

The plot in Figure 4 shows the difference between the asymptotic limit of the Davis et al model and the Black-Scholes call option value. This is the bold curve. It has two components, the $O(\epsilon^{2/3})$ part and the $O(\epsilon)$ part, and these two curves are also shown in the figure. The bold curve is the sum of the other two curves. Note that the $O(\epsilon^{2/3})$ and the $O(\epsilon)$ curves are similar in magnitude. This is because they differ by a factor of order $O(\epsilon^{1/3})$ which for $\epsilon = 0.002$ is 0.13 and not very small.

This plot (and Figure 5) has required the solution of (18). The solution shown in Figure 4 was computed by a simple explicit finite-difference scheme and thus took approximately the same time to run as the binomial solution of an American option.

In Figure 5 we plot the time dependence of the difference between the asymptotic limit of the Davis et al model and the Black-Scholes value, for the same parameters as in Figure 4 with $S = 19$. This is the bold curve and is the sum of the lower two curves. Again these $O(\epsilon^{2/3})$ and $O(\epsilon)$ curves are similar in magnitude. Nevertheless this asymptotic solution shows very good agreement with the numerical results of Davis et al, also plotted.

To finish this paper, let us recall the model of Leland and Hoggard, Whalley & Wilmott. In that model it is assumed that a delta-hedged portfolio is rehedged every fixed time period δt. The option is then valued so as to give the hedged portfolio the same expected return as that from a bank. With the same cost structure as above it is readily found that for a short position

$$V_t + rSV_S + \frac{\sigma^2 S^2}{2} V_{SS} - rV = -\sqrt{\frac{2}{\pi \delta t}}\, \epsilon \sigma S^2 |V_{SS}|.$$

By writing $V(S,t) = W(S,t) + \epsilon V_2(S,t) + \cdots$ we have

$$V_{2_t} + rSV_{2_S} + \frac{\sigma^2 S^2}{2} V_{2_{SS}} - rV_2 = -\sqrt{\frac{2}{\pi \delta t}}\, \sigma S^2 |W_{SS}|, \qquad (22)$$

with $V_2(S,T) = 0$.

Now recall the model of Whalley & Wilmott. In that model the investor delta hedges with rehedging determined by 'market movements'. If the difference between the delta and the number of assets actually held becomes greater than $d(S,t)/S$ then the portfolio is rehedged to the delta value giving the portfolio the minimum variance. The function $d(S,t)$ which specifies the hedging bandwidth must be prescribed by the investor. The option value is again determined by assuming that the expected return is equal to the risk-free rate. With $V(S,t) = W(S,t) + \epsilon V_2(S,t) + \cdots$ it is found that this time the correction term for a short position satisfies

$$V_{2_t} + rSV_{2_S} + \frac{\sigma^2 S^2}{2} V_{2_{SS}} - rV_2 = -\frac{\sigma^2 S^4}{d^{1/2}} W_{SS}^2, \qquad (23)$$

with $V_2(S,T) = 0$. This models a strategy commonly used in practice.

Now we can see the similarities between the three different models. All of them give the Black-Scholes value to leading order with a smaller order correction. This correction differs between models, but in all cases satisfies an 'inhomogeneous Black-Scholes-type equation', where the extra term resulting from the transaction costs depends on some power of the Black-Scholes option gamma (W_{SS}).

In Whalley & Wilmott (1993) many issues arising from such equations are discussed. Briefly, these include the following.

1. Nonlinearity. Since the right-hand side of the V_2 equation is in each case a non-linear function of the Black-Scholes value of gamma, W_{SS}, there will inevitably be different values for short and long positions. Also portfolios of options must be treated as a whole and not as the sum of individually valued components.

2. Negative option prices. With the more general costs structure discussed in Hoggard, Whalley & Wilmott and Whalley & Wilmott (not simply bid-offer spread) it is possible to arrive at negative option prices. (To see this consider the commission component of costs. If a fixed amount is paid at each rehedge then for small asset values the call option can have a negative value.) This suggests modifying hedging strategies to allow the possibility of not rehedging if to rehedge would make the option value negative. This introduces a free boundary below which (for a call) the option should not be rehedged. However, it is unlikely that the simple bid-offer spread considered here would lead to negative option prices.

3. American options. As also mentioned in Davis et al it is the owner of the American option who controls its exercise. It is difficult to optimally value an American option unless the owner's hedging and exercise strategy is known. This entails at least knowing all of his estimates of the parameters.

From the point of view of the numerical solution of these equations we can say that the inhomogeneous equations will not take significantly longer to solve by finite-difference methods than the basic inhomogeneous Black-Scholes equation. Thus, by performing this simple asymptotic analysis of the Davis et al model, we have made its use a practical possibility.

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References

Atkinson, C. & Wilmott, P. 1993 Portfolio management with transaction costs: an asymptotic analysis. To appear in Math. Fin.

Black, F. & Scholes, M. 1973 The pricing of options and corporate liabilities. J. Pol. Econ. 81 637{54.

Boyle, P.P. & Vorst, T. 1992 Option replication in discrete time with transaction costs. J. Fin. 47 271.

Davis, M.H.A., Panas, V.G. & Zariphopoulou, T. 1993 European option pricing with transaction costs. SIAM J. Control & Optim. 31 470{493.

Fleming, W.H., Grossman, S.D., Vila, J.-L. & Zariphopoulou, T. 1990 Optimal portfolio rebalancing with transaction costs. Working paper, Division of Applied Mathematics, Brown University.

Hinch, E.J. 1991 Perturbation methods CUP

Hodges, S.D. & Neuberger, A. 1993 Optimal replication of contingent claims under transaction costs. Rev. Fut. Mkts. 8 222-239.

Hoggard, T., Whalley, A.E. & Wilmott, P. 1994 Hedging option portfolios in the presence of transaction costs. Adv. Fut. Opt. Res. 7 21.

Leland, H.E. 1985 Option pricing and replication with transaction costs. J. Fin. 40 1283{301.

Mohamed, B. 1994 Simulations of transaction costs and optimal rehedging. Appl. Math. Fin. 1 49{63.

Morton, A. & Pliska, S. 1993 Optimal portfolio management with ¯xed transaction costs. To appear in Math. Fin.

Shreve, S.E. & Soner, H.M. 1994 Optimal nvestment and consumption with transaction costs. *Ann. Appl. Prob.* 4 680–692.

Whalley, A.E. & Wilmott, P. 1993 A hedging strategy and option valuation model with transaction costs. OCIAM Working paper, Mathematical Institute, Oxford.

Caption to figures

Figure 1: A schematic diagram of (S, y) space showing the buy, sell and no-transaction regions.

Figure 2: The hedge ratio and no-transaction band as functions of S without the option liability. See text for details of parameters.

Figure 3: The hedge ratio and no-transaction band as functions of S for the problem with the option liability. See text for details of parameters.

Figure 4: The difference between the asymptotic limit of the Davis et al model and the Black-Scholes value for a European call. The bold curve is the sum of the other two curves. See text for details of parameters.

Figure 5: The difference between the asymptotic limit of the Davis et al model and the Black-Scholes value for a European call as a function of time to expiry. The bold curve is the sum of the other two curves. Numerical results taken from Davis et al are also shown.