An American in Paris

Richard J. Haber
Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB

Philipp J. Schönbucher
University of Bonn, Statistical Department,
Adenaueralle 24-42, 53115 Bonn, Germany

Paul Wilmott
Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB

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1. Introduction

Parisian options are barrier options for which the knock-in/knock-out feature is only activated after the price process has spent a certain prescribed, consecutive time beyond the barrier. This speci…cation is motivated by the need to make the option more robust against short-term movements of the share price, a single outlier cannot trigger the barrier. In particular, it is far harder to a¤ect the triggering of the barrier by manipulation of the underlying (see Taleb [4]). Classical barrier options present hedging problems close to the barrier because their Gamma becomes very large. To some extent, these problems are reduced, or at least 'smoothed', in the Parisian contract.

We present a ‡exible approach to valuing such options using the numerical solution of a partial di¤erential equation. This approach can price a variety of modi…cations of the basic Parisian contract including Parasian options (activation of the barrier conditional on the total time spent above the barrier), American early exercise rights and general payo¤s. The approach readily accommodates features, such as early exercise, that render the traditional Monte Carlo approach impractical. To demonstrate the ‡exibility of this method we have written a program for valuing many types of Parisian contract. This program may be downloaded free of charge from http://www.oxford…nancial.co.uk/oxford.

In January 1997 Risk Magazine published an article by Chesney, et al.[1] on the pricing of Parisian contracts by Laplace transform methods. That article provided the motivation for the present work since we believe our method to be superior for at least four reasons:

- It is relatively simple to understand, requiring no more than a knowledge of elementary partial differential equations.
- It is flexible, and can be extended to price many more general contracts.
- It is easy to program. Great care is needed with the numerical inversion of the Laplace transform but the finite difference method is robust.
- It is fast.

2. The State Space

The crucial point to note about Parisian options is that they are strongly path dependent; not only does the payoff depend on the value of the underlying at expiry but also on the path taken to get there. Yet this path dependence is perfectly manageable within the partial differential equation framework: we do not need to know all the details of the path taken. The only further information we require is the value of the new state variable τ, defined as the length of time the asset price has been beyond the barrier,

$$\tau := t - \sup\left\{ t' \leq t \mid S(t') \leq \overline{S} \right\}; \tag{2.2.1}$$

for the case of an 'up' barrier, where $\overline{S}$ is the barrier level. The analogous expression for a 'down' barrier is obtained by reversing the second inequality in relation (2.2.1). The dynamics of τ, for the 'up' barrier, are given by the simple expression

$$d\tau = \begin{cases} dt & \text{if } S > \overline{S}; \\ -\tau^{-} & \text{if } S = \overline{S}; \\ 0 & \text{if } S < \overline{S}; \end{cases} \tag{2.2.2}$$

where τ^{-} is the value of τ before it jumps to zero. τ increases at the same rate as t if the share price S is beyond the 'up' barrier $S > \overline{S}$, it is reset to zero if the share price hits the barrier $S = \overline{S}$, and it does not change if the share price is below the barrier $S < \overline{S}$. The new state variable τ can be viewed as a clock that starts ticking as soon as the share price crosses the barrier level, $\overline{S}$, and is immediately reset when the share price returns below $\overline{S}$. The knock-in/knock-out feature is only activated if $\tau \geq \overline{T}$, where $\overline{T}$ is the barrier-time triggering parameter. This specification exactly covers the covenants in the Parisian option's payoff.

The other two state variables needed are the share price S, and time t. We assume that the share pays dividends at rate D, and its price follows a lognormal Brownian motion given by

$$dS = \mu S\,dt + \sigma S\,dW \tag{2.2.3}$$

where dW denotes the increment of a standard Brownian motion. A typical sample path of the state vector in the state space for an 'up' option is shown in figure 2.

In terms of the state variables, the specification of a Parisian option with a knockout is as follows. The option has a payoff $F(S)$ at expiry T, unless at some time before T the state variable τ reaches an upper barrier $\overline{T}$, in which case the option expires worthless.

Given this setup we can write the value of a Parisian option V as a function of the three state variables S, t and τ as $V(S; t; \tau)$. The governing equation can be derived formally but here we will just state it and give an intuitive justification. There are two distinct situations to consider. The first is when the asset is below the barrier, $S < \overline{S}$. In this case the variable τ remains unchanged and we must solve the basic Black–Scholes equation with a continuous dividend rate D

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0. \tag{2.2.4}$$

The second case occurs when the asset rises above the barrier, $S > \overline{S}$, and so the clock, τ, is ticking. Here we must solve

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV + \frac{\partial V}{\partial \tau} = 0; \tag{2.2.5}$$

where the new state variable τ gives rise to a modified form of the Black–Scholes equation. We will show below how the solutions are linked in these two regions. Both regions are shown in figure 1.

3. Parisians and Parasians

In a standard Parisian option the clock variable τ is reset to zero once the share price moves below $\overline{S}$. This will be the case even if the excursion lasts for a very short time. For example, in the case of an 'up' barrier the share price can reside above the barrier for almost the entire time without triggering the option, provided that it returns below $\overline{S}$ often enough. However, the constraint on the time τ being consecutive may defeat one of the original intentions of the Parisian option which was to make the triggering of the knock-in/knock-out less susceptible to one-off outliers and very short-term movements of the share price (or even manipulation). With the given specification the triggering of the barrier is fairly robust, but the resetting of τ is still very much subject to short-term price movements.

In the light of this it seems natural to introduce a variation of the Parisian contract, an option where τ is not reset and where the knock-in/knock-out feature is only activated if the cumulative time spent beyond $\bar{S}$ exceeds some prescribed value. This aggregation feature resembles closely the averaging feature of an Asian option which is why we call this contract the Parasian option. Note that in all cases that the Parasian contract knocks in/out, the equivalent Parisian will have done so too.

This variation requires only a minor modification in the model, we just have to change the definition (2.2.2) of the dynamics of τ so that it does not reset at the boundary $S = \bar{S}$:

$$d\tau_a = \begin{cases} dt & \text{if} \quad S \geq \bar{S}; \\ 0 & \text{if} \quad S < \bar{S}: \end{cases} \tag{3.3.1}$$

where we have called the non-reset clock τ_a.

It is easy to imagine many other possible specifications of the clock τ. There could be another, lower barrier $\underline{S}$ and the time S spent below $\underline{S}$ would be subtracted from τ, or the speed with which τ changes could be proportional to the distance the share price S is beyond the barrier (thus weighting large deviations beyond the barrier more strongly). We invite the reader to find other variations himself and to find the appropriate specification of $d\tau$.

4. Boundary Conditions

At $S = \bar{S}$ we have to impose pathwise continuity of V, which will mean for Parisians (where τ is reset to zero at $S = \bar{S}$)

$$V(\bar{S}; t; \tau) = V(\bar{S}; t; 0): \tag{4.4.1}$$

However, in the case of a Parasian option V does not jump at $\bar{S}$[1], thus invalidating condition (4.4.1).

The exact specification of the option enters our model via the boundary conditions that we specify. These conditions are to be applied at the boundaries of the state space, $t = T$, $S = 0$, $S \to \infty$, and $\tau = \bar{T}$ (see Figure 4.1). In its most general form a Parisian-type option is specified as follows:

- If the knock-in/knock-out has not been triggered by expiry T, then the option has the share price contingent payoff $F(S)$ at expiry. This payoff might also depend on τ and would then be given by $F(S,\tau)$. For example, a boost option is an option whose payoff is proportional to the time the share price spent beyond the barrier. This option would then have a payoff of $F(S,\tau) = c\tau$.
- If the knock-in/knock-out has been triggered during the lifetime of the option, the option pays off $G(S)$ at expiry.

All common variations of the option can be accommodated within this framework, e.g.

1. 'In' Boundaries: One gets a European Call of maturity T and exercise price E as soon as the knock-in is triggered. Set $F(S) = 0$ and $G(S) = (S - E)^+$.
2. 'Out' Boundaries: One gets a European Call of maturity T and exercise price E unless the knock-out is triggered. Set $F(S) = (S - E)^+$ and $G(S) = 0$.

In this framework 'ins' and 'outs' are treated the same. The boundary conditions to specify are

$$\begin{aligned} V(S,T,\tau) &= F(S,\tau) \qquad 0 \le \tau < \bar{T}, \\ V(S,T,\bar{T}) &= G(S). \end{aligned} \tag{4.4.2}$$

5. An American in Paris

The pricing of American-style options in the pde framework could not be simpler, either conceptually or from a numerical analysis point of view.

We shall be very general in setting an American-style Parisian option, simply stating that the option gives its holder the additional right to exercise the option at any time prior to expiry and thereby receive an amount $A(S,t,\tau)$. From the simplest of arbitrage considerations we must have

$$V(S,t,\tau) \ge A(S,t,\tau). \tag{5.5.1}$$

Since we can exercise whenever we want, we should act to maximise the value of the contract to us. This optimality amounts to insisting that $\partial V/\partial S$, the option delta, is continuous. In solving the US-style contract numerically by an explicit finite-difference scheme, discussed below, all we need do is to add to our code a line representing (5.5.1)

6. Numerical Algorithm

The numerical solution to equations (2.2.4) and (2.2.5) is implemented using an explicit finite difference scheme. For simplicity we choose to only discuss the case of an 'up' barrier.

The share price S, time t and barrier time τ are discretised[1] as ΔS, Δt and $\Delta\tau$, respectively, where for convenience we set $\Delta\tau = \Delta t$. We denote with $V_{i,j}^{n}$ the numerical approximation to the option value $V(S,t,\tau)$ at share price $S = i\Delta S$, time from expiry $T - t = n\Delta t$, and time from the knockin/knockout $\bar{T} - \tau = j\Delta\tau$. We call $\bar{\imath}$ the discrete barrier in the share price (i.e. $\bar{\imath}\Delta S = \bar{S}$) and $\bar{\jmath}$ is the value of the clock if τ is at zero(i.e. $\bar{\jmath}\Delta\tau = \bar{T}$).

The explicit finite difference approximation to the spacial part of the Black-Scholes equation (2.2.4) is defined for fixed times n, j by the operator L as

$$L_{i,j}^{n} = \frac{i^2\sigma^2}{2}\left(V_{i+1,j}^{n} - 2V_{i,j}^{n} + V_{i-1,j}^{n}\right) + \frac{i(r-D)}{2}\left(V_{i+1,j}^{n} - V_{i-1,j}^{n}\right) - rV_{ij}^{n}. \tag{6.6.1}$$

There are two regions to consider (see Figure 6.1): The area where the 'clock' τ is running (i.e. beyond the barrier) and the area where the 'clock' τ stands still. For an up barrier the domain is defined by $i > \bar{\imath}$ and the time stepping is accomplished by the difference equation

$$V_{i,j+1}^{n+1} = V_{i,j}^{n} + \Delta t \cdot L_{i,j}^{n} \qquad \forall i > \bar{\imath}. \tag{6.6.2}$$

Note that in equation (6.6.2) we had to increase j since the 'clock' τ is ticking. The system can be visualized as diffusively propagating in time along a diagonal plane, with normal vector $(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ in (S, t, τ) space, as shown in Figure 6.1. In the second region, $i < \bar{\imath}$, the 'clock' τ does not change. Hence the system evolves along the vertical plane, with normal vector $(0, 0, 1)$, in accordance to the time-stepping scheme

$$V_{i,j}^{n+1} = V_{i,j}^{n} + \Delta t \cdot L_{i,j}^{n} \qquad \forall i < \bar{\imath}. \tag{6.6.3}$$

Until now we have neglected the dynamics at the the boundary $i = \bar{\imath}$ which link the upper and lower regions of the state space. In fact, it is this condition that differentiates between the Parisian and Parasian option. A Parisian options require the resetting of the 'clock' at $i = \bar{\imath}$. Thus we first use (6.6.3) to calculate $V_{\bar{\imath},\bar{\jmath}}^{n+1}$ for $i \le \bar{\imath}$ and then set the remaining boundary values, at time step $n + 1$, for all $j < \bar{\jmath}$ to the former (i.e. $V_{\bar{\imath},j}^{n+1} = V_{\bar{\imath},\bar{\jmath}}^{n+1}$, $j < \bar{\jmath}$). Next we proceed to calculate $V_{i,j}^{n+1}$ for $i > \bar{\imath}$ according to the scheme (6.6.2). In contrast, for a Parasian option

[1] For stability of the scheme Δt has to be chosen small enough.

we simply apply (6.6.3) over the domain $i \leq \hat{\imath}$, and then use (6.6.2) for $i > \hat{\imath}$ without resetting the values at the boundary.

The payoff at knockin or knockout enters the scheme through the specification of the value of the payoff at $j = 0$, $\tau = \hat{T}$ by way of boundary condition

$$V^n_{i;0} = \hat{G}(i\Delta S; T - n\Delta t) \qquad \forall i; \tag{6.6.4}$$

where $\hat{G}(S; t)$ is the (time t)-value of receiving $G(S)$ at time T.

The final payoff is included in the scheme by starting it off with

$$V^0_{i;j} = F(i\Delta S; \hat{T} - j\Delta t) \qquad \forall i; j > 0: \tag{6.6.5}$$

Finally, for the case of an American exercise feature we have to check, before updating $V^{n+1}_{i;j}$, to determine if the newly determined value is larger than the US exercise payoff $A(i\Delta S; T - (n+1)\Delta t; \hat{T} - j\Delta t)$. If not, we simply set the two equal by way of the conditional relation $V^{n+1}_{i;j} = A(i\Delta S; T - (n+1)\Delta t; \hat{T} - j\Delta t)$.

The program, which may be downloaded from www.oxfordfinancial.co.uk/oxford, solves for the price and hedging variables by way of the aforementioned explicit finite-difference scheme. Although Explicit finite-difference schemes are similar in spirit to the binomial numerical method they are more general and thus flexible. The dowloadable program is fast but certainly not optimal. Had speed had been our prime concern we would have used an implicit method such as Crank-Nicolson. This, along with other methods, is discussed by Dewynne & Wilmott in Risk, March 1993[3] and in great depth (with sample code) in Wilmott et al. [5]. The finite-difference solution of financial partial differential equations is finally becoming accepted as the most time-efficient method of pricing and hedging certain types of contract. The method is time efficient because it is extremely easy to program and the programs run very quickly. It is suitable for many types of contract including most common path-dependent derivatives and is trivially—with one extra line of code—extended to American-style exercise. It easily outperforms Monte Carlo simulation, the other popular choice for path-dependent contracts. The downloadable program has the following inputs and outputs

Inputs		Outputs
Underlying	Barrier type	
Spot Volatility Dividend yield Interest rate	In/Out Call/Put Strike Expiry Reset (Y/N) Trigger time	Option value Delta Gamma Theta

The outputs are arrays (against the underlying).

7. Results and Discussion

We will first consider the simple case of a Parisian, European, down-and-in put on an asset with no dividends, an expiration time of $T = 0.25$ years, a volatility of $\sigma = 0.20$, an interest rate of $r = 0.08$; strike $E = 10.0$; barrier $\overline{S} = 8.0$, and barrier time $\overline{T} = 0.05$. Figure 3 depicts the option value V versus price S and time t for the previously defined parameter values. As expected, the option is worthless for values of time less than the barrier time $\overline{T}$. However for time greater than the barrier time the function appears quite smooth. This is validated by examining the hedge value Δ versus price S and time t, as seen in Figure 4. In essence, the diffusion, prior to the barrier time, act to smooth the data such that the hedge ratio remains reasonably manageable, compared to a traditional knock-out barrier.

Next we consider the more sophisticated example of a Parisian, American, up-and-out call with dividend rate $D = 0.04$; an expiration time of $T = 0.25$ years, a volatility of $\sigma = 0.20$, an interest rate of $r = 0.08$; strike $E = 8.0$; barrier $\overline{S} = 10.0$, and barrier time $\overline{T} = 0.05$. The resulting plots of option value V and hedge ratio Δ of this more complicated example are shown in Figures 5 and 6, respectively.

Finally, we contrast the results of a Parisian option and a Parasian option for both an ‘in’ and ‘out’ barriers with European exercise, zero dividends, an expiration time of $T = 0.25$ years, a volatility of $\sigma = 0.20$, an interest rate of $r = 0.08$; strike $E = 10.0$; barrier $\overline{S} = 8.0$, and barrier time $\overline{T} = 0.10$. Figure 7 presents the results of value V versus price S for a down-and-out put at time $t = 0$. The Parisian option retains a higher premium for all values of S but particularly dominates its Parasian counterpart for values near the barrier $\overline{S}$. Intuitively this makes goods sense since the cumulative effect on τ of the Parasian amplifies the ‘out’ feature of the option. Similarly, as seen in Figure 8, for a down-and-in put the Parisian becomes worthless near the barrier value because of the resetting feature of τ.

In conclusion, out method provides an extremely flexible, fast, easy to program method for evaluating more sophisticated variations of traditional barrier options such as the Parisian and Parasian options under various exercise and payoff structures. This work clearly demonstrates the versatility and ease of utilizing the pde framework for the practical implementation of exotic option pricing.

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8. About the Authors

Richard Haber is a postdoctoral researcher in the Mathematical Institute, Oxford.

Philipp Schönbucher is a research assistant in the Financial Markets Group at the London School of Economics and the Department of Statistics at Bonn University.

Paul Wilmott is a Royal Society University Research Fellow in the Mathematical Institute, Oxford, and the Department of Mathematics, Imperial College, London.

9. List of Figures

Figure 1 The domain for a Parisian ‘down’ option.

Figure 2 Characteristics for an ‘up’ ParAsian option.

Figure 3 Value V versus time t and asset price S for a Parisian down–and-in European put with no dividends, an expiration time of T = 0:25 years, a volatility of ¾ = 0:20, an interest rate of r = 0:08; strike E = 10:0; barrier $\overline{S}$ = 8:0, and barrier time $\overline{T}$ = 0:05.

Figure 4 Hedge ratio ¢ versus time t and asset price S for a Parisian down–and-in European put with no dividends, an expiration time of T = 0:25 years, a volatility of ¾ = 0:20, an interest rate of r = 0:08; strike E = 10:0; barrier $\overline{S}$ = 8:0, and barrier time $\overline{T}$ = 0:05.

Figure 5 Value V versus time t and asset price S for a Parisian up-and-out American call with dividend rate D = 0:04; an expiration time of T = 0:25 years, a volatility of ¾ = 0:20, an interest rate of r = 0:08; strike E = 8:0; barrier $\overline{S}$ = 10:0, and barrier time $\overline{T}$ = 0:05.

Figure 6 Hedge ratio ¢ versus time t and asset price S for a Parisian up-and-out American call with dividend rate D = 0:04; an expiration time of T = 0:25 years, a volatility of ¾ = 0:20, an interest rate of r = 0:08; strike E = 8:0; barrier $\overline{S}$ = 10:0, and barrier time $\overline{T}$ = 0:05.

Figure 7 Value V of a Parisian and Parasian option (at initial time t = 0) versus asset price S for a down-and-out put with European exercise, zero dividends, an expiration time of T = 0:25 years, a volatility of ¾ = 0:20, an interest rate of r = 0:08; strike E = 10:0; barrier $\overline{S}$ = 8:0, and barrier time $\overline{T}$ = 0:10

Figure 8 Value V of a Parisian and Parasian option (at initial time t = 0) versus asset price S for a down-and-in put with European exercise, zero dividends, an expiration time of T = 0:25 years, a volatility of ¾ = 0:20, an interest rate of r = 0:08; strike E = 10:0; barrier $\overline{S}$ = 8:0, and barrier time $\overline{T}$ = 0:10