The best hedging strategy

by

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1. Introduction: In October 1993 we wrote an article in *Risk* magazine on pricing options under different hedging strategies (Whalley & Wilmott, 1993a). We discussed several commonly used strategies[1] such as hedging at fixed intervals and the market movement model, that is, hedging when the delta moves by a prescribed amount. Both of these strategies are used in practice and we showed how the option price in each case is the solution of a non-linear version of the Black-Scholes equation. This had important consequences for issues such as hedging portfolios as opposed to hedging individual options. The one question that we did not discuss was, What is the *best* hedging strategy?

2. Optimal hedging strategies: Before we can determine the best hedging strategy we must decide what we mean by 'best'. We must choose criteria for optimality. Obviously, different people will have different definitions for what is best. Part of the problem is having a systematic way of deciding which is the best out of a choice of investments. When buying shares or hedging options discretely in time, an investor is necessarily taking on some risk. This risk is not accounted for in the Black-Scholes world. Some hedging strategies are based on bounding risk as measured by the variance of a portfolio. One such strategy is to rehedge when the option's delta moves by a specified amount. Another way of deciding on what is an optimal investment is to employ a 'utility function'. Loosely put, this is a method of relating risk and return in a quantifiable fashion. An advantage of this approach is that it is possible to put all investments, both risky and risk free, on the same footing. So the method can handle both pure risky investments, like a stock, and a potentially risk-free investment, like a hedged option, in a systematic fashion. In our last *Risk* article we discussed strategies based on bounding risk. In the present article our criterion for optimality is the 'maximisation of expected utility'.

The seminal work in this area, combining both utility theory and transaction costs, was by Hodges & Neuberger (1989) (HN), with Davis, Panas & Zariphopoulou (1993) (DPZ) making some improvements to the underlying philosophy. HN explain that their philosophy is to assume that a financial agent holds a portfolio that is already optimal in some sense but then has the opportunity to issue an option and hedge the risk using the underlying. However, since rehedging is costly, they must define their strategy in terms of a 'loss function'. They thus aim to maximise expected utility. This entails the investor specifying a utility function. The case considered in most detail by HN and DPZ is of the exponential utility function. This has the nice property of constant risk aversion. Mathematically, such a problem is one of stochastic control and the differential equations involved are very similar to the Black-Scholes equation.

In HN and DPZ the value of the option and, most importantly, the hedging strategy are given in terms of the solution of a three-dimensional free boundary problem. The variables in the problem are asset price S, time t, as always, and also y, the number of shares held in the hedged portfolio. In the absence of any transaction costs the investor can perfectly hedge by holding a special number of shares, the delta. With costs present the investor cannot continuously hedge. In other words, the writer of the option holds *approximately* the Black-Scholes delta of the underlying but with some latitude: it turns out that there is a 'hedging bandwidth' around the Black-Scholes delta within which no transactions are made. What happens when the edge of this boundary is reached depends on the nature of the transaction costs. If costs are entirely proportional to volume traded then shares are bought or sold to remain at the edge. If there is a fixed cost component then shares are traded to position the number of shares to be at some 'optimal rebalance point'. This is illustrated schematically in Figure 1. If y is too large then the portfolio contains too many shares and some must be sold. Conversely, if y is too small some more must be bought to maintain the optimal portfolio.

HN give some indications as to how to find the hedging bandwidth and the optimal rebalance points. However, it is computationally very time consuming to solve a three-dimensional free boundary problem. In this article we describe a particularly important limit of this problem that allows us to find the details of the hedging strategy simply and quickly. We give explicit equations for the bandwidth and rebalance points in cases of practical importance. To do all this we must make one simple assumption: we shall assume that costs are small! This is not such a major assumption since even in very illiquid markets transaction costs are only a small percentage.

3. Transaction costs: We shall now describe the hedging strategy model but would advise the interested reader to see the original papers by HN, DPZ and Whalley & Wilmott (1993b, 1994) (WW) for all of the details.

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[1]This was based on the work of Leland (1985), Boyle & Vorst (1992), Hoggard, Whalley & Wilmott (1992) and Whalley & Wilmott (1992).

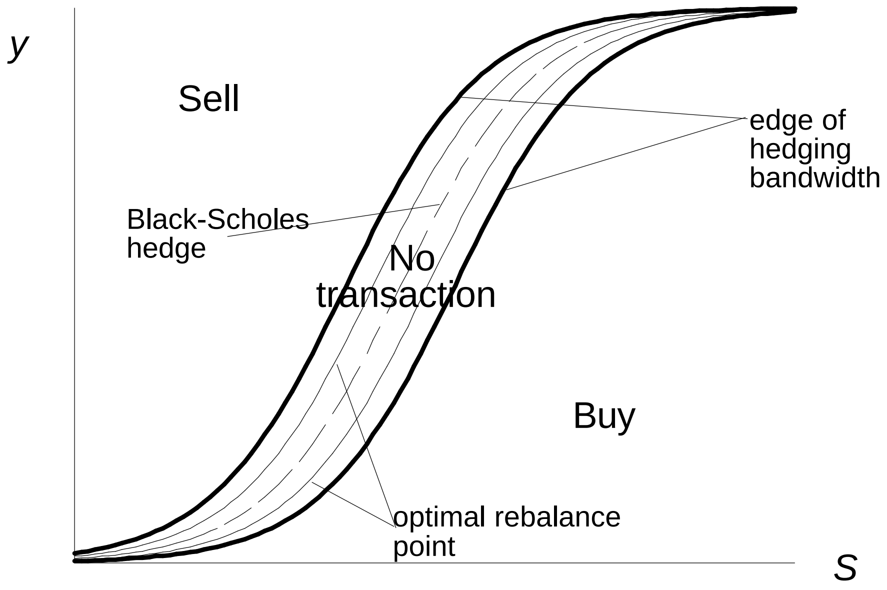

Figure 1: A schematic diagram of the transaction regions.

The problem of HN and DPZ is one of stochastic control. That is, the price of the underlying asset varies randomly, which would normally require a continual rehedging, as in the Black-Scholes analysis. Here though there is a cost associated with this rebalancing and this leads to the control part of the problem: we must determine when to rehedge, that is, when to assert the control. HN and DPZ find the natural result that hedging should only take place when the delta moves by a certain amount. This amount is a function of both time and asset price. It is not a simple expression but can be determined as part of a free boundary problem. This free boundary problem for the option value $V(S,t)$ involves the function $Q(S,y,t)$ and takes the form

$$V(S,t)=\frac{e^{-r(T-t)}}{\gamma}\log Q(S,0,t)$$

where

$$\frac{\partial Q}{\partial t}+rS\frac{\partial Q}{\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 Q}{\partial S^2}=0,$$

S is the asset price, t is time, r is the interest rate, T the expiry date and γ the index of risk aversion in the investor's exponential utility function.[2] This last parameter is chosen by the investor; the larger γ the more risk averse he is. The above equations must be solved subject to a final condition that depends on the payoff of the option. The option value must also satisfy a constraint, and this is where transaction costs come in. This constraint is the mathematical statement of the requirement that any trade in the underlying on rehedging cannot increase the value of a portfolio after allowing for transaction costs. The hedging bandwidth and optimal rebalance points follow from optimising the portfolio's value. This corresponds to requiring certain continuity and smoothness conditions. This is the problem presented by HN and DPZ.

This free boundary problem is quite complicated and as such it is difficult to solve numerically and to draw any intuitive results. We have found that it is possible to solve this free boundary problem asymptotically, that is, to take advantage of the typical size of the costs.

Asymptotic analysis is a mathematical technique for exploiting the magnitude of parameters in an otherwise complicated problem. For example, in the present problem, as costs decrease in size the resulting option value should be the Black-Scholes value. We can say that in the presence of costs the option value will be the Black-Scholes value plus a *small* correction. Asymptotic analysis is a formal method for finding a simpler problem for that correction, and the accuracy of that correction improves as costs get smaller.

The details of the asymptotic analysis that follows can be found in WW. Part of the asymptotic analysis is to realise that the hedging bandwidth is a narrow region around the Black-Scholes delta. The size of this region depends on the size of the transaction costs. A rough guide to the width is the fourth root of a typical individual rehedge. It then transpires that the correction to the Black-Scholes value is the square root of this cost. The assumption of small transaction costs means that instead of solving a three-dimensional free boundary problem we need only solve a much simpler one-

[2]The growth rate of the asset should appear in these equations but we have taken this to be the interest rate since it simplifies the algebra slightly.

dimensional free boundary problem. In Table 1 we give the algorithm for finding the hedging bandwidth and optimal rebalance points.[3] For arbitrary transaction cost structure this must be done numerically. For some simple structures, however, the solution may be found explicitly. We now give some illustrative examples of realistic cost structures.

3.1 Proportional costs

When the cost structure takes the form

$$K(S,U) = a|U|S,$$

that is, the cost of a trade of U shares with price S, we find that the hedging bandwidth takes the form

$$\Delta - \left(\frac{3aSe^{-r(T-t)}\Gamma^2}{2g}\right)^{1/3} \le y \le \Delta + \left(\frac{3aSe^{-r(T-t)}\Gamma^2}{2g}\right)^{1/3}.$$

Rehedging takes place *to the edge* of the bandwidth: the hedging bandwidth and the optimal rebalance points coincide. Here Δ and Γ are the Black-Scholes delta and gamma respectively. Notice how the bandwidth is large for large gamma e.g. near the strike price at expiry, and small for small gamma e.g. a long way in or out of the money.

This is the hedging strategy tested by Mohamed (1994). He compared the three strategies of (a) rehedging at fixed intervals, (b) rehedging at fixed movements in the delta and (c) the above, with a bandwidth proportional to the gamma to the power 2/3. His criterion for optimality was different from ours and involved the requirement that in 95% of simulations the strategy should make a profit. Despite the different criteria, he found that strategy (c), hedging with a bandwidth proportional to the gamma to the power 2/3, was the best out of the three. As a rough guide he found a cost saving of approximately 20% over the next best strategy.

3.2 Fixed costs

When costs takes the form

$$K(S,U) = b,$$

that is, the cost of any trade is the same amount, we find that the hedging bandwidth takes the form

$$\Delta - \left(\frac{12be^{-r(T-t)}\Gamma^2}{g}\right)^{1/4} \le y \le \Delta + \left(\frac{12be^{-r(T-t)}\Gamma^2}{g}\right)^{1/4}.$$

On rehedging, the trade is *to the centre* of the bandwidth, that is, to the Black-Scholes delta. After all, if you are going to rehedge you may as well hedge to the best position since the charge is fixed. Although we have deliberately not discussed the pricing of options only the hedging, we note that in the case of fixed cost the option price is simply represented by a volatility adjustment. This is similar to the Leland modification except that, of course, he had proportional costs and traded at fixed intervals; we have fixed costs but flexible trading periods. Again observe the behaviour of the bandwidth, where it is wide and where it is narrow.

3.3 Proportional plus fixed

When costs take the form

$$K(S,U) = a|U|S + b,$$

we must solve non-linear equations for the hedging bandwidth and the rebalance points. These equations are

$$AB(A+B) = \frac{3aSe^{-r(T-t)}\Gamma^2}{g}$$

and

$$(A-B)^3(A+B) = \frac{12be^{-r(T-t)}\Gamma^2}{g},$$

with $0 \le B \le A$. And then the hedging bandwidth is given by

$$\Delta - A \le y \le \Delta + A.$$

and the rebalance points are

$$y = \Delta \pm B.$$

It is a very simple matter to solve these polynomial equations for A and B. Any textbook on numerical analysis will describe how to do this.

4. Conclusion:

In the idealised Black-Scholes world all investors agree on the value of an option. In the less than ideal real world it is perfectly reasonable for different investors to give different values to the same option. This is intuitively obvious: if two people have different hedging strategies then they will have different levels of both risk and accumulated costs. In the

[3]This table assumes that costs are the same whether buying or selling. The more general case can be found in Whalley & Wilmott (1994 b).

above model for optimal hedging the investor must choose his risk aversion parameter γ. As we have said, the larger γ, the less risk the investor is willing to take and thus the narrower will be his hedging bandwidth and the larger the difference between his option value and the Black-Scholes value.

In this article we have given a taste of some recent work on optimal hedging strategies, the underlying methodology, the mathematics involved and some of the key results. Unfortunately, for want of space, we have had to gloss over many of the details. Some of the issues we have not addressed are valuing the option, the role of the asset price growth rate and the treatment of portfolios. These issues are addressed in the original papers to which we have referred. Again, we strongly urge the interested reader to study these carefully.

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Algorithm for finding the hedging bandwidth and
optimal rebalance points for arbitrary symmetric cost structure

Transaction costs are of the form

$$K(S,U)$$

i.e. this is the cost of a trade (either buying or selling) of *U* shares with price *S*.

The hedging bandwidth is given by

$$\Delta - A(S,t) \le y \le \Delta + A(S,t),$$

where

$$\Delta = \frac{\partial V}{\partial S}, \text{ the Black-Scholes delta.}$$

The optimal rebalance points are given by

$$y = \pm B(S,t).$$

To find the positions *A* and *B* solve

$$\frac{\gamma}{3\delta\Gamma^2} AB(A+B) = \frac{\partial K}{\partial U}(S, A-B)$$

$$\frac{\gamma}{12\delta\Gamma^2}(A+B)^3(A-B) = K(S, A-B)$$

where

$$\delta = e^{-r(T-t)}$$

and

$$\Gamma = \frac{\partial^2 V}{\partial S^2}, \text{ the Black-Scholes gamma,}$$

and γ is the index of risk aversion in the investor's utility function.

Table 1: The algorithm for finding the hedging bandwidth and rebalance points.