High Dimensional Radial Barrier Options

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Abstract

Pricing high dimensional American options is a difficult problem in mathematical finance. Many simulation methods have been proposed, but Monte Carlo is numerically intensive, and therefore slow. We derive an analytic expression for a new type of multi-asset barrier option using Laplace transform methods. The solution is assumed to be radially symmetric in the normalized non dimensional variables, hence the name "Radial Barrier Options". In the single-asset case our results reduce to published results for American binary barrier options.

1 Introduction

Pricing American options is a difficult problem in mathematical finance (Myneni [1992]). Tree based methods (Cox et al. [1979]) and finite difference methods (Brennan and Schwartz [1977]) work well for a single underlying asset as it is possible to solve backwards through time from the payoff condition at the expiry of the option. However, these methods quickly become impractical for more than three underlying assets (Tavella [2002]). Monte Carlo methods (Boyle [1977], Boyle et al. [1997]) have good convergence properties for higher dimensional problems and were used for European options in Barraquand [1993]. American options are more difficult to value with simulation as the paths are generated forwards through time, so it is non-trivial to determine the optimal exercise strategy for the option.

An important simulation method for American options was Tilley (Tilley [1993]). This and other early methods such as (Barraquand and Martineau [1995], Fu and Hu [1995], Bossaerts [1989], Broadie and Glasserman [1997a,b]) are reviewed in (Boyle et al. [1997]). Since then the stochastic mesh method (Broadie and Glasserman [1997b]) has been modified to use low-discrepancy sequences (Boyle et al. [2000]). Other methods include; the parameterization of the optimal exercise boundary (Bossaerts [1989], Fu and Hu [1995], Ibáñez and Zapatero

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[2001]), a quantization tree algorithm (Bally et al. [2002]), wavelets (Dempster et al. [2000], Dempster and Eswaran [2001]), and an irregular grid approximation (Berridge and Schumacher [2002]). Regression methods include (Longstaff and Schwartz [2001], Carriere [1996], Tsitsiklis and Van Roy [2001]). Primal-Dual representations of the American option problem allow both an upper and lower bound to be calculated (Andersen and Broadie [2001], Haugh and Kogan [2001], Rogers [2002]). Papers using a Malliavin calculus approach include (Fournié et al. [1999, 2001], Lions and Regnier [2001], Bally et al. [2003]). A comparison of some approaches can be found in (Fu et al. [2001]).

While Monte Carlo methods provide an approach to pricing American option problems they are numerically intensive and therefore slow. As there are no closed form solutions to the American put pricing problem analytic approximations have been proposed to improve the speed of pricing single-asset options (Geske and Johnson [1984], Barone-Adesi and Whaley [1987], Bunch and Johnson [1992], Ju [1998], Bjerksund and Stensland [2002]. For multi-asset options there has been some research for European options (Stultz [1982], Johnson [1987]), however in higher dimensions these formulae have to be computed numerically (Boyle and Tse [1990]). There has been only limited research into analytic solutions for multi-asset American or compound options (Broadie and Detemple [1997], Buchen and Skipper [2003]).

We look for multi-asset American style options that can be solved analytically, and therefore quickly. We call this class of options "Radial Barrier Options" as they depend on the assumption of radial symmetry in the solution method. These options payoff if a barrier, defined as a function of the parameters describing the process for the underlying assets, is hit. Radial options may be useful in the financial market place themselves, or it may be possible to use them to approximate other, actively traded, financial products.

In Section 2 we formulate the problem, and detail the reduction, via a series of transformations, of the multi-asset Black–Scholes equation to the standard high dimensional heat equation. In Section 3 we find radially symmetric solutions to this problem, by using Laplace transforms. We generalize the boundary conditions for these solutions in Section 4 using the Laplace convolution theorem. In Section 5 we reverse the transformations we have made to find the analytic value of these options in the original financial variables and verify the results in the case of one asset. Finally, we conclude and suggest possible future directions for this work in Section 6.

2 The multi-asset Black Scholes equation

We consider a Black–Scholes economy for each asset (Black and Scholes [1973]). The partial differential equation for the value, V, of an option that depends on the evolution of n different underlying assets with price $0 < S_i < \infty$, where $i = 1 \ldots n$, is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sum_{ij}^{n} \sigma_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i}^{n} (r - q_i) S_i \frac{\partial V}{\partial S_i} - rV = 0\,, \qquad (1)$$

where r is the risk free rate, q_i is the dividend yield of the i^{th} asset, t is time, and σ_{ij} is the covariance of the i^{th} asset with the j^{th} asset. Throughout this paper bold face capital letters, such as $\mathbf{X}$, represent matrices, and bold face lower case letters, such as $\mathbf{x}$, represent column vectors. The covariance matrix with elements σ_{ij}, denoted by $\mathbf{COV}$, is symmetric, $\sigma_{ij} = \sigma_{ji}$. The element σ_{ii} is the volatility squared of the i^{th} asset, σ_i^2. We write the volatility as a diagonal matrix, $\mathbf{\Sigma}$, with the σ_i on the diagonal and zeros off the diagonal. The correlation between assets i and j is written ρ_{ij}, we write this as a symmetric matrix, $\mathbf{P}$, with unity on the diagonal and ρ_{ij} as the off diagonal entries. The covariance, volatility and correlation are related by $\sigma_{ij} = \sigma_i\, \rho_{ij}\, \sigma_j$ We write this in matrix notation as $\mathbf{COV} = \mathbf{\Sigma}\,\mathbf{P}\,\mathbf{\Sigma}$.

We non-dimensionalize in a similar manner to Wilmott et al. [1993]. Let E be some representative price scale of the option, and let σ be a representative volatility. We transform the value function and variables using

$$v = \frac{V}{E}\,, \qquad x_i = \log\frac{S_i}{E} \quad \text{and} \quad \tau = \tfrac{1}{2}\sigma^2 (T - t)\,, \qquad (2)$$

where T is the expiry date of the option. Therefore τ is the risk remaining until expiry. We also non-dimensionalize the parameters by

$$\alpha_{ij} = \frac{\sigma_{ij}}{\sigma^2}\,, \quad k_0 = \frac{r}{\frac{1}{2}\sigma^2} \quad \text{and} \quad k_i = \frac{r - q_i}{\frac{1}{2}\sigma^2} \quad i > 0\,, \qquad (3)$$

where σ is a representative volatility. This gives

$$\frac{\partial v}{\partial \tau} = \sum_{ij}^{n} \alpha_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i}^{n} (k_i - \alpha_{ii}) \frac{\partial v}{\partial x_i} - k_0 v\,, \qquad (4)$$

where $0 < \tau < \frac{1}{2}\sigma^2 T$ and $-\infty < x < \infty$. Note that the normalized covariance matrix of the α_{ij}, let us call it $\mathbf{A}$, is symmetric positive definite, as dividing by σ^2 does not affect the symmetry or positive definiteness.

2.1 Fixed boundary

We now reduce this equation to the heat equation. One way to achieve this is to make the transformation

$$v(\mathbf{x}, \tau) = e^{\mathbf{a}.\mathbf{x} - b\tau} u(\mathbf{x}, \tau)\,. \qquad (5)$$

This allows us to eliminate the u and $\partial u/\partial x_i$ terms by solving equations for $\mathbf{a}$ and b. Making this substitution we get the equation,

$$\begin{aligned} -bu + \frac{\partial u}{\partial \tau} &= \textstyle\sum_{ij} \alpha_{ij} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} + a_j \frac{\partial u}{\partial x_i} + a_i \frac{\partial u}{\partial x_j} + a_i a_j u \right) \\ &+ \textstyle\sum_i (k_i - \alpha_{ii}) \left(a_i u + \frac{\partial u}{\partial x_i} \right) - k_0 u\,. \end{aligned}$$

We use the fact that α_{ij} is symmetric, and that we can swap the order of summation to write this as,

$$\begin{aligned}
-bu+\frac{\partial u}{\partial \tau} &= \textstyle\sum_{ij}\alpha_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \\
&+ \textstyle\sum_i \left(\left(2\sum_j \alpha_{ij}a_j\right)+k_i-\alpha_{ii}\right)\frac{\partial u}{\partial x_i} \\
&+ \left(\left(\textstyle\sum_{ij} a_i a_j \alpha_{ij}\right)+\left(\sum_i (k_i-\alpha_{ii})a_i\right)-k_0\right)u\,.
\end{aligned}$$

Setting the coefficients of $\partial u/\partial x_i$ and u to zero gives n equations for the coefficients of the first derivatives and a single equation for the coefficient of u. First consider the coefficients of $\partial u/\partial x_i$. For each $i=1,\ldots,n$, we have,

$$\sum_j \alpha_{ij}a_j = -\tfrac{1}{2}\left(k_i-\alpha_{ii}\right)\,.$$

We can write this in terms of matrices as

$$\mathbf{A}\mathbf{a} = -\tfrac{1}{2}\tilde{\mathbf{k}}\,,$$

where $\tilde{\mathbf{k}}$ is the column vector whose i^{th} entry is $k_i-\alpha_{ii}$. We can solve this for $\mathbf{a}$ by inverting $\mathbf{A}$,

$$\mathbf{a} = -\tfrac{1}{2}\mathbf{A}^{-1}\tilde{\mathbf{k}}\,. \tag{6}$$

Now let us consider the equation for the coefficient of u,

$$b = -\sum_{ij} a_i\alpha_{ij}a_j - \sum_i \left(k_i-\alpha_{ii}\right)a_i + k_0\,.$$

Writing this as matrices gives,

$$b = -\mathbf{a}^T\mathbf{A}\mathbf{a} - \mathbf{a}^T\tilde{\mathbf{k}} + k_0\,,$$

which, as $\mathbf{A}$ is symmetric, can be written,

$$b = -\tfrac{1}{4}\tilde{\mathbf{k}}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1}\tilde{\mathbf{k}} + \tfrac{1}{2}\tilde{\mathbf{k}}^T\mathbf{A}^{-1}\tilde{\mathbf{k}} + k_0\,,$$

hence,

$$b = \tfrac{1}{4}\tilde{\mathbf{k}}^T\mathbf{A}^{-1}\tilde{\mathbf{k}} + k_0\,. \tag{7}$$

Therefore we have reduced (4) to an n dimensional heat equation,

$$\frac{\partial u}{\partial \tau} = \sum_{ij}^{n}\alpha_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\,. \tag{8}$$

Now, as $\mathbf{A}$ is symmetric positive definite we can write

$$\mathbf{A} = \mathbf{Q}^T\,\mathbf{D}^{\mathbf{2}}\,\mathbf{Q} \tag{9}$$

where $\mathbf{Q}$ is a rotation with $\mathbf{Q}\,\mathbf{Q}^T=1$ and $\mathbf{Q}^T$ has columns which are the eigenvectors of $\mathbf{A}$. $\mathbf{D}$ is a diagonal matrix of the corresponding n eigenvalues, d_i. Therefore we can rotate to a new basis

$$\bar{\mathbf{z}} = \mathbf{Q}\mathbf{x}\,, \tag{10}$$

where $\mathbf{x}$ is the column vector of elements x_i. which, when combined with a scaling, $z_i = d_i \bar{z}_i$ (in matrix notation $\mathbf{z} = \mathbf{D}\,\bar{\mathbf{z}}$), will convert our x_i basis into an orthonormal one, z_i. This gives the result,

$$\frac{\partial u}{\partial \tau} = \sum_i^n \frac{\partial^2 u}{\partial {z_i}^2} \,. \tag{11}$$

2.1.1 The one dimensional problem

We shall be able to compare our general results with published results for the case of a single underlying asset. To this end it will be useful to note that

$$k_0 = \frac{r}{\frac{1}{2}\sigma^2} \quad \text{and} \quad k_1 = \frac{r-q}{\frac{1}{2}\sigma^2} \,.$$

We find the one dimensional versions of (6) and (7),

$$a = -\tfrac{1}{2}(k_1 - 1) \quad \text{and} \quad b = \tfrac{1}{4}(k_1 - 1)^2 + k_0 \,. \tag{12}$$

Writing the constants a and b in financial variables gives

$$a = -\frac{1}{2}\left(\frac{2(r-q)}{\sigma^2} - 1\right) \quad \text{and} \quad b = \frac{1}{4}\left(\frac{2(r-q)}{\sigma^2} - 1\right)^2 + \frac{2r}{\sigma^2} \,. \tag{13}$$

2.2 Moving boundary

Another way to obtain an n dimensional heat equation is to transform the non-dimensionalized Black-Scholes equation (4) using,

$$v = e^{-k_0 \tau} w \,, \tag{14}$$

to obtain

$$\frac{\partial w}{\partial \tau} = \sum_{ij}^n \alpha_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_i^n (k_i - \alpha_{ii}) \frac{\partial w}{\partial x_i} \,.$$

Next we remove the drifts by

$$y_i = x_i + (k_i - \alpha_{ii})\tau \tag{15}$$

to get

$$\frac{\partial w}{\partial \tau} = \sum_{ij}^N \alpha_{ij} \frac{\partial^2 w}{\partial y_i \partial y_j} \,.$$

Note that this transformation will cause a fixed boundary in x space to move in S space. Also, an option payoff which is a function of x in x space will be a function of time in S space.

This is in the same form as (8), but in the y_i basis, rather than the x_i basis. Again we use the eigenvector - eigenvalue decomposition (9) to find a new basis

$$\bar{\bar{\mathbf{z}}} = \mathbf{Q}\mathbf{y} \,,$$

where the elements of the vector $\mathbf{y}$ are y_i, which, when combined with one final transformation $z_i = d_i \bar{\hat{z}}_i$ (in matrix notation $\mathbf{z} = \mathbf{D}\,\bar{\hat{\mathbf{z}}}$), gives the result

$$\frac{\partial w}{\partial \tau} = \sum_i^n \frac{\partial^2 w}{\partial \hat{z}_i^2}\,.$$

For the rest of this paper we will only consider the fixed boundary case.

3 The radial problem for u

We look for radially symmetric solutions, $u(\rho, \tau)$, which depend on the radial distance, $\rho^2 = \sum z_i^2$, and τ. In radial co-ordinates we have

$$\frac{\partial u}{\partial \tau} = \frac{1}{\rho^{n-1}} \frac{\partial}{\partial \rho} \left(\rho^{n-1} \frac{\partial u}{\partial \rho} \right) = \frac{\partial^2 u}{\partial \rho^2} + \frac{n-1}{\rho} \frac{\partial u}{\partial \rho}\,. \tag{16}$$

We choose boundary conditions that allow this problem to be solved analytically,

$$\begin{aligned} u(1,\tau) &= 1 && (17)\\ u(\infty,\tau) &= 0 && (18)\\ u(\rho,0) &= 0\,. && (19) \end{aligned}$$

We solve the problem where the sphere $\rho = 1$ initially has value 1. The rest of the region outside is initially 0. We then let our time-like variable evolve.

This is an option which pays 1 when the boundary is hit before the expiry date, but otherwise expires worthless. For the case of one underlying asset ($n = 1$) the solution

$$u(\rho,\tau) = \operatorname{erfc}\left(\frac{\rho - 1}{2\sqrt{\tau}} \right)\,, \tag{20}$$

is well known Carslaw and Jaeger [1959].

In three dimensions the problem can be transformed to the the one dimensional case. The transformation

$$F = u\rho\,,$$

reduces the problem to the heat equation

$$\begin{aligned} \frac{\partial F}{\partial \tau} &= \frac{\partial^2 F}{\partial \rho^2} && (21)\\ F(1,\tau) &= 1 && (22)\\ F(\infty,\tau) &= 0 && (23)\\ F(\rho,0) &= 0\,. && (24) \end{aligned}$$

Section 9.10 of Carslaw and Jaeger [1959] gives the solution valid for $\rho \geq 1$, the region bounded internally by the sphere $\rho = 1$. We find the result for an initial value of zero, and a constant surface value of 1, by using the 1-dimensional solution (20), to find,

$$u(\rho,\tau) = \frac{1}{\rho} \operatorname{erfc}\left(\frac{\rho - 1}{2\sqrt{\tau}} \right)\,. \tag{25}$$

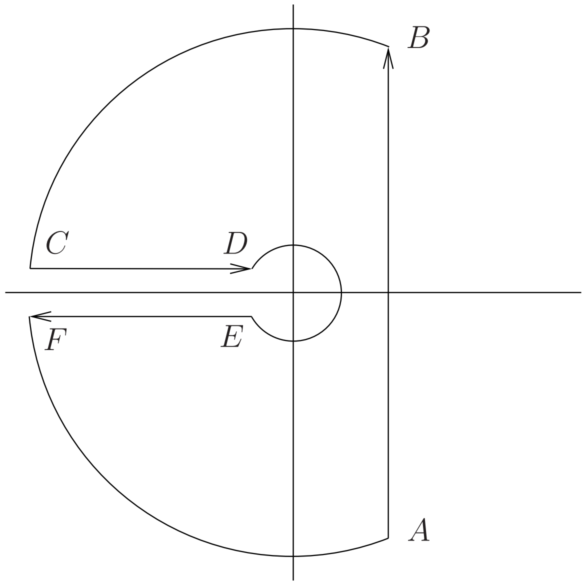

Figure 1: The keyhole contour for inverting the Laplace transform

3.1 Solution in n dimensions

We can solve the problem (16) – (19) in n dimensions using the Laplace transform method. In general the solution involves Bessel functions. We solve the problem in the region $\rho \geq 1$, with $u(\rho, \tau) = 1$. The rest of space initially having value 0. The solution for the two dimensional case can be found in Carslaw and Jaeger [1959] 13.5(I). We take the Laplace transform in time and find the subsidiary equation,

$$\frac{d^2\bar{u}}{d\rho^2} + \frac{n-1}{\rho}\frac{d\bar{u}}{d\rho} - s\,\bar{u} = 0\,, \tag{26}$$

for the region $\rho \geq 1$, where s is the Laplace parameter. The surface $\rho = 1$ is held at a constant value 1. We set $\nu = 1 - n/2$. We require that $\bar{u}$ be finite as $\rho \to \infty$. The transformed boundary condition is

$$\bar{u}(1, s) = 1/s\,.$$

The general solution to this modified Bessel equation is

$$\bar{u} = AI_\nu(\sqrt{s}\,\rho) + BK_\nu(\sqrt{s}\,\rho)\,,$$

where A and B are constants. We require a finite solution as $\rho \to \infty$, so $A = 0$. We find B from the transformed boundary condition and obtain the solution

$$\bar{u}(\rho, s) = \frac{\rho^\nu K_\nu(\sqrt{s}\,\rho)}{s\,K_\nu(\sqrt{s})}\,. \tag{27}$$

Next we invert this solution using the Laplace Inversion Theorem, so

$$u = \frac{\rho^\nu}{2\pi\,i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda\tau}\,\frac{K_\nu(\sqrt{\lambda}\,\rho)}{K_\nu(\sqrt{\lambda})}\,\frac{d\lambda}{\lambda}\,. \tag{28}$$

Note that our series of Bessel functions, for increasing n, is

$$K_{1/2}, K_0, K_{-1/2}, K_{-1}, K_{-3/2}, \dots$$

As $K_{-\nu}(z) = K_\nu(z)$, we can see that the $n = 1$ and $n = 3$ cases will differ only in factors of ρ, which is indeed the case.

The zeros of $K_\nu(z)$ are discussed in Watson [1944] 15.7. We find that there are no zeros for $|\arg z| \leq \frac{1}{2}\pi$. The Bessel function in the denominator of the integrand in (28) has argument $\sqrt{\lambda}$, therefore we know that the integrand has no poles within the contour in Figure 1.

The integrand has a branch point at $\lambda = 0$, so we use the contour in Figure 1. From Watson [1944] we know that there are no zeros of $K_\nu(\sqrt{\lambda})$ within this contour. The integral along AB, that we wish to calculate is equal to the integral

$$u = \frac{\rho^\nu}{2\pi\, i}\left(AF + FE + ED + DC + CB\right) .$$

We now calculate these integrals:

- AF: We write $\lambda = Re^{i\theta}$ so the integral along the arc AF, from $-\pi/2$ to $-\pi$ becomes

$$I_{AF} = \int_{-\pi}^{-\pi/2} e^{R\tau e^{i\theta}} \frac{K_\nu(\sqrt{R}\, e^{i\theta/2}\rho)}{K_\nu(\sqrt{R}\, e^{i\theta/2})} i\, d\theta .$$

  We wish to prove that this integral goes to zero as $\rho \to \infty$. This is a standard exercise in integration in the complex plane. We consider the modulus of the integral I_{AF}, then we move the modulus inside the integral and consider the modulus of each term. We can expand $K_\nu(z)$ for large z, and expand the resulting fraction using a Taylor series. We find that the second term in our integral and be bounded above by some value M.

  We find that,

$$|I_{AF}| \leq M \int_{-\pi}^{-\pi/2} e^{R\tau\cos\theta}\, d\theta .$$

  Make the change of variable $-\phi = \theta + \pi/2$ and use Jordan's inequality to verify that as $R \to \infty$ we have $I_{AF} \to 0$ for $\tau > 0$, as required.

- FE: To calculate the integral over FE we put $\lambda = \zeta^2 e^{-i\pi}$ which gives,

$$2\int_\infty^0 e^{-\zeta^2\tau} \frac{K_\nu(\rho\zeta e^{-i\pi/2})}{K_\nu(\zeta e^{-i\pi/2})} \frac{d\zeta}{\zeta} = -2\int_0^\infty e^{-\zeta^2\tau} \frac{J_\nu(\zeta\rho) + iY_\nu(\zeta\rho)}{J_\nu(\zeta) + iY_\nu(\zeta)} \frac{d\zeta}{\zeta} ,$$

  since

$$K_\nu(ze^{\pi i/2}) = \tfrac{1}{2}\pi i H_\nu^{(1)}(z) = \tfrac{1}{2}\pi i[J_\nu(z) + iY_\nu(z)] .$$

- ED: We write $\lambda = \epsilon e^{i\theta}$ so the integral around the circle ED, from $-\pi$ to π, becomes
$$\int_{-\pi}^{\pi} e^{\epsilon\tau e^{i\theta}} \frac{K_\nu(\sqrt{\epsilon}\, e^{i\theta/2}\rho)}{K_\nu(\sqrt{\epsilon}\, e^{i\theta/2})} i\, d\theta\,.$$
We take the limit as $\epsilon \to 0$ to find that the integral evaluates to $2\pi i$.
- DC: For the integral on DC we put $\lambda = \zeta^2 e^{i\pi}$, and use
$$K_\nu(ze^{\pi i/2}) = -\tfrac{1}{2}\pi i H_\nu^{(2)}(z) = -\tfrac{1}{2}\pi i[J_\nu(z) - iY_\nu(z)]\,,$$
since we have a complex argument of $\pi/2$ rather than $-\pi/2$ as in the integral FE, and obtain,
$$2\int_0^\infty e^{-\zeta^2\tau} \frac{K_\nu(\rho\zeta e^{i\pi/2})}{K_\nu(\zeta e^{i\pi/2})}\frac{d\zeta}{\zeta} = 2\int_0^\infty e^{-\zeta^2\tau} \frac{J_\nu(\zeta\rho) - iY_\nu(\zeta\rho)}{J_\nu(\zeta) - iY_\nu(\zeta)}\frac{d\zeta}{\zeta}\,,$$
which is minus the complex conjugate of the integral that we found for FE.
- CB: The argument is similar to the case AF, hence we find that as $R \to \infty$, $I_{CB} \to 0$ for $\tau > 0$.

Combining these results we obtain
$$u = \rho^{\min[0,2-n]} + \frac{2\rho^\nu}{\pi}\int_0^\infty e^{-\zeta^2\tau} \frac{J_\nu(\zeta\rho)Y_\nu(\zeta) - J_\nu(\zeta)Y_\nu(\zeta\rho)}{J_\nu(\zeta)^2 + Y_\nu(\zeta)^2}\frac{d\zeta}{\zeta}\,, \quad (29)$$
which agrees with the solution for the two dimensional case in Carslaw and Jaeger [1959] 13.5(I). Note, if we leave the integrand as modified Bessel functions we have
$$u = \rho^{2\nu} + \frac{\rho^\nu}{i\,\pi}\int_0^\infty e^{-\zeta^2\tau}\left\{\frac{K_\nu(\rho\zeta\, i)}{K_\nu(\zeta\, i)} + \frac{K_\nu(-\rho\zeta\, i)}{K_\nu(-\zeta\, i)}\right\}\frac{d\zeta}{\zeta}\,.$$

3.1.1 Spherical Bessel functions

It is informative to note that in one dimension the Bessel functions in the integrand have order of a half and therefore can be rewritten as trigonometric functions. We can verify our solution before and after the Laplace inversion in this way.

We can also do this in the three dimensional case.

In five dimensions the situation is slightly more complicated. However, we can write the Bessel functions in (29) as
$$\frac{\zeta(\rho-1)\cos(\zeta - \zeta\rho) + (\rho\zeta^2+1)\sin(\zeta - \zeta\rho)}{(\zeta^2+1)\rho^{3/2}}\,,$$
we split this into three parts and find one relatively simple integral, and two others that may be found in Gradshteyn and Ryzhik [2000] 3.954. Combining these and simplifying (29) we find
$$u(\rho,\tau) = \frac{1}{\rho^3}\left[\operatorname{erfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) + (\rho-1)\, e^{(\rho-1)+\tau}\operatorname{erfc}\left(\sqrt{\tau} + \frac{\rho-1}{2\sqrt{\tau}}\right)\right]\,, \quad (30)$$
which we can verify satisfies the partial differential equation for $n = 5$ dimensions.

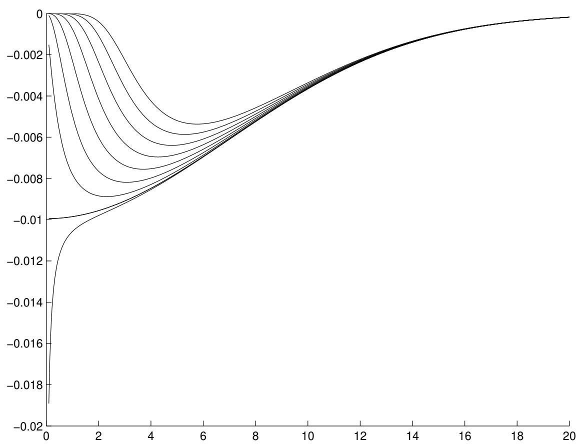

Figure 2: A graph of the integrand in (29) with $\rho = 1.01$ and $\tau = 0.01$ over a range of values for the dummy variable u. The lowest line is for dimension $n = 1$ proceeding to the topmost line where $n = 10$.

3.1.2 Even dimensions

When we try to solve the integral in even dimensions we have to expand the Bessel functions of integer order. It has been proven Watson [1944] that this is not possible in finite terms. A finite expansion is only possible when the order of the Bessel functions is half an odd integer.

3.1.3 Numerical solution

We can calculate the value of the function u from (29). As the integrand appears to be fairly well behaved we used the `quad` routine in MATLAB as a first approximation to the integral. We find good agreement with the analytic solutions for $n = 3$ and $n = 5$ dimensions, as shown in Figure 3. The fact that the $n = 1$ dimension exact solution crosses the $n = 2$ dimension numerical solution requires investigation.

Note that as the dimension increases the value of u for a particular value of ρ decreases, which is as we expect. All lines come together at 1 when $\rho = 1$, also as required, and all lines decay toward zero as ρ increases. As we increase the time the influence of the fixed value diffuses away from $\rho = 1$. For small times the diffusion is less. For $\tau < 0.1$ we find numerical instability in the solution.

3.1.4 Asymptotic solution for small values of τ

Unfortunately we can see that (29) does not converge well for small values of τ. To find solutions for small τ we make an asymptotic expansion, as in Carslaw and Jaeger [1959] 13.5. Expanding (27), and

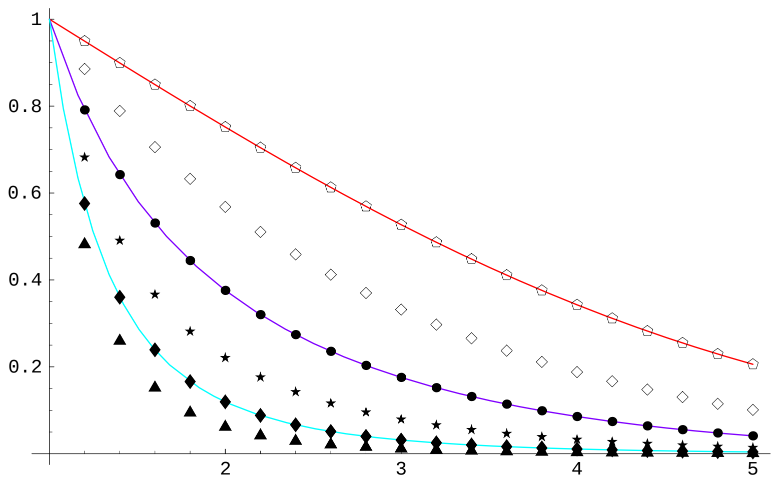

Figure 3: A graph of the function u for a range of dimensions. The integral in (29) was evaluated over $[0,\infty)$ using Mathematica, $\tau = 5$, $u_0 = 1$. The horizontal axis is ρ and the vertical u. The top line is for one dimension, the dimension increases until $n = 6$ for the bottom line. The smooth lines are the exact solutions, the points are the numerical solution.

keeping $\nu = 1 - (n/2)$, we have

$$\bar{u} = \frac{1}{s\rho^{-\nu+(1/2)}} e^{-\sqrt{s}\,(\rho-1)} \left\{1 + \frac{(4\nu^2-1)}{8\rho\sqrt{s}} + \frac{(4\nu^2-1)((4\nu^2-9))}{2(8\rho)^2 s} + \mathcal{O}(\frac{1}{s^{3/2}})\right\} \Bigg/ \left\{1 + \frac{(4\nu^2-1)}{8\sqrt{s}} + \frac{(4\nu^2-1)((4\nu^2-9))}{2(8)^2 s} + \mathcal{O}(\frac{1}{s^{3/2}})\right\},$$

which we can expand as

$$\bar{u} = \frac{1}{s\rho^{-\nu+(1/2)}} e^{-\sqrt{s}\,(\rho-1)} \left\{1 - \frac{(4\nu^2-1)(\rho-1)}{(8\rho)\sqrt{s}} + \frac{(4\nu^2-1)[\rho^2(4\nu^2+7) - 2\rho(4\nu^2-1) + (4\nu^2-9)]}{2(8\rho)^2 s} + \mathcal{O}(\frac{1}{s^{3/2}})\right\}.$$

Inverting the Laplace transform term by term gives

$$u = \frac{1}{\rho^{(n-1)/2}} \operatorname{erfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) - \frac{\sqrt{\tau}\,(4\nu^2-1)(\rho-1)}{2^2\rho^{(n+1)/2}}\, \mathrm{i}^1\operatorname{erfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) + \frac{\tau\,(4\nu^2-1)[4\nu^2(\rho-1)^2 + 7\rho^2 + 2\rho - 9]}{2.2^4\rho^{(n+3)/2}}\, \mathrm{i}^2\operatorname{erfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) + \dots, \tag{31}$$

where $\mathrm{i}^n\operatorname{erfc}(.)$ is the iterated error function (Abramowitz and Stegun [1974]). This approximation is valid, according to Carslaw and Jaeger [1959], for $\tau < 0.02$ and ρ not small. We know that $\rho \geq 1$ so the second condition is not restrictive.

4 Convolutions

In this section we use the Laplace Convolution Theorem

$$\mathcal{L}[g * h] = \mathcal{L}[g]\mathcal{L}[h]$$

to allow us to change the boundary condition when the radial barrier at $\rho = 1$ is hit from a unit payoff to a payoff of the form $\phi(\tau)$.

We have a solution $u(\rho, \tau)$ to the one dimensional problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \rho^2} + \frac{n-1}{\rho}\frac{\partial u}{\partial \rho} \tag{32}$$

$$u(1, \tau) = 1 \tag{33}$$

$$u(\infty, \tau) = 0 \tag{34}$$

$$u(\rho, 0) = 0\,. \tag{35}$$

We would like to have a solution to this problem, but with the boundary condition at $\rho = 1$ being a function, $\phi(\tau)$, of τ. Let us call the solution to this modified problem $\hat{u}(\rho, \tau)$.

Let us shift the space variable by

$$\rho = \xi + 1\,. \tag{36}$$

We have the partial differential equation,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} + \frac{n-1}{\xi+1}\frac{\partial u}{\partial \xi}\,,$$

but the boundary condition at $\rho = 1$ is now the condition $u(0, \tau) = 1$ at $\xi = 0$. If we make the same change of variable (36) in the problem for $\hat{u}(\rho, \tau)$ we have the heat equation

$$\frac{\partial \hat{u}}{\partial \tau} = \frac{\partial^2 \hat{u}}{\partial \xi^2} + \frac{n-1}{\xi+1}\frac{\partial \hat{u}}{\partial \xi}\,,$$

with boundary condition $\hat{u}(0, \tau) = \phi(\tau)$. The second and third boundary conditions remain unchanged.

Taking the Laplace transform in τ of the u problem we can find the solution $\bar{u}$, using bars to represent the Laplace transform. We also take the Laplace transform of the $\hat{u}$ problem. Let s be the transform variable, we can write the solution to the $\bar{\hat{u}}$ problem in terms of $\bar{u}$

$$\bar{\hat{u}}(\xi, s) = s\,\overline{\phi}(s)\,\bar{u}(\xi, s)\,.$$

Using the properties of Laplace transforms, and the convolution theorem, this is equal to

$$\bar{\hat{u}}(\xi, s) = \overline{\phi}(s)\,\overline{\frac{\partial u}{\partial \tau}}(\xi, s) = \mathcal{L}\left[\phi(\tau) \,*\, \frac{\partial u}{\partial \tau}(\xi, \tau)\right]\,,$$

hence

$$\hat{u}(\xi, \tau) = \phi(\tau) \,*\, \frac{\partial u}{\partial \tau}(\xi, \tau) = \int_0^\tau \phi(\tau - \eta)\,\frac{\partial u}{\partial \tau}(\xi, \eta)\,d\eta\,.$$

Alternatively we could choose to use

$$\hat{u}(\xi, \tau) = \frac{\partial \phi}{\partial \tau}(\tau) \,*\, u(\xi, \tau) = \int_0^\tau \frac{\partial \phi}{\partial \tau}(\eta)\,u(\xi, \tau - \eta)\,d\eta\,.$$

4.1 One dimension

In one dimension we have, (20)

$$u(\xi,\tau) = \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right).$$

We would like to set $\phi(\tau) = e^{\alpha\tau}$ to simplify the option after we invert the non-dimensionalizing transforms, hence

$$\hat{u}(\xi,\tau) = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau e^{\alpha(\tau-\eta)}\ \eta^{-3/2}\, e^{-\xi^2/4\eta}\, d\eta\,,$$

in which we substitute $y = \eta^{-1/2}$ to obtain

$$\hat{u}(\xi,\tau) = \frac{\xi e^{\alpha\tau}}{\sqrt{\pi}} \int_{1/\sqrt{\tau}}^\infty \exp\left[-\frac{\xi^2 y^2}{4} - \frac{\alpha}{y^2}\right] dy\,.$$

We can solve this integral to obtain,

$$\frac{e^{\alpha\tau}}{2}\ \left[e^{\xi\sqrt{\alpha}} \operatorname{erfc}(d_+) + e^{-\xi\sqrt{\alpha}} \operatorname{erfc}(d_-)\right],$$

where

$$d_\pm = \frac{\xi}{2\sqrt{\tau}} \pm \sqrt{\alpha\tau}\,. \tag{37}$$

Writing this in terms of cumulative normals, $\mathrm{N}(.)$, instead of error functions gives

$$\hat{u}(\xi,\tau) = e^{\alpha\tau}\ \left[e^{\xi\sqrt{\alpha}}\, \mathrm{N}\left(-\sqrt{2}\, d_+\right) + e^{-\xi\sqrt{\alpha}}\, \mathrm{N}\left(-\sqrt{2}\, d_-\right)\right]. \tag{38}$$

4.2 Three dimensions

In three dimensions we have

$$u(\xi,\tau) = \frac{1}{\xi+1} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right),$$

which is simply the one dimensional solution divided by a factor of $\xi + 1$, so for the boundary condition $\phi(\tau) = e^{\alpha\tau}$ we can solve the convolution as in the one dimensional case to obtain,

$$\hat{u}(\xi,\tau) = \frac{e^{\alpha\tau}}{\xi+1}\ \left[e^{\xi\sqrt{\alpha}}\, \mathrm{N}\left(-\sqrt{2}\, d_+\right) + e^{-\xi\sqrt{\alpha}}\, \mathrm{N}\left(-\sqrt{2}\, d_-\right)\right], \tag{39}$$

where $d_\pm$ is as in (37). We can verify that this satisfies (16) when $n = 3$.

4.3 Five dimensions

In five dimensions we convolve the solution (30) with the boundary condition $\phi(\tau) = e^{\alpha\tau}$ and find the solution,

$$\begin{aligned}\hat{u}(\xi,\tau) = \frac{1}{2(\alpha-1)(\xi+1)^3} \Bigg(&- 2\,\xi\, e^{\xi+\tau} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\tau}\right) \\ &+ (\alpha - 1 + \sqrt{\alpha}\,\xi + \alpha\xi)\, e^{\alpha\tau+\xi\sqrt{\alpha}} \operatorname{erfc}(d_+) \\ &+ (\alpha - 1 - \sqrt{\alpha}\,\xi + \alpha\xi)\, e^{\alpha\tau-\xi\sqrt{\alpha}} \operatorname{erfc}(d_-)\Bigg),\end{aligned} \tag{40}$$

5 Reverse transformations

Now that we have the solution to some high dimensional problems we transform our results back to the financial variable space. We investigate what classes of financial products we have obtained valuation equations for.

In the case of a single underlying asset we can compare our results with published results for single-asset American binary barrier options, by which we mean an option that pays a cash amount at the time a fixed barrier value is hit during the life of the option. Otherwise the option expires worthless.

5.1 Single-asset American binary barrier option

If we take the convolution solution in one dimension (38) and reverse the change of variable (36), the transformation (5), the non-dimensionalizations (2) and then choose $\alpha = b$ we cause the $b\tau$ term in the exponent of (5) to cancel. By inspection of (13), b is always positive, so the square roots that appear are not problematic. After simplification we find the solution in financial variables to be

$$V(S,t) = E\,e^{a}\left[\left(\frac{E\,e}{S}\right)^{\mu_+}\mathrm{N}\left(z\right) + \left(\frac{E\,e}{S}\right)^{\mu_-}\mathrm{N}\left(z - 2\sigma\sqrt{b(T-t)}\right)\right].$$

where

$$\mu_\pm = -a \pm \sqrt{b} \quad \text{and} \quad z = \frac{\log(Ee/S)}{\sigma\sqrt{T-t}} + \sigma\sqrt{b(T-t)}\,.$$

This is the solution to a cash-at-hit binary barrier option which pays $\$Ee^a$ if the barrier located at $\$Ee$ is hit. This agrees with Rubinstein and Reiner [1991].

5.2 General radial barrier option

Recall the transform (5),

$$v(\mathbf{x},\tau) = e^{\mathbf{a}.\mathbf{x}-b\tau}u(\mathbf{x},\tau)\,.$$

We have obtained radially symmetric solutions for u. We made additional transformations to change $\mathbf{x}$ into an orthonormal basis $\mathbf{z}$. We have found solutions $u(\rho,\tau)$ where $\rho^2 = \sum z_i^2$. In matrix notation this is $\rho^2 = \mathbf{z}^T\mathbf{z}$. In terms of the $\mathbf{x}$ basis we have

$$\rho^2 = (\mathbf{DQx})^T(\mathbf{DQx}) = \mathbf{x}^T\mathbf{A}\mathbf{x}\,,$$

using (9) and the facts that $\mathbf{Q}^T\mathbf{Q} = 1$ and $\mathbf{A}$ is symmetric.

Now, we can write

$$\mathbf{a}.\mathbf{x} = |\mathbf{a}||\mathbf{x}|\cos\theta$$

where θ is the angle between the two vectors $\mathbf{a}$ and $\mathbf{x}$.

We have defined our option to have value $u = 1$ when $\rho = 1 = \mathbf{x}^T \mathbf{A}^2 \mathbf{x} = |\mathbf{A}\mathbf{x}|$. This is a barrier option which pays out when the barrier is hit. Therefore, reversing the transformation we have, on the boundary,

$$v(|\mathbf{A}\mathbf{x}| = 1, \tau) = e^{|\mathbf{a}||\mathbf{x}| \cos\theta - b\tau}.$$

Using Duhamel's theorem we can construct options with payoffs of the form

$$V = E\, e^{|\mathbf{a}||\mathbf{x}| \cos\theta},$$

when the radial barrier is hit. Modifying the payoff to something more financially intuitive is left for future research.

5.3 Three asset radial option

We take the three asset solution (39), and reverse the transformations, as in the previous section. Note, from (7), that b is always a scalar so we can choose $\alpha = b$, and obtain,

$$V = \frac{E\, e^{\mathbf{a}.\mathbf{x}}}{\sqrt{\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}}} \left[e^{\sqrt{b\, \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}}} \mathrm{N}\,(z) + e^{-\sqrt{b\, \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}}} \mathrm{N} \left(z - 2\sigma\sqrt{b(T-t)} \right) \right],$$

where

$$z = \frac{\sqrt{\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}} - 1}{\sigma\sqrt{T-t}} + \sigma\sqrt{b(T-t)}\,.$$

6 Conclusion

We have derived analytic valuation expressions for multi-asset American style financial options. These "Radial Barrier Options" may be of use in the financial markets, as benchmark cases for other numerical methods for high dimensional options, or as an approximation to other, actively traded, financial options.

Hedging these options may be an interesting problem. In the case of a single underlying asset they have the same hedging difficulties as standard single-asset barrier options (Shaw [1998]). Investigating how this problem applies in the multi-asset case may be worthwhile, as may static hedging strategies.

Future research could also involve further modification of the boundary conditions to define more intuitive financial options. It is also possible to price the radial equivalent of a double binary barrier option using this approach.

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