A Model of Entrepreneurial Finance

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Abstract

A wealth-constrained entrepreneur seeks financing from a financial institution. Because the entrepreneur has a greater preference for continuing the firm over liquidating it, and for aggressive continuation strategies over conservative strategies, the institution must monitor the firm and exercise some control over its decisions. The institution's own liquidity concerns make it prefer less exposure to the firm's risk, subject to meeting its incentive and break-even constraints. The optimal contract is either debt or convertible debt; moreover, for firms with limited financial slack, convertible debt is only used if the institution monitors more actively, and debt is used otherwise. Convertible debt and active monitoring are more likely to be optimal if: (1) the firm faces greater uncertainty in its choice of continuation strategies; (2) the aggressive continuation strategy is not too profitable on average; and (3) the firm's cash flow distribution is more skewed, with low probability of success, low liquidation value, and high returns if successful. These results mirror entrepreneurial firms' choice between bank finance and venture capital finance.

For firms with somewhat higher financial slack and high strategic uncertainty, a third option may be optimal: convertible debt with passive monitoring by the institution. This resembles the circumstances and structure of so-called 'mezzanine finance'.

Introduction

Although start-ups and venture capital finance are often linked in the public eye, bank loans are a more common source of finance for entrepreneurial firms. Both sources share some common features. Because entrepreneurial firms are usually small and have high risk of failure, both venture capital and bank loans require careful monitoring of borrowers. Both types of finance use covenants to restrict the borrower's behavior and provide additional levers of control in the event that the firm performs poorly. These covenants often restrict the ability of the firm to seek financing elsewhere, which ties to yet another common feature: the use of capital rationing through staged financing and credit limits as means of controlling borrowers' ability to continue and grow their business.

Despite these similarities, there are significant differences between these two types of financing. Whereas banks lend to a wide variety of firms, firms with venture capital finance tend to have very skewed return distributions, with a high probability of weak or even negative returns and a small probability of extremely high returns (see Sahlman (1990), and Fenn et al. (1995))[1]. Whereas bank loans usually take the form of pure debt, venture capitalists almost always employ convertible securities or a combination of debt and equity (see Kaplan and Stromberg (2001)). Finally, banks' monitoring and control rights are typically far less intensive than those of venture capitalists, and focus on avoiding or minimizing bad outcomes. Banks mostly monitor for covenant violations, deteriorating performance, or worsening collateral quality that might jeopardize their loan; they exercise control by threatening to force default and possible liquidation. By contrast, as documented by Sahlman (1990) and Kaplan and Stromberg (2001), venture capitalists often hold seats on the borrowing firm's board and voting rights far in excess of their cash flow rights, and may have the contractual right to replace the entrepreneur with a new manager if covenants are violated. Along with these rights, venture capitalists monitor borrowers more frequently than banks do and play an active role in most of the firm's major decisions[2].

[1] Sahlman (1990) quoting various sources reports that more than one-third of the investments made by venture capital funds result in absolute losses. However, returns on successful investments more than offset this loss, with some investments resulting in payoffs more than ten times the initial investment.

[2] Gorman and Sahlman (1989) survey venture capitalists and find that lead venture capitalists visit their portfolio companies an average of 18.7 times per year. By contrast, Blackwell and Winters (1997) find that most bank loans to smaller firms are monitored once or twice a year; the most risky loans are monitored at least quarterly and sometimes more frequently. Similarly, Hellmann and Puri (2002) find that venture-capital-backed firms are more likely to have higher measures of professionalization than start-up firms that rely on other types of financing, which is consistent with venture capitalists playing a more active role in the firm's management.

In this paper, we develop a simple theoretical model to explain these differences. When an institution finances an entrepreneur, there is a well-known tension between the entrepreneur's desire to keep the firm going in order to maintain her control benefits and the institution's desire to liquidate poorly-performing investments. If this were the end of the story, debt would generally be an optimal contract; by giving the institution a senior claim to liquidation proceeds, it gives the institution incentive to monitor enough to see if liquidation is desirable. Indeed, as shown by Winton (2002), debt is uniquely optimal once the institution's own liquidity concerns are incorporated. Debt is less risky than equity, and so the institution's assets are less affected by its private information about the firm, reducing adverse selection problems when the institution itself needs additional funding. Since these costs are passed on to the entrepreneur in her costs of funds, she shares this preference for debt, all else equal.

This simple picture is complicated by the fact that, even if it is optimal to keep the entrepreneur's firm going, there may be additional choices to be made. Should the firm expand conservatively or aggressively? Should the firm attempt an IPO or settle for sale to another corporation? Again, there is a tension between the entrepreneur and the institution, because the entrepreneur may "excessively" prefer aggressive or risky decisions that maintain or expand her control benefits even when such decisions do not always maximize contractible cash flows. If the institution is to make an informed decision about whether or not the firm should pursue a more risky strategy, additional monitoring is required. But this in turn requires that the institution's gain from more informed decisions offsets the cost of additional monitoring.

For many firms, the level of strategic uncertainty may not be very high, or the cost of intensive monitoring may be so high that the entrepreneur is unwilling to reimburse the institution for this cost. In these cases, debt works well; the institution can insist on a covenant the rules out the risky action unless it agrees, and it can then condition agreement on the less precise knowledge it gleans from more passive monitoring.

If instead the impact of the choice between risky and conservative strategies varies greatly with the firm's precise situation, the institution must be given incentive to monitor more actively. This requires that the institution gains greatly from having the firm pursue a risky strategy when conditions are favorable and otherwise gains greatly from a conservative strategy. In general, debt will not accomplish this in a cost-effective manner. Although the institution's promised payment can be set so high that it bears most of the firm's cash flow risk, this implies that the institution effectively buys much of the firm initially. To the extent that this is more than the firm's required investment, the institution takes on excessive exposure to the firm's risk, needlessly increasing the illiquidity of its position. A position that combines debt with equity (either a convertible security or joint holdings of debt and equity securities) can give the institution the necessary exposure to the firm's strategic decision with much less overall risk.

This situation requires a number of special factors. First, as already noted, strategic uncertainty must be high [3]. Second, expected profits from the risky continuation strategy cannot be too high. Otherwise, the institution can recoup its investment even if the firm unconditionally pursues the risky strategy, which is the manager's preference. Third, the firm's cash flow distribution must be sufficiently skewed; i.e., the probability of success must be low, the value of the firm in liquidation low, and the firm's cash flows in success higher. Greater skewness means that the institution can only recoup its investment by taking high payments when the firm is successful, which in turn means that it gains more from more careful monitoring of the firm's strategic decisions [4].

Our paper presents an explanation for the existence of different modes of start-up financing. We address the issue of why some start-up firms get financed by bank loans, while some others obtain their funds from venture capitalists by issuing convertible securities or a combination of debt and equity. We relate this choice to the ex-ante distribution characteristics of the firm's returns.

Our analysis assumes that the firm's 'financial slack' - the difference between its expected cash flows and its required investment and passive monitoring costs - is relatively low, so that it is not feasible to pay the entrepreneur enough to voluntarily choose the safe continuation strategy. If the firm's financial slack is somewhat higher, a third financing option arises. In this, the institution holds convertible debt but monitors only passively; the debt payment is set low enough that the entrepreneur prefers to choose the safe strategy when this maximizes contractible cash flows in continuation, and the risky strategy otherwise. In addition to requiring more slack, this is only optimal if the firm's strategic uncertainty is sufficiently high; otherwise, the entrepreneur will continue to prefer the risky strategy regardless.

This scenario resembles 'mezzanine finance'. In comparison with firms that receive venture capital, firms that receive mezzanine finance are typically better-established and more profitable but still face high strategic uncertainty going forward. Moreover, although mezzanine finance is usually structured as convertible debt, it is monitored less actively than venture capital. Thus, a simple extension of our model allows us to capture this alternative form of financing.

Although there are several papers on various aspects of venture capital financing, few researchers have addressed the issue of what determines the mode of

[3] As evidence for this, Hellmann and Puri (2000) find that innovator firms are more likely to obtain venture capital financing than are imitator firms. Since strategic uncertainty is higher for newer products, this is consistent with our prediction.

[4] Also as noted in Winton (2002), the liquidity costs of bearing additional risk (through either a conversion feature or a mix of debt and equity) cannot be too high. This means that an institution either must be well-insulated against liquidity needs, or else cannot hold too much equity or convertible debt in its overall portfolio.

financing[5]. Dewatripont et al. (2002) develop a theory of start-up financing using a model of observable but nonverifiable effort, where contracts can be renegotiated after the entrepreneur has exerted his effort. They show that when the entrepreneur has the bargaining power in renegotiation, debt and convertible debt are often optimal. However, in their paper, convertible debt is the renegotiation-proof equivalent of pure debt. In contrast, we argue that in comparison with pure debt, convertible debt provides the investor with incentives to monitor the firm more actively.

Landier (2001) offers an alternative theory of start-up financing using a career concerns model. In his model, contractual incompleteness arises because the entrepreneur and the investor can hold up each other once the venture is under way. The optimal form of financing balances the terms of bargaining in case of a hold-up. The relative bargaining powers in the hold-up problem depends on the 'stigma' associated with failure of the venture, which in turn depends on the strategies of other entrepreneurs in equilibrium. The higher the stigma associated with failure, the easier it becomes to enforce entrepreneur's discipline. If the stigma of failure is high, the optimal form of financing looks like bank debt. If, however, the stigma associated with failure is low, the optimal form of financing looks like venture capital characterized by high monitoring and staging of finance. While this explains features like staged financing and high monitoring associated with venture capital, it does not shed any light on the differences between banks and venture capitalists in terms of financial securities employed and exercise of control. Moreover, Landier assumes that entrepreneurs endogenously choose the risk of their business. By contrast, our paper makes cross-sectional predictions on the use of bank loans and venture capital.

Ueda (2000) offers an alternative explanation based on intellectual property rights protection. Her findings hinge on two critical assumptions. First, venture capitalists have a technological expertise over banks when it comes to screening projects. Second, this also enables them to 'steal' the project, i.e., continue the project without the original entrepreneur.

Lastly, our paper is also related to the literature on convertible debt. In a model with asymmetric information, Green (1984) argues that convertible debt affects the inclination of the entrepreneur to engage in risky projects. Using asymmetric information and high costs of financial distress, Stein (1992) argues that corporations may use convertible bonds as an indirect method for getting equity into their capital structures in situations where conventional

[5] For example, Hellmann (1998) focuses on contracting between a venture capitalist and entrepreneur. He shows that, due to wealth constraints, the entrepreneur may voluntarily give up control rights ex ante, which in turn may lead to excessive loss of control relative to the first-best. Nevertheless, these features are not unique to venture capital finance. Our model seeks to motivate more detailed differences across various types of financing.

stock issue might be unattractive. Cornelli and Yosha (1997) argue that when financing is staged, convertible debt provides incentives for the entrepreneur to not indulge in short-term window dressing. We provide an additional reason for using convertible debt, namely that, it can cheaply provide a private investor with incentives to actively monitor the firm.

The rest of the paper is organized as follows. Section 1 outlines our model and basic assumptions. Section 2 examines how exercise of control by the institution at date-1 depends on the type of monitoring that it engages in. Section 3 examines the interaction between contract structure and equilibrium monitoring intensity. We then introduce the notion of liquidity costs following Winton (2002). Out of the several contracts that can implement the equilibrium outcomes, we identify those that have the lowest liquidity costs. We then show that pure debt is optimal for inducing passive monitoring, while convertible debt or a debt-equity combination is optimal for inducing active monitoring. In section 4, we endogenize the choice of monitoring by relating it to the ex-ante return distribution of the firm. Section 5 concludes the paper.

1 The Model

The Firm: A firm operates over 3 dates 0, 1 & 2. At date 0, the manager of the firm makes an investment I. Since the manager is wealth constrained, she must obtain the necessary funds by issuing claims to an institution. The firm yields cash flows at date 2, which we denote by X. The firm can be of two types; 'good' or 'bad'. 'Good' firms can further be of two subtypes; 'medium' and 'high'. We denote the subtype of a 'good' firm by $i \in \{m, h\}$. The cash flow generated by a firm at date 2 depends on its type and subtype and on the actions it takes at date 1. These are described in detail below.

At date 1, the firm can either be liquidated or allowed to continue operating through date 2. If the firm is liquidated at date 1, its assets have a liquidation value $L < I$. If the firm is to continue to operate till date 2, an action $a \in \{a_S, a_R\}$ needs to be taken at date 1. We interpret action a as a crucial strategic decision regarding the future direction of the firm. One example is deciding whether or not to commit to a major expansion. Another is the choice between a risky IPO and sale of the firm to a larger rival.

If a 'bad' firm continues until date 2, it yields $X = 0$ with certainty, irrespective of the action a taken. By contrast, for a 'good' firm, X depends on i and a, as follows: If the safe action a_S is chosen, then the firm yields a date 2 cash flow of $X_S > I$ with certainty. If the risky action a_R is chosen, the firm yields a cash flow X_R with probability p_i, where $i \in \{m, h\}$, and 0 otherwise. The date 1 actions are taken by whichever party is in control.

In addition to cash flows, the manager also gets non-pecuniary or otherwise noncontractible (see Diamond (1993)) private benefits of control from operating the firm. If the firm is liquidated at date 1, the manager does not derive any control benefits. If the firm continues to operate through date 2, the manager receives control benefits that are valued at C_1 if action a_S is chosen, and C_1+C_2 if action a_R is chosen, $C_1, C_2 > 0$ [6] . All else equal, the manager will not voluntarily choose to liquidate the firm at date 1, unless she is compensated for her loss of control benefits.

The value of the firm at date 2 is equal to the cash flow generated plus the value of control benefits. The table below summarizes the cash flows and private benefits arising from the different decisions:

Action	**Cash flows**	**Control benefits**
Liquidation at date-1	$L < I$ at date-1. $X = 0$.	0
Continuation with action a_S	$X = X_s > I$ with certainty	C_1
Continuation with action a_R	$X = \begin{cases} X_R \text{ w.p. } p_i, i \in \{m, h\} \\ 0 \quad \text{otherwise} \end{cases}$	$C_1 + C_2$

Information Structure: At time-0, it is common knowledge that the firm is 'good' with probability θ and 'bad' with probability $1-\theta$. Also, it is common knowledge that a 'good' firm is of subtype $i = h$ with probability ϕ and subtype $i = m$ with probability $(1 - \phi)$. At an intermediate date 1/2, the manager of the firm freely observes the type and subtype of the firm. In order to simplify the analysis, we define:

$$q \equiv \Pr\left(X = X_R | a = a_R,\ \text{type=good}\right) = \phi p_h + (1 - \phi)\, p_m \tag{1}$$

The institution can only learn the type and subtype by engaging in costly monitoring. There are two levels of monitoring that the institution can engage in. We call these 'passive' and 'active' monitoring. We use the term active institution to denote an institution that monitors actively, and similarly for the term passive institution. We assume that passive monitoring costs m_P, while active monitoring costs $m_A > m_P$ at date 0. By engaging in passive monitoring, the institution learns the firm 'type' but not its 'subtype'. In this sense, passive monitoring is imperfect. An active institution, however, can learn the subtype of a 'good' firm by incurring an additional cost m_g at date 1.

The institution decides whether to monitor actively or passively at date 0.

[6] In the context of an expansion decision, a larger firm may offer greater managerial perquisites or prestige. In the context of our IPO vs acquisition example, after an acquisition, the entrepreneur would either be replaced or would end up as a manager of a division of the acquiring firm, whereas after an IPO she would still be the CEO of an independent firm.

This choice is observed by the manager. Monitoring and the information it reveals cannot be verified by any outside agency, so it cannot be contracted upon. This means that at date 0 it is not possible to write a complete contract specifying action as a function of firm type. Therefore, the institution must have the option of exercising control rights at date 1. We assume that all realized cash flows are verifiable, so the ex-ante contract can specify a rule for sharing these cash flows. The actions taken by the firm are also verifiable.

Assumptions: We make the following assumptions:

Assumption 1: $p_m X_R < X_S < p_h X_R$.

In other words, when $i = m$, the expected date-2 cash flow is maximized by taking the safe action a_S. When $i = h$, the expected date-2 cash flow is maximized by taking the risky action a_R.

Assumption 2: The following conditions hold:

(a) $\max\{L, \theta X_S, \theta q X_R\} < I$
(b) $(1-\theta) L + \theta (1-\phi) (X_S - C_2) + \theta \phi p_h X_R < I + m_P$
(c) $(1-\theta)(L - C_1) + \max\{\theta X_S, \theta q X_R\} < I$

Assumption 2(a) states that neither unconditional liquidation nor unconditional continuation allows the institution to break even. This assumption guarantees that the agency problem is non-trivial, i.e., it can only be overcome by an institution that monitors and exercises control.

Assumption 2(b) states that the institution cannot break even if it monitors passively and bribes the manager to choose the safe action in state m by compensating her for her loss of control benefits C_2.

Assumption 2(c) states that the institution cannot break even if it does not monitor and bribes the manager to liquidate by compensating her for her loss of control benefits C_1.

Finally, we assume that,

Assumption 3: $(1-\theta) L + \theta X_S > I + m_P$

In other words, the institution can break even if it monitors passively and forces the firm to choose the safe strategy.

Ex-ante contract: As discussed earlier, the date-0 contract cannot specify actions depending on the firm's type and subtype, because these are unverifiable. The contract can, however, specify a rule for sharing the firm's cash flows subject to limited liability constraints. Since there are 4 possible cash flows – 0, L, X_S and X_R – that the firm can generate, the date-0 contract would be of the form (S_L, S_S, S_R) satisfying $0 \leq S_L \leq L$, $0 \leq S_S \leq X_S$ and $0 \leq S_R \leq X_R$. S_L, S_S and S_R represent the cash flow to the institution when $X = L, X_S$ and X_R respectively, with the residual cash flows going to the manager. Notice that when $X = 0$, limited liability implies that the cash flow to the institution is also 0.

If the institution is to break even, the contract must give it the ability to force liquidation if the firm proves to be bad. The institution may also want the ability to control the firm's choice between risky and conservative strategies. The easiest way to do this is to have the contract require repayment from the entrepreneur at date 1. Since the firm has no cash inflows at this date, it cannot make such a payment. (One can also show that uninformed investors would not be willing to refinance the firm; essentially, asking for refinancing is at best a neutral signal, and the firm is not worth funding on an uninformed basis.) For debt, this structure can be accomplished by having the debt mature at date 1. For preferred stock, this structure can be accomplished by requiring a dividend at date 1, along with a covenant granting the institution control if the dividend is not paid.

We use S to denote the payoff to the institution, and V to denote the firm's value, i.e., the sum of cash flows generated by the firm and the manager's control benefits.

Renegotiation: At date 1, if the institution decides to exercise control, the manager can try to renegotiate with the institution so as to choose the risky action a_R. For simplicity, we assume that the manager has the bargaining power in renegotiation. The manager makes an offer to the institution, which the institution can either accept or decline. If the institution declines, then the ex-ante contract remains valid.

Timing of events: At date 0, the manager signs a contract with the institution in exchange for cash. The manager then makes an investment I. The institution also chooses its level of monitoring. At date $\frac{1}{2}$, the manager freely observes the type and sub-type of the firm. The institution observes the result of its monitoring. At date 1, the institution either forces liquidation or allows the firm to continue. In the event of continuation, the institution can exercise control to force action a_S. The manager can try to renegotiate with the institution. At date 2, cash flow X is realized.

2 Exercise of control at date 1

In this section, we analyze the institution's exercise of control and possible renegotiation with the manager, taking the institution's level of monitoring and contractual payments as given. This allows us to state the expected value of the firm (cash flows plus control benefits) and the expected payoff to the institution as a function of monitoring level and contract structure.

We begin with the case of passive monitoring. In this case, the institution knows only whether the firm is 'good' or 'bad.' If the institution knows that the firm is 'bad,' then so long as the institution's payment in liquidation (S_L) exceeds zero it will use its control rights to liquidate the firm; after all, allowing the firm to continue would only give the institution zero for sure.

If instead the institution knows that the firm is 'good,' it can be talked out of any threat to liquidate the firm; after all, the firm can offer expected cash flows (either X_S or qX_R) that exceed total liquidation proceeds L. This also means that without loss of generality we can assume that the institution's payment S_S if the firm's cash flow is X_S satisfies $S_S \geq S_L$. With this in hand, the next Lemma establishes the equilibrium behavior of both institution and manager in this case.

Lemma 1 ***(Passive monitoring):*** *If the firm is 'good', then the institution exercises control to choose $a = a_S$ if and only if $S_S > qS_R$. The manager then renegotiates to choose $a = a_R$ if and only if $S_S \leq qX_R$. If the institution does not exercise control, the manager will choose $a = a_R$.*

Since the passive institution does not know whether the firm is 'medium' or 'high', it must rely on its ex-ante beliefs regarding the firm's subtype when deciding whether or not to exercise control. The institution knows that if it doesn't exercise control, the manager will surely choose the risky action. Thus, the institution's expected payoff from not exercising control is qS_R, and so it exercises control if and only if its cash flow under the safe action S_S exceeds qS_R. If the institution does choose to exercise control, the manager will try to renegotiate and choose a_R by offering the institution a higher payment $S_R' = \frac{S_S}{q}$, so that the institution gets the same expected payoff under the risky action that it would get from the safe action. Since the manager has all the bargaining power, she claims all the gains from renegotiation; however, the firm's maximum cash flow X_R may limit the possibility of successful renegotiation.

Let $E_P(V)$ and $E_P(S)$ denote the expected value of the firm and the expected payments to the institution, respectively, under passive monitoring. Then, from Lemma 1,

$$E_P(V) = \begin{cases} (1-\theta)L + \theta(C_1 + C_2 + qX_R) & \text{if} \quad qX_R \geq S_S \\ (1-\theta)L + \theta(C_1 + X_S) & \text{otherwise} \end{cases} \tag{2}$$

$$E_P(S) = (1-\theta)S_L + \theta \max\{qS_R, S_S\} \tag{3}$$

We now turn to the case where the institution has monitored actively. As with passive monitoring, if the firm is 'bad', the institution exercises control and liquidates the firm. If the firm is 'good', the institution can learn its subtype by incurring a cost m_g. If it chooses not to incur m_g, the active institution has the same information as a passive institution. Since $m_A > m_P$, this strategy is strictly dominated by passive monitoring, so the institution would not choose active monitoring in the first place. Therefore, an active institution will also incur m_g at date 1 if the firm is 'good' and learn whether the firm's subtype is 'medium' or 'high'.

Lemma 2 ***(active monitoring):*** *For a 'good' firm:*

- *If $i = m$: The institution exercises control to choose $a = a_S$ if and only if $S_S > p_m S_R$. The manager then renegotiates with the institution to choose $a = a_R$ if and only if $S_S \leq p_m X_R$.*
- *If $i = h$: The institution exercises control to choose $a = a_S$ if and only if $S_S > p_h S_R$. The manager always renegotiates successfully with the institution and chooses $a = a_R$.*

As in Lemma-1, the institution exercises control if and only if it expects to get a higher payoff under the safe action, but its information is now more precise. It follows that an active institution is less likely to exercise control than a passive institution when $i = h$ and more likely to exert control when $i = m$. To see this, suppose $i = m$ and the institution exerts control. For renegotiation to be successful, the manager must give away a higher share of cash flows when $X = X_R$ to the active institution than she would have given a passive institution ($\frac{S_S}{p_m} > \frac{S_S}{q}$). On the other hand, if $i = h$, renegotiation is easier with an active institution than with a passive institution.

Let $E_A(V)$ and $E_A(S)$ denote the expected value of the firm and the expected payment to the institution, respectively, under active monitoring. Then:

$$E_A(V) = \begin{cases} (1-\theta)\,L + \theta\,(C_1 + C_2 + qX_R) & \text{if } p_m X_R \geq S_S \\ (1-\theta)\,L + \\ \theta\,[C_1 + (1-\phi)\,X_S + \phi\,(C_2 + p_h X_R)] & \text{otherwise} \end{cases} \tag{4}$$

$$E_A(S) = (1-\theta)\,S_L + \theta\,(1-\phi)\max\{p_m S_R, S_S\} + \theta\phi\max\{p_h S_R, S_S\} \tag{5}$$

Finally, note that whether the institution monitors passively or actively, the manager's net payoff is $E(V) - E(S)$.

3 Equilibrium Monitoring

We now begin to analyze the equilibrium choices of the institution and the manager at date 0. First, we define equilibrium. We then show that, generically, it is never optimal to have the institution randomize its monitoring choice. We then turn to the specific conditions required for passive and active monitoring to obtain in equilibrium. Finally, we discuss how the inclusion of liquidity costs as in Winton (2002) affects optimal contract structures for a given type of monitoring. These results form the basis for the manager's overall choice of contracts and monitoring levels, which we deal with in Section 4.

Before stating the formal definition of equilibrium, we make some preliminary observations. First, note that, given a set of contractual payments S, the institution's expected date 0 payoff after it has invested is its expected payments less any expected monitoring costs; i.e., $E_P(S) - m_P$ if it monitors passively and $E_A(S) - m_A - \theta m_g$ if it monitors actively.

Second, because we assume that the manager has all power in bargaining, the manager receives all rents initially as well as subsequently. If the institution's expected payments are $E(S)$ and its expected monitoring costs are $E(m)$, then it will invest precisely $E(S) - E(m)$ initially, breaking even on average. If the firm is to be funded initially, $E(S) - E(m)$ must weakly exceed I.

The manager's date-0 expected utility is the sum of her expected net payoffs as of date 1 and any cash she receives initially over and above the required investment I. Since the manager's expected payoff at date 1 is given by $E(V) - E(S)$, and her initial excess cash is given by $E(S) - E(m) - I$, it follows that, in equilibrium, the manager's expected utility at date 0 is $E[V] - I - E[m]$.

Third, equilibrium requires that, given the contractual payments it receives, the institution prefers its monitoring choice at date 0 over alternative choices. In principle, the institution could randomize its choice of monitoring strategy, monitoring actively with probability α, monitoring passively with probability β, and not monitoring at all with probability $1 - \alpha - \beta$. Nevertheless, we can immediately rule out the case where the institution does not monitor with positive probability ($1 - \alpha - \beta > 0$), and restrict our attention to equilibria where the institution chooses only between active and passive monitoring; i.e., $\beta = 1 - \alpha \in [0, 1]$.[7]

Given these observations, the formal definition of an equilibrium in this game consists of monitoring strategy $\hat{\alpha}$ for the institution and a contract (S_L, S_S, S_R) offered by the manager such that:

- $\hat{\alpha}$ is incentive compatible given (S_L, S_S, S_R), i.e.,

$$\hat{\alpha} = \arg\max \ \alpha \left[E_A(S) - m_A - \theta m_g\right] + (1 - \alpha)\left[E_P(S) - m_P\right] \tag{6}$$

- (S_L, S_S, S_R) maximizes the manager's ex-ante utility subject to the financing

[7] If the institution does not monitor, it can only choose a completely unconditional strategy for the firm. Assumption 2(a) immediately shows that the institution could not possibly break even on the ground investment I. But for the institution to be willing to randomize between monitoring and not monitoring, it must be indifferent between its payoffs under the different choices. Thus, its expected payoff from monitoring would have to be the same as that from not monitoring, so that its *overall* expected payoff $E(S) - E(m)$ would also be less than I.

constraint given $\hat{\alpha}$, i.e., (S_L, S_S, S_R) is a solution to the following problem:

$$\max_{S_L,S_S,S_R} U_M = \alpha \left[E_A(V) - m_A - \theta m_g\right] + (1-\alpha)\left[E_P(V) - m_P\right] - I$$

subject to the financing constraint:

$$\alpha \left[E_A(S) - m_A - \theta m_g\right] + (1-\alpha)\left[E_P(S) - m_P\right] \geq I \tag{7}$$

We now show that, generically, it is never optimal for the manager to have the institution to randomize its choice between monitoring actively and monitoring passively (that is, choose $\alpha \in (0,1)$). To see this, first note that if the institution randomizes, then the institution must be indifferent between its choices. For such an α to be an equilibrium, then, the manager must also be indifferent between these two monitoring choices. This requires that $\frac{\partial U_M}{\partial \alpha} = 0$. But,

$$\frac{\partial U_M}{\partial \alpha} = E_A(V) - E_P(V) - (m_A + \theta m_g - m_P)$$

Since $E_A(V)$ and $E_P(V)$ are functions only of the underlying parameters of the model, this derivative will be either positive or negative except on a set of measure zero. Thus, generically, randomized monitoring will not be optimal.

We can, therefore, focus our attention on pure strategy equilibria. In the following two subsections, we characterize pure strategy equilibria involving passive and active monitoring, respectively. We discuss the feasibility conditions under which these equilibria can be implemented, and also the financial contracts that can optimally implement them. In Section 4, we deal with the choice of optimal contracts and equilibria.

3.1 Equilibria with passive monitoring:

From equation (2), we can see that there are two possible outcomes with passive monitoring, differing in terms of the continuation action chosen for a 'good' firm. In one ("P1"), the institution allows the firm to choose the risky action a_R; in the other ("P2"), the institution forces the safe action a_S. We use the term $U_{M,P1}$ to denote the manager's utility under outcome P1, and similarly for the term $U_{M,P2}$. Both these outcomes can occur only when the following constraints are satisfied:

$$(1-\theta) S_L + \theta \max\{qS_R, S_S\} - m_P \geq I \tag{8}$$

$$\begin{aligned}\theta \max\{qS_R, S_S\} - m_P \geq\ & \theta(1-\phi)\max\{p_m S_R, S_S\} \\ & + \theta\phi \max\{p_h S_R, S_S\} - m_A - \theta m_g\end{aligned} \tag{9}$$

Here, (8) is the institution's financing constraint and (9) is its incentive compatibility constraint; the institution must prefer passive monitoring over active monitoring.

Outcome P1: $U_{M,P1} = (1-\theta)\,L + \theta\,(C_1 + C_2 + qX_R) - m_P - I$**.** Outcome P1 is implemented by any contract (S_L, S_S, S_R) that satisfies the following renegotiation constraint in addition to (8) and (9) above:

$$qX_R \geq S_S \tag{10}$$

Lemma 3 ***(Feasibility):*** *Outcome P1 can be implemented if and only if* $(1-\theta)\,L + \theta qX_R \geq I + m_P$.

In outcome P1, the manager of the 'good' firm always renegotiates and gives the institution a new claim $S'_R = \frac{S_S}{q}$, in return for which she is allowed to choose $a = a_R$. Without loss of generality, this outcome can be implemented by a contract that gives the institution a flat payoff irrespective of whether $X = X_S$ or X_R, such that $S_S = S_R \leq qX_R$. The incentive compatibility constraint is then satisfied because a passive institution realizes the same expected value of claims as an active institution, but at a lower cost.

Note that any debt contract with face value $D \leq qX_R$ that satisfies the financing constraint can implement outcome P1. The feasibility condition in Lemma 3 above ensures that such a contract exists.

Outcome P2: $U_{M,P2} = (1-\theta)\,L + \theta\,(C_1 + X_S) - m_P - I$**.** Outcome P2 is implemented by any contract (S_L, S_S, S_R) that satisfies the following renegotiation constraint in addition to (8) and (9) above:

$$qX_R < S_S \tag{11}$$

If condition (11) did not hold, the manager would always renegotiate to outcome P1, which she prefers.

Lemma 4 ***(Feasibility):*** *Outcome P2 can be implemented if and only if* $qX_R < X_S$.

Using our intuition above in case of outcome P1, one can see that a debt contract will work here as long as $qX_R < X_S$, because otherwise the renegotiation and limited liability constraints cannot be satisfied simultaneously. Any debt contract with face value $qX_R < D \leq X_S$ that satisfies the financing constraint can implement outcome P2. Assumption 3 ensures that such a contract exists.

3.2 Equilibria with active monitoring:

From equation (4), we can see that there are two possible outcomes with active monitoring. We refer to these outcomes as A1 and A2. These outcomes differ in terms of the continuation action chosen for a 'good' firm. In outcome A1, the institution allows a 'good' firm's manager to choose the risky action a_R regardless of the firm's subtype i; in outcome A2, the institution forces the safe action a_S when $i = m$, and allows the risky action when $i = h$. We use the term $U_{M,A1}$ to denote the manager's utility under outcome A1, and similarly for the term $U_{M,A2}$. Both these outcomes can occur only when the following constraints are satisfied:

$$(1-\theta) S_L + \theta [(1-\phi) \max \{p_m S_R, S_S\} + \phi \max \{p_h S_R, S_S\} - m_g] \\ -m_A \geq I \tag{12}$$

$$\theta [(1-\phi) \max \{p_m S_R, S_S\} + \phi \max \{p_h S_R, S_S\} - m_g] - m_A \\ \geq \theta \max \{q S_R, S_S\} - m_P \tag{13}$$

Here (12) is the institution's financing constraint, and (13) is its incentive compatibility constraint.

Claim 1 *Outcomes A1 and A2 can only be implemented by contracts satisfying $p_m S_R < S_S < p_h S_R$.*

Incentive compatibility requires that, if the institution chooses active monitoring, then the expected value of its claims under active monitoring must exceed the expected value of its claims under passive monitoring by more than the incremental monitoring costs it incurs. If $S_S \geq p_h S_R$, the institution always threatens to force action a_S regardless of subtype. On the other hand, if $S_S \leq p_m S_R$, the institution always allows the manager to choose the risky action regardless of subtype. In either case, the expected value of an active institution's claims would equal the expected value of a passive institution's claims, so the institution would have no incentive to monitor actively. This gives the intuition behind Claim 1. Thus, we can rewrite the above financing and incentive compatibility constraints as follows:

$$(1-\theta) S_L + \theta [(1-\phi) S_S + \phi p_h S_R - m_g] - m_A \geq I \tag{14}$$

$$\theta [(1-\phi) S_S + \phi p_h S_R - m_g] - m_A \geq \theta \max \{q S_R, S_S\} - m_P \tag{15}$$

Outcome A1: $U_{M,A1} = (1-\theta) L + \theta (C_1 + C_2 + q X_R - m_g) - m_A - I$. Outcome A1 is implemented by any contract (S_L, S_S, S_R) that satisfies the following renegotiation constraint, in addition to (14) and (15) above:

$$p_m X_R \geq S_S \tag{16}$$

For convenience, we define $\Delta \equiv m_A + \theta m_g - m_P$. Δ is the incremental cost involved in active monitoring. Note that Δ depends on θ and that $\Delta > 0$.

Lemma 5 ***(Feasibility):*** *Outcome A1 can be implemented if and only if the following conditions are satisfied:*

(i) $\Delta \leq \frac{\theta\phi(1-\phi)(p_h-p_m)}{q} p_m X_R$, *and*

(ii) $(1-\theta)\, L + \theta q X_R - \frac{\phi p_h}{(1-\phi)p_m}\Delta \geq I + m_A + \theta m_g$.

Note that $U_{M,A1} < U_{M,P1}$. Therefore, outcome P1 Pareto dominates outcome A1. Also, from Lemmas (5) and (3), outcome P1 is feasible whenever outcome A1 is feasible, but not the other way round. Therefore, it follows that *outcome A1 never occurs in equilibrium.*

Outcome A2: $U_{M,A2} = (1-\theta)\, L + \theta\left[C_1 + (1-\phi)\, X_S + \phi\left(C_2 + p_h X_R\right) - m_g\right] - m_A - I$**.** Outcome A2 is implemented by any contract (S_L, S_S, S_R) that satisfies the following renegotiation constraint, in addition to (14) and (15) above:

$$p_m X_R < S_S \tag{17}$$

Lemma 6 ***(Feasibility):*** *Outcome A2 can be implemented if and only if the following conditions are satisfied:*

(1) $\Delta \leq \frac{\theta\phi(1-\phi)(p_h-p_m)}{q} \min\{X_S, qX_R\}$, *and*

(2) Either of the following holds:

(a) $X_S \leq qX_R$, *or*

(b) $(1-\theta)\, L + \theta\,(1-\phi) \min\left\{X_S, p_h X_R - \frac{\Delta}{\theta\phi}\right\} + \theta\phi p_h X_R \geq I + m_A + \theta m_g$.

Incentive compatibility requires that the incremental cash flow to the institution from active monitoring must exceed the incremental cost of active monitoring, Δ. The incremental cash flow from active monitoring is $\theta\left[(1-\phi)\, S_S + \phi p_h S_R - \max\{S_S, qS_R\}\right]$. Requiring that this amount exceed Δ can be shown to imply condition (1) in Lemma 6. When Condition 2(a) holds, incentive compatibility guarantees that the financing constraint is met; otherwise, Condition 2(b) is required to guarantee this.

Can pure debt implement outcome A2? From Claim 1, we know that $S_R > S_S$. Therefore, the only way pure debt can induce active monitoring is if the face value of debt exceeds X_S. Indeed, we show in the proof of Lemma 6 that, if $X_S \leq qX_R$, then the contract $\left(L, X_S, \frac{X_S}{q}\right)$, which is equivalent to debt with face value $\frac{X_S}{q}$, can implement A2. However, this implies that the institution effectively buys much of the firm initially. By providing cash far in excess of the required investment I, the institution is needlessly increasing the illiquidity of its position, as we now show.

3.3 Liquidity costs:

Thus far, our analysis has taken the institution's claim on the firm in isolation from the rest of the institution's business. This simplification is neither realistic nor harmless. As Winton (2002) discusses, the fact that a financial institution is in the business of investing "other people's money" means that the institution faces ongoing needs for funds as other firms come to it for financing or investors need to withdraw their funds. For banks, such "liquidity needs" might involve meeting demand for additional loans, takedowns under existing credit lines and loans commitments, or meeting higher than usual demands for repayment from depositors and other investors. Similarly, venture capital funds may have other portfolio companies that are attractive and require funds in a timely fashion. In all of these cases, failure to meet liquidity needs hurts the institution's business, creating costs.

If the institution's liquidity needs are sufficiently high, it may be forced to seek financing from its own investors, using its claim on the firm as collateral. Winton (2002) shows that, because the institution has private information about the value of its claim on the firm, this leads to adverse selection. On average, the institution cannot access the full value of its assets, so some of its liquidity needs go unmet, creating liquidity costs. The greater the risk of the institution's claim on the firm, the greater the adverse selection and so the greater the expected liquidity costs. Because the institution demands recompense for expected liquidity costs, this preference is passed through to the firm's manager: all else equal, she prefers to reduce the risk of the institution's claim on her firm. Winton uses these costs to motivate cross-sectional differences in the asset structures of financial institutions.

Because our goal in this paper is simply to analyze the impact of these costs on entrepreneurial financing structures, we introduce these liquidity costs in a reduced form that is derived in the appendix. For further simplicity, we use these costs as a tie-breaking rule only; i.e., for a given optimal outcome, we assume that the manager chooses the structure that gives the institution the lowest liquidity costs. Explicitly including these costs in the manager's optimization problem would clutter analysis without changing our qualitative results.

Accordingly, we assume that an institutional institution whose claim has values S_L, S_S, S_R and which monitors with level α faces expected liquidity costs $\Lambda\left(S_L, S_S, S_R; \alpha\right)$. These liquidity costs are linked to adverse selection that the institution faces if it seeks funds before date 1 from uninformed institutions using its claim on the entrepreneur as collateral. It is easy to show (see appendix) that $\Lambda\left(.\right)$ has the following properties:

Assumption L1: $\Lambda\left(\gamma S_L, \gamma S_S, \gamma S_R; \alpha\right) = \gamma\Lambda\left(S_L, S_S, S_R; \alpha\right)$ for all $\gamma > 0$. (Liquidity costs are proportional to the size of the payoffs.)

Assumption L2: $\Lambda\left(S_L, S_S, S_R; \alpha\right)$ is decreasing in S_L and increasing in S_R.

Assumption L3: $\Lambda\left(S_L, S_S, S_R; \alpha = 0\right)$ is increasing in S_S.

Assumption L4: Define z as a change in payments such that $\frac{dS_L}{dz} = 0$, $\frac{dS_S}{dz} > 0$, $\frac{dS_R}{dz} > 0$, and $\frac{dS_S}{dz} \leq p_h \frac{dS_R}{dz}$. Then $\Lambda\left(S_L, S_S, S_R; \alpha = 1\right)$ is increasing in z.

These assumptions follow from the nature of adverse selection problem that the institution faces. The greater the risk of the institution's claims on the firm, the greater the adverse selection problem that the institution faces when it seeks to use these claims as collateral to get funding from uninformed institutions. Since the lowest value of institution's claims at date-1 is S_L, raising S_L raises the 'worst-case' value of the institution's claims, increasing its ability to get financing in the event of a liquidity need (Assumption L2). Conversely, as the maximum value of the institution's claims increases, the gap between this 'best-case' value and the worst-case value increases, and the institution faces greater adverse selection costs (Assumption L2).

If the institution monitors passively, its claims have two possible values: either S_L or $\max\{S_S, qS_R\}$. An increase in S_S unambiguously increases the maximum value of the institution's claims, increasing the gap between best- and worst-case values and thus increasing adverse selection costs (Assumption L3). If the institution monitors actively, there are now three possible values of the institution's claims – S_L, S_S and $p_h S_R$ – and analysis is more complex. An increase in S_S increases adverse selection costs when the institution knows that its claim has value S_S rather than S_L, but it tends to reduce adverse selection costs when the institution knows that its claim has value $p_h S_R$. (Decreasing the difference between the best and second-best types makes it cheaper for the best type to separate itself from the second-best type.) Nevertheless, if the best-case value itself increases by the same amount as the second-best type, the net effect is an increase in adverse selection costs across the board.

Corollary: Under these assumptions, all else equal, the entrepreneur prefers that: (i) the institution receives as high a payment in liquidation (S_L) as possible; (ii) for a given level of monitoring, S_S and S_R be as small as possible subject to financing, incentive, and renegotiation constraints.

3.3.1 Optimal contracts

As noted above, rather than explicitly incorporate liquidity costs, in what follows, we use them as a tie-breaking rule. We have seen that there exist a variety of feasible contracts that can implement outcomes P1, P2 and A2[8]. For each outcome, we now examine which of these feasible contracts is optimal, i.e., which has the lowest liquidity cost.

Lemma 7 ***(Optimal Contract):***

(1) If outcome P1 is feasible, then an optimal contract that implements this outcome is $S_L^* = L$, $S_S^* = S_R^* = \frac{I+m_P-(1-\theta)L}{\theta}$.

(2) If outcome P2 is feasible, then an optimal contract that implements outcome P2 is $S_L^* = L$, $S_S^* = S_R^* = \max\left\{\frac{I+m_P-(1-\theta)L}{\theta}, qX_R\right\}$.

(3) Any optimal contract that can implement outcome A2 will satisfy $S_L^* = L$ *and* $S_S^* < X_S^*$.

[8] We have already shown that outcome A1 will never be implemented in equilibrium.

We have seen earlier that outcomes P1 and P2 can be implemented by contracts resembling pure debt. The optimal contract that can implement these outcomes is a standard debt contract with face value $D = \frac{I+m_P-(1-\theta)L}{\theta}$. Under this contract, the financing constraint is binding, so the institution need not provide any excess funds to the manager. We have already seen that pure debt can implement outcome A2 only if the institution invests in excess of the required investment I into the firm. Such a claim would involve higher liquidity costs, and so would not be optimal. Lemma 7 states that the optimal contract that can implement outcome A2 is convertible debt, or a combination of debt and equity [9] .

The upshot is that the presence of institutional liquidity costs makes a pure debt structure optimal unless the manager's preferred outcome is one that involves active monitoring (outcome A2). When active monitoring is preferred, the institution must be given incentive to differentiate between high and low medium levels of success, which in turn requires that the institution's promised payments must differ across the safe and risky strategies. Convertible debt or a mix of debt and equity accomplishes this with the lowest total exposure to the firm, minimizing the institution's liquidity costs.

4 Optimal start-up finance

In the previous section, we showed that there are two possible equilibrium outcomes involving passive monitoring, and one equilibrium outcome of interest with active monitoring. We now endogenously determine the optimal outcome chosen by the manager given the ex-ante return distribution of the firm (parameters θ, ϕ, p_h, p_m, X_S and X_R) and her own control benefits (C_1 and C_2).

Claim 2 *Either P1 or P2 is always feasible.*

From Lemma 3 and Assumption 3(a), it follows that if P1 is not feasible then $qX_R < X_S$. Lemma 4 then implies that P2 is feasible. Similarly, if P2 is not feasible, it can be shown that P1 is feasible. Therefore, either of outcomes P1 or P2 is always feasible.

Proposition 8 *The manager chooses P1 whenever it is feasible.*

The manager's control benefits are the highest under P1. Therefore, if P1 is feasible, the manager would choose P2 if and only if the cash flow under P2 was higher than the cash flow and incremental control benefits under P1. But then, a strategy of passive monitoring and bribing the manager for her loss of control benefits would work, violating Assumption 2(b). Thus, if P1 is feasible, it is always preferred over P2. A similar argument shows that P1 is always preferred over A2.

[9] $S_L^* = L$ and $S_S^* < X_S$ rule out pure equity.

Proposition 8 implies that the manager chooses P2 or A2 if and only if P1 is not feasible. Claim 2 tells us that if P1 is not feasible, then P2 is. It follows that A2 is the optimal outcome if and only if the following conditions are satisfied:

(1) P1 is not feasible, i.e., from Lemma 3:

$$\theta q X_R < I + m_P - (1-\theta) L \tag{18}$$

(2) A2 is feasible, i.e., combining $qX_R < X_S$ (since P1 is not feasible) with Lemma 6,

$$\Delta \leq \theta\phi (1-\phi)(p_h - p_m) X_R \tag{19}$$

and

$$(1-\theta) L + \theta (1-\phi) \min\left\{X_S, p_h X_R - \frac{\Delta}{\theta\phi}\right\} + \theta\phi p_h X_R \geq I + m_A + \theta m_g \tag{20}$$

(3) $U_{M,A2} > U_{M,P2}$, which simplifies to

$$\Delta < \theta\phi (C_2 + p_h X_R - X_S) \tag{21}$$

Similarly, P2 is the optimal outcome if and only if constraint (18) is satisfied, and at least one of constraints (19), (20), or (21) is violated.

Clearly, A2 is optimal only when several restrictive conditions are simultaneously satisfied. We now characterize how the ex-ante return distribution of the firm and the manager's control benefits influence these constraints.

We first consider condition (18), which guarantees that outcome P1 is infeasible.

Substituting $q = (1-\phi) p_m + \phi p_h$, and rearranging terms, constraint 18 can be rewritten as follows:

$$\phi < \frac{I + m_P - (1-\theta) L - \theta p_m X_R}{\theta (p_h - p_m) X_R} \leq \frac{X_S - p_m X_R}{(p_h - p_m) X_R} \tag{22}$$

where the second inequality follows from Assumption 3(a) ($I + m_P - (1-\theta) L \leq \theta X_S$). Thus, the infeasibility of P1 sets an upper bound on ϕ, the probability that a 'good' firm is of the 'high' subtype. [10] .

Next, consider condition (19). The left-hand side of (19) is the increase in monitoring costs that results from monitoring actively rather than passively. The right-hand side is the gain in cash flows. This gain is directly related to the strategic uncertainty of the firm's risky continuation strategy a_R, conditional on the information revealed by active monitoring, the variance of cash flows under this strategy is $\phi(1-\phi)(p_h - p_m)^2 X_R^2$. Intuitively, this makes sense; the

[10] Note that $X_S < p_h X_R$ implies that $\frac{X_S - p_m X_R}{(p_h - p_m) X_R} < 1$.

value of active monitoring is the value of the option to use better information to choose between the safe and risky strategies. As the conditional variance of risky cash flows increases, the value of this option increases, making active monitoring more attractive. This is consistent with the stylized fact that firms that receive venture capital finance typically exhibit higher strategic uncertainty than those that rely on bank finance.

Another feature of venture capital targets is their highly skewed returns, with low chances of success and low liquidation values, but extremely high payoffs if the firm is in fact successful. We can parameterize this type of skewness as follows:

Definition 1: For $\beta > 1$, consider the transformation $(\theta, X_S, X_R, C_1, C_2, L) \rightarrow \left(\beta\theta, \frac{X_S}{\beta}, \frac{X_R}{\beta}, \frac{C_1}{\beta}, \frac{C_2}{\beta}, \left[\frac{1-\theta}{1-\beta\theta}\right] L\right)$. We define this transformation as a decrease in the firm's risk and skewness.

Under this transformation, the expected value of the firm's cash flows and control benefits under any monitoring and liquidation or continuation strategy is unchanged; however, success is more likely, liquidation value is higher, and cash flows and control benefits under continuation are lower. Thus an increase in β reduces the firm's risk and skewness. Our next result shows that such a decrease in risk makes it less likely that A2 is the optimal outcome.

Proposition 9 *Consider a decrease in the firm's risk and skewness as defined above. Then it is less likely that A2 is the optimal outcome.*

To see this result, first note that an increase in β does not affect condition (18) (the infeasibility of outcome P1); this condition depends only on expected cash flows, which are by definition unchanged. By contrast, conditions (19)-(21) are less likely to be met as Δ increases, and an increase in β increases Δ. Intuitively, as the firm's returns are less skewed, success is more likely and so the firm is more likely to be allowed to continue. This increases the odds that the firm will be allowed to continue, making an active monitoring strategy more costly: the institution is more likely to have to make the decision between safe and risky continuation strategies, which is when the additional monitoring cost m_g must be incurred. Since the expected cash flows under continuation are unchanged, active monitoring is less attractive as the firm's risk and skewness decrease.

Thus, in this section we have shown that outcome A2, which mirrors venture capital finance, is only optimal under a restricted set of circumstances: the firm's strategic uncertainty and the risk and skewness of its returns must be sufficiently high, and its financial slack cannot be too high. Otherwise, debt finance with passive monitoring (outcome P1 or P2) is optimal.

5 Mezzanine Finance

One of our critical assumptions is Assumption 2(b), which rules out getting the manager to choose the safe continuation strategy over the risky one. In this section, we show that relaxing this assumption leads to new type of financing choice in which the institution monitors passively, holds a security that resembles convertible debt, and relies on the manager to make the choice between safe and risky continuation strategies. This contract is feasible only if the firm's financial slack is sufficiently high, and optimal only if the firm's strategic uncertainty is sufficiently high. As we will discuss, this constellation of contract and circumstances resembles so-called "mezzanine debt" or "mezzanine finance", which is typically provided to firms that are either late-stage venture capital targets or more established firms.

More formally, the new outcome we have described is as follows: the institution monitors passively and liquidates the firm when the firm's type is 'bad'. When the firm's type is 'good', the manager is allowed to choose the firm's continuation strategy: the risky action a_R if the firm's sub-type is 'high' ($i = h$), and a_S if the sub-type is 'medium' ($i = m$). We refer to this outcome as P3.

It follows that, under outcome P3, the firm's ex-ante expected value (including control benefits and netting out the required investment I and passive monitoring costs m_P) is

$$U_{M,P3} = (1-\theta)\,L + \theta\left[C_1 + (1-\phi)\,X_S + \phi\,(p_h X_R + C_2)\right] - m_P - I \qquad (23)$$

From Lemma 1, we know that under passive monitoring the manager always chooses the risky continuation action unless the institution forces her to choose the safe action, regardless of her firm's sub-type. This analysis follows from Assumption 2(b): the institution cannot break even under any contract which compensates the manager for her loss of control benefits C_2 when she chooses the safe action over the risky action. Thus Assumption 2(b) must be violated if outcome P3 is to be feasible. This in turn requires that the firm has higher financial slack. Nevertheless, violation of Assumption 2(b) is not sufficient, as our next lemma shows.

Lemma 10 *Outcome P3 can be implemented if and only if the following conditions are satisfied:*

(1) $X_S - C_2 \geq 0$, *and*
(2) $(1-\theta)\min\{L, X_S - C_2\} + \theta\,(1-\phi)\,(X_S - C_2) + \theta\phi p_h X_R \geq I + m_P$.

In outcome P3, the manager must sometimes voluntarily choose the safe action over the risky action. Thus, she must be compensated for her loss of control benefits, which requires that the institution's payment under the safe action, S_S, must be less than or equal to $X_S - C_2$. This accounts for condition (1) in Lemma 10.

Another constraint follows from the fact that, in outcome P3, the manager's choice of continuation strategy reveals the firm's sub-type. Thus, the institution may make selective threats to liquidate the firm even though it knows that the firm is 'good'. This implies that we can effectively restrict contracts to those in which the institution's payment in liquidation, S_L, is less than or equal to S_S and $p_h S_R$. It follows that $S_L \leq \min\{L, X_S - C_2\}$, which is why this term appears in condition (2) in Lemma 10. (We need not worry about the constraint on $p_h S_R$, since if this is less than S_L the financing constraint cannot be met.)

As noted previously, violation of Assumption 2(b) is necessary but not sufficient for Lemma 10. In principle, Assumption 2(b) can be violated even if condition (1) in the lemma does not hold; moreover, the financing constraint (condition (2) in the lemma) must incorporate the institution's ability to threaten liquidation.

If the conditions in Lemma 10 are satisfied, then outcome P3 can be implemented by the contract $S_L = \min\{L, X_S - C_2 - p_m\varepsilon\}$, $S_S = X_S - C_2 - p_m\varepsilon$ and $S_R = X_R - \varepsilon$, where $\varepsilon \geq 0$ is chosen such that Condition 2 binds.

Optimal start-up finance: In Section 4 of the paper, we analyzed the optimal form of start-up finance chosen by the manager, in a model where Assumption 2(b) held. If the feasibility conditions identified in Lemma 10 are satisfied, then we can have three possible equilibrium outcomes involving passive monitoring – P1, P2 and P3 – and one possible outcome involving active monitoring, A2. Similar to the analysis in Section 4, we endogeneously determine the optimal outcome chosen by the manager given the ex-ante return distribution of the firm and her control benefits.

Notice that $U_{M,P3} > U_{M,A2}$. Outcome P3 results in the same ex-ante expected firm value as outcome A2, but at a lesser monitoring cost, because $m_P < m_A + \theta m_g$. Therefore, if outcome P3 is feasible, the manager strictly prefers it to outcome A2.

Also, recall that,

$$U_{M,P1} = (1-\theta)\,L + \theta\,(qX_R + C_1 + C_2) - m_P - I \tag{24}$$

$$U_{M,P2} = (1-\theta)\,L + \theta\,(X_S + C_1) - m_P - I \tag{25}$$

$U_{M,P3} - U_{M,P2} = \theta\phi\,(p_h X_R + C_2 - X_S) > 0$ (by Assumption 1). Therefore, if outcome P3 is feasible, the manager strictly prefers it to outcome P2.

$U_{M,P1} - U_{M,P3} = \theta\,(1-\phi)\,(p_m X_R + C_2 - X_S)$. Therefore, outcome P1 is preferred to outcome P3 if and only if $p_m X_R + C_2 - X_S \geq 0$.

Proposition 11 *Suppose outcome P3 is feasible. Then, the manager chooses outcome P3 if* $p_m X_R + C_2 - X_S < 0$*, and outcome P1 if* $p_m X_R + C_2 - X_S \geq 0$.

Proposition 11 follows directly from the discussion preceding it. We have shown that if outcome P3 is feasible, then the manager will never choose outcomes P2 or A2. That leaves only outcomes P1 and P3 to choose from. If $p_m X_R + C_2 - X_S < 0$, then $U_{M,P3} - U_{M,P1} > 0$, and the manager chooses P3. Otherwise, $p_m X_R + C_2 - X_S \geq 0$, combined with the feasibility of P3, implies that outcome P1 is feasible. $p_m X_R + C_2 - X_S \geq 0$ also implies that outcome P1 is preferred to outcome P3; hence, the manager will choose outcome P1 in this case.

Suppose that P3 is optimal. Since the institution's liquidity costs are reduced by making its payment in liquidation, S_L, as high as possible, liquidity considerations will push this payment to $\min\{L, X_S - C_2\}$. Thus, the institution's payment looks like convertible debt: a payment of L or $X_S - C_2$ (whichever is lower) if the firm is liquidated, a payment of $X_S - C_2$ when the firm continues under the safe strategy, and a higher payment when the firm continues under the risky strategy.

Thus, we have shown that when Assumption 2(b) is violated, a third passive monitoring outcome P3 is feasible if the firm's financial slack is sufficiently high. Whenever P3 is feasible, it dominates the active monitoring outcome A2, which resembles venture capital finance. Nevertheless, the manager only prefers P3 over other passive outcomes if $p_m X_R + C_2 - X_S$ is negative. This is equivalent to $X_S - p_m X_R > C_2$; i.e., the cash flow value of the option to choose the safe action over the risky action when the firm's sub-type is 'medium' must be sufficiently high. This requires that the firm has high strategic uncertainty. Finally, the optimal contract resembles convertible debt.

This combination of features – convertible debt financing with less active monitoring, high strategic uncertainty but also more financial slack – resembles the circumstances of firms that receive so-called mezzanine financing or mezzanine debt. In the context of entrepreneurial firms, recipients of mezzanine finance are typically better-established firms that may have received traditional venture capital earlier on. Thus, extending our model to allow for violation of Assumption 2(b) allows us to incorporate mezzanine finance as an optimal outcome under plausible circumstances.

6 Concluding Remarks: Implications and Extensions

In our introduction, we suggested that venture capital differs from bank finance by greater use of equity features and by more active monitoring, particular when the firm is choosing continuation strategies. In this concluding section, we discuss our model's implications for the choice between these two financing structures and some possible extensions of our analysis.

One point that our model emphasizes is that, from the manager's viewpoint, active monitoring is often a necessary evil. Thus, if the 'good' firm's risky continuation strategy is lucrative enough on average (i.e., qX_R is high enough), the manager prefers debt and passive monitoring over convertible debt and active monitoring. In this case, liquidating only when the firm is an out-and-out failure is enough to allow the institution to break even on its investment. Active monitoring only worsens matters: it is more costly for the institution (which will demand recompense) and it limits the manager's benefits of control. Thus, firms with higher financial slack are less likely to prefer active monitoring.

Even when the risky strategy is less profitable, on an ex ante basis the manager may prefer being forced to hew to the safer continuation strategy regardless, even though ex post she would prefer active monitoring that would sometimes allow her to opt for the risky strategy. Once again, debt and passive monitoring does the job. It follows that convertible debt (or a debt-equity mix) and active monitoring are feasible and preferred only in a limited range of circumstances.

First, strategic uncertainty $\phi(1-\phi)(p_h - p_m)X_R$ must be high. This is fairly intuitive – active monitoring adds most value when the possible outcomes of the risky continuation strategy are most uncertain, since this increases the option value of choosing between risky and safe strategies.

Second, even if the firm is successful, the average profitability of the risky continuation strategy cannot be too high. If this strategy is very profitable, then if the firm pursues this strategy whenever it is allowed to continue, it has more than enough cash to allow the institution to recoup its investment. Because the manager gets higher control benefits from the risky strategy than from the safe strategy, she prefers this outcome, and she can implement it by giving the institution pure debt and relatively low monitoring incentives.

Finally, a more subtle point is borne out by our discussion of the impact of decreasing the firm's risk by simultaneously increasing the initial chance of success θ, reducing values in continuation strategies, and increasing value under liquidation. Such a transformation makes it more likely that passive monitoring is preferred. Conversely, it follows that active monitoring is most preferred for firms that are long shots – firms with low liquidation values and low chances of success, but high values if and when success occurs. This "long-shot" aspect is increased by our result that ϕ, the chance that the successful firm does best under a risky continuation strategy, cannot be too high. These results accord very well with the stylized facts of venture capital targets discussed in the introduction.

These results apply to firms with limited financial slack. As we showed in Section 5, if the firm has enough slack to pay the entrepreneur to voluntarily give up her control benefits by choosing the safe strategy over the risky strategy, a third possibility arises: convertible debt with passive monitoring. This structure resembles that used in 'mezzanine finance', which targets better-established firms than those receiving venture capital. Moreover, like mezzanine finance, this structure is only optimal if the firm faces high strategic uncertainty.

Although thus far we have interpreted passive monitoring and debt as bank debt, in reality, other financial institutions often make loans to privately-held firms. Finance companies often extend shorter-term loans on a collateralized basis, and life insurers invest in privately-placed bonds (see Carey et al. (1993)). We are currently pursuing extensions of our basic model that should allow us to capture differences between these types of monitored debt finance; a basic sketch follows.

Suppose that the decision to monitor passively need not be made at date 0, but instead can be deferred to some time between date $\frac{1}{2}$ and date 1. By contrast, we continue to assume that active monitoring must begin at date 0, reflecting the more intensive scrutiny of the firm that this represents. Suppose also that the timing of liquidation matters: there is a chance that, if liquidation is delayed until date 1, the liquidation value is lower than L. Intuitively, the longer one waits, the greater the chance that the firm's investment is more irreversible, at which point it is more attractive to let the firm continue.

In this changed setting, the precise maturity of the debt matters for a passive institution. If the maturity is fairly short-term, the institution knows it will get L if it monitors passively and learns that the firm is 'bad'. If the maturity is slightly longer, there is a chance that the liquidation value of the firm has been reduced. This reduces the institution's desire to monitor at that date, and one can easily create a situation in which the institution only monitors if investment is still reversible. This leads to partial monitoring by the institution - it only monitors with some probability.

Such partial monitoring may be attractive to the entrepreneur, precisely because it increases the odds that the firm will be allowed to continue, increasing her control rents. To be feasible, this requires that the firm has sufficient financial slack; otherwise, insufficient liquidation does not allow the institution to break even on the required investment. If it is feasible, the manager generally prefers partial monitoring on an ex ante basis. The only case in which this is not true is when the firm's liquidation value L is high relative to the control benefits from continuation C_1 and C_2, the chance of success θ is relatively low, and the cost of passive monitoring m_P is not too high.

Contrasting banks and finance companies, Carey et al. (1998) show that finance companies lend to riskier firms than banks do, but there is no difference in information asymmetry between these borrowers (the cost of monitoring is similar). Finance company loans are more likely to be heavily secured (asset-based) than are bank loans, and monitoring by asset-based lenders is intensive and focuses on collateral quality and current cash flow. This accords with the (limited) circumstances under which the manager prefers passive monitoring to be complete rather than partial.

If the institution monitors partially, it must charge a higher interest rate in success (all else equal). This tends to increase the spread between the loan's best- and worst-case outcomes, increasing its liquidity costs, and the increase is more marked the lower the amount of monitoring that is chosen. As discussed in Winton (2002), such riskier loans entail higher liquidity costs, making them more attractive to institutions with lower liquidity needs. Since life insurers have longer liability structures than commercial banks and (unlike banks) do not specialize in liquidity provision, loans where monitoring is especially infrequent (through long maturity) may be more attractive to life insurers.

In conclusion, our relatively simple model is consistent with the circumstances under which venture capital financing structures are preferred to debt-based financing structures and those under which 'mezzanine finance' structures are preferred. Although our current model lumps all such debt finance under the rubric of commercial bank loans, we suggest that some modest extensions will permit us to make finer distinctions that are consistent with the firm's choice between various types of lenders.

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Appendix

Proof. (of Lemma-1): Given that the firm is good: (i) We will first prove (by contradiction) that if the institution does not exercise control, the manager will strictly prefer $a = a_R$, irrespective of whether $i = m$ or $i = h$. Suppose not, i.e., suppose $i = m$ and the manager prefers $a = a_S$ (Note that if manager chooses a_R when $i = m$, then she will choose a_R when $i = h$ as well). It must then be true that $C_1 + C_2 + p_m (X_R - S_R) \leq C_1 + X_S - S_S$. Financing constraint would require that $(1-\theta) S_L + \theta (1-\phi) S_S + \theta\phi p_h S_R \geq I + m_p$. Now, $S_L \leq L$ and $S_R \leq X_R$. Therefore, it must be true that $(1-\theta) L + \theta (1-\phi) S_S + \theta\phi p_h X_R \geq I + m_p$. Combining this with Assumption 2(b), and cancelling common terms, we are left with $S_S > X_S - C_2 \Rightarrow C_2 > X_S - S_S$. Adding $p_m (X_R - S_R)$ to the left hand side, we obtain $C_2 + p_m (X_R - S_R) > X_S - S_S$ (this is true because $X_R - S_R \geq 0$). Contradicts our assumption that manager prefers a_S when $i = m$.

(ii) From (i), it follows that, if the institution exercises control, its payoff S_S, and if it doesn't, its payoff is qS_R (because the manager will choose $a = a_R$). Therefore, the institution will exercise control if and only if $S_S > qS_R$.

(iii) We will now prove that if the institution decides to exercise control, the manager will renegotiate if and only if $S_S \leq qX_R$. If the institution decides to exercise control, the manager can renegotiate with the institution to choose $a = a_R$ by offering $\frac{S_S}{q}$ to the institution when output X_R is realized, i.e., the expected payoff to the institution under renegotiation is still S_S. Limited liability would require that $\frac{S_S}{q} \leq X_R$ (*necessity*).

Sufficiency can be proved by contradiction. Suppose $\frac{S_S}{q} \leq X_R$, and the manager chooses not to renegotiate. It must then be true that $C_2 + p_i \left(X_R - \frac{S_S}{q} \right) < X_S - S_S$. As shown in (i) above, this will contradict Assumption 2(b). ■

Proof. (of Lemma-2): (i) We will first prove (by contradiction) that, for $i \in \{m, h\}$, if the institution does not exercise control, the manager will strictly prefer $a = a_R$. It is enough to prove this for $i = m$, because $p_h > p_m$ (so, if a manager of subtype m prefers a_R, so will a manager of subtype h). Suppose not, i.e., suppose the manager with $i = m$ prefers $a = a_S$. It must then be true that $C_2 + p_m (X_R - S_R) \leq X_S - S_S$. When $i = h$, the institution's payoff is either S_S or $p_h S_R$, both of which are less than $p_h X_R$. Financing constraint would require that $(1-\theta) S_L + \theta (1-\phi) S_S + \theta\phi p_h X_R \geq I + m_A + \theta m_g > I + m_p$. Since $S_L \leq L$, we have that $(1-\theta) L + \theta (1-\phi) S_S + \theta\phi p_h X_R > I + m_p$. Combining this with Assumption 2(b), and cancelling common terms, we are left with $S_S > X_S - C_2 \Rightarrow C_2 > X_S - S_S$. Adding $p_m (X_R - S_R)$ to the left hand side, we obtain $C_2 + p_m (X_R - S_R) > X_S - S_S$ (this is true because $X_R - S_R \geq 0$). Contradicts our assumption that manager of subtype $i = m$ prefers a_S. As noted above, this also implies that the manager of subtype $i = h$ also strictly prefers the risky action.

(ii) The institution could exercise control and get a payoff of S_S, or it he could let the manager choose $a = a_R$, and get a payoff of $p_i S_R$ $(i = m, h)$. Therefore, the institution will exercise control if and only if $S_S > p_i S_R$ $(i = m, h)$.

(iii) When $i = m$, if the institution decides to exercise control, the manager could renegotiate by offering $\frac{S_S}{p_m}$ to the institution when output X_R is realized. Limited liability requires that $\frac{S_S}{p_m} \leq X_R \Rightarrow S_S \leq p_m X_R$ (necessity part). Sufficiency can be proved by contradiction. Suppose $\frac{S_S}{p_m} \leq X_R$, and the manager chooses not to renegotiate$\Rightarrow C_2 + p_m \left(X_R - \frac{S_S}{p_m} \right) < X_S - S_S$. As shown in (i) above, this will contradict Assumption 2(b).

(iv) When $i = h$, if the institution decides to exercise control, the manager could renegotiate by offering $\frac{S_S}{p_h}$ to the institution when output X_R is realized. Limited liability is satisfied because $S_S \leq X_S < p_h X_R$ (Assumption 1(b)).The manager is clearly better off renegotiating (as argued in (iii) above). Therefore, the manager will always renegotiate. ■

Proof. (of Lemma-3): (Necessity:) This can be proved by contradiction. Suppose $(1-\theta) L + \theta q X_R < I + m_P$. The financing constraint (8) requires that $(1-\theta) S_L + \theta \max\{qS_R, S_S\} \geq I + m_P$. Also, the renegotiation constraint (10) and the limited liability constraint $S_R \leq X_R \Rightarrow \max\{qS_R, S_S\} \leq qX_R$. Combining this with the financing constraint (8), we obtain that $I + m_P \leq (1-\theta) S_L + \theta q X_R$. But $S_L \leq L \Rightarrow I + m_P \leq (1-\theta) L + \theta q X_R$, which contradicts our assumption.

(Sufficiency:) Suppose the feasibility condition $(1-\theta) L + \theta q X_R \geq I + m_P$ is satisfied. Consider a contract with $S_L = L$ and $S_R = S_S = \min\{qX_R, X_S\}$. The renegotiation constraint (10) is then automatically satisfied. Also, note that $S_S > qS_R$ and $S_S > p_i S_R$ $(i = m, h)$. Incentive compatibility constraint (9) then simplifies to $\Delta > 0$, which holds by assumption. Lastly, Assumption 3(a) and the above feasibility condition imply that the financing constraint (8) is also satisfied. ■

Proof. (of Lemma-4): (Necessity:) We prove this by contradiction. Suppose $qX_R \geq X_S$. Combining this with the renegotiation constraint (11), we get $X_S < S_S$, which contradicts limited liability. Therefore, it is necessary that $qX_R < X_S$.

(Sufficiency:) We prove this by construction. Suppose the feasibility condition $qX_R < X_S$ is satisfied. Then the contract $S_L = L$, $S_S = S_R = X_S$ satisfies the financing, incentive compatibility, and renegotiation constraints, and hence can implement outcome P2. ■

Proof. (of claim-1): Recall that $p_m < q < p_h$. If $S_S \geq p_h S_R$ or if $S_S \leq p_m S_R$, the incentive compatibility constraint (13) reduces to $m_P - (m_A + \theta m_g) \geq 0$, which contradicts the fact that $m_A > m_P$. Therefore, we require $p_m S_R < S_S < p_h S_R$. ■

Proof. (of Lemma-5): (Necessity:) We prove this by contradiction. Recall that from claim-1, we know that $p_m S_R < S_S < p_h S_R$.

Part-I) *Suppose* $\Delta > \frac{\theta\phi(1-\phi)(p_h-p_m)}{q} p_m X_R$.

The incentive compatibility constraint (15) can be written as $\theta(1-\phi)S_S + \theta\phi p_h S_R - \theta\max\{qS_R, S_S\} \geq \Delta$. Since $\max\{qS_R, S_S\} \geq S_S$, this implies:

$$\theta\phi(p_h S_R - S_S) \geq \Delta \tag{26}$$

Similarly, $\max\{qS_R, S_S\} \geq qS_R$ implies that:

$$\theta(1-\phi)(S_S - p_m S_R) \geq \Delta \tag{27}$$

$S_R \leq X_R$ and $(1-\phi)*(26)+\phi*(27) \Rightarrow \theta\phi(1-\phi)(p_h-p_m)X_R \geq \Delta$. Similarly, $S_S \leq p_m X_R$ (renegotiation constraint (16)) and $(1-\phi)p_m*(26)+\phi p_h*(27) \Rightarrow \theta\phi(1-\phi)(p_h-p_m)\frac{p_m X_R}{q} \geq \Delta$. Since, $\frac{p_m}{q} \leq 1$, the two inequalities we have just obtained, can be combined into the single inequality $\theta\phi(1-\phi)(p_h-p_m)\frac{p_m X_R}{q} \geq \Delta$. *Contradiction.*

This proves the necessity of the first feasibility condition that $\Delta \leq \frac{\theta\phi(1-\phi)(p_h-p_m)}{q} p_m X_R$.

Part-II) *Now suppose that* $(1-\theta)L + \theta q X_R - \frac{\phi p_h}{(1-\phi)p_m}\Delta < I + m_A + \theta m_g$.

(i) Suppose we choose S_R and S_S such that $S_S \geq qS_R \Rightarrow S_R \leq \frac{S_S}{q}$. Plugging this upper bound for S_R into the financing constraint (14), along with the upper bounds for S_L and S_S (from the renegotiation constraint (16)), we have that $(1-\theta)L + \theta(1-\phi)p_m X_R + \theta\phi p_h \frac{p_m X_R}{q} \geq I + m_A + \theta m_g$. Combining this with our assumption above, it must be true that $\theta(1-\phi)p_m X_R + \theta\phi p_h \frac{p_m X_R}{q} > \theta q X_R - \frac{\phi p_h}{(1-\phi)p_m}\Delta$. Rearranging the terms, this inequality simplifies to $\Delta > \frac{\theta\phi(1-\phi)(p_h-p_m)}{q} p_m X_R$. This *Contradicts* the first feasibility condition.

(ii) Instead, suppose we choose S_R and S_S such that $S_S < qS_R$. The incentive compatibility constraint (15) now simplifies to $\theta(1-\phi)(S_S - p_m S_R) \geq \Delta$. ¿From the renegotiation constraint (16), we know that $S_S \leq p_m X_R$. Combining this with the incentive compatibility constraint (15), we obtain that $\theta(1-\phi)(p_m X_R - p_m S_R) \geq \Delta \Rightarrow S_R \leq X_R - \frac{\Delta}{\theta(1-\phi)p_m}$. Plugging this upper bound for S_R into the financing constraint (14), along with the upper bounds for S_L and S_S, we obtain that $(1-\theta)L + \theta(1-\phi)p_m X_R + \theta\phi p_h\left(X_R - \frac{\Delta}{\theta(1-\phi)p_m}\right) \geq I + m_A + \theta m_g$. Combining terms, the inequality simplifies to $(1-\theta)L + \theta q X_R - \frac{\phi p_h}{(1-\phi)p_m}(\Delta) \geq I + m_A + \theta m_g$. *Contradiction.*

This proves the necessity of the second feasibility condition.

(Sufficiency:) We prove this by construction. Suppose both the necessary feasibility conditions are satisfied. Then the contract $S_L = L$, $S_S = p_m X_R$ and $S_R = X_R - \frac{\Delta}{\theta(1-\phi)p_m}$ satisfies (14), (15) and (16). ■

Proof. (of Lemma-6): (Necessity:)

Part-I) *Suppose* $\Delta > \frac{\theta\phi(1-\phi)(p_h-p_m)}{q} \min\{X_S, qX_R\}$.

The proof of this part is very similar to that in case of Lemma-5 above. Just as above, $S_R \leq X_R$ and $(1-\phi)*(26)+\phi*(27) \Rightarrow \theta\phi(1-\phi)(p_h-p_m)X_R \geq \Delta$. Similarly, $S_S \leq X_S$ and $(1-\phi)p_m*(26)+\phi p_h*(27) \Rightarrow \theta\phi(1-\phi)(p_h-p_m)\frac{X_S}{q} \geq \Delta$.

The two inequalities that we have obtained can be combined into the single inequality $\frac{\theta\phi(1-\phi)(p_h-p_m)}{q} \min\{X_S, qX_R\} \geq \Delta$. This *contradicts* our assumption above. Therefore, it must be true that $\Delta \leq \frac{\theta\phi(1-\phi)(p_h-p_m)}{q} \min\{X_S, qX_R\}$.

Part-II) Suppose $X_S > qX_R$ and $(1-\theta)L+\theta(1-\phi)\min\left\{X_S, p_hX_R - \frac{\Delta}{\theta\phi}\right\}+\theta\phi p_hX_R < I+m_A+\theta m_g$.

(i) Suppose we choose S_R and S_S such that $S_S \geq qS_R$. IC simplifies to $\theta\phi(p_hS_R - S_S) \geq \Delta \Rightarrow S_R \geq \frac{1}{p_h}\left(\frac{\Delta}{\theta\phi}+S_S\right)$. Limited liability requires that $S_R \leq X_R$. Combining these two inequalities, we have that $\frac{1}{p_h}\left(\frac{\Delta}{\theta\phi}+S_S\right) \leq X_R \Rightarrow S_S \leq p_hX_R - \frac{\Delta}{\theta\phi}$ (an upper bound for S_S). Substituting the upper bounds for S_S and S_S into the financing constraint (14), it must be true that $(1-\theta)L+\theta(1-\phi)\left(p_hX_R - \frac{\Delta}{\theta\phi}\right)+\theta\phi p_hX_R \geq I+m_A+\theta m_g$. Thus $(1-\theta)L+\theta(1-\phi)X_S+\theta\phi p_hX_R < I+m_A+\theta m_g$. But, this contradicts the financing constraint (14).

(ii) Instead, suppose we choose S_R and S_S such that $S_S < qS_R \Rightarrow S_R > \frac{S_S}{q}$. Combining this with limited liability constraint ($S_R \leq X_R$), we obtain that $S_S < qX_R$. Substituting these upper bounds into the financing constraint (14), we obtain $(1-\theta)L+\theta(1-\phi)qX_R+\theta\phi p_hX_R > I+m_A+\theta m_g$. Combining this with our assumption at the beginning of part-II, and cancelling common terms, it must be true that $qX_R > \min\left\{X_S, p_hX_R - \frac{\Delta}{\theta\phi}\right\}$. If $\min\left\{X_S, p_hX_R - \frac{\Delta}{\theta\phi}\right\} = X_S$, the inequality simplifies to $qX_R > X_S$ *(Contradiction)*. If $\min\left\{X_S, p_hX_R - \frac{\Delta}{\theta\phi}\right\} = p_hX_R - \frac{\Delta}{\theta\phi}$, the inequality simplifies to $qX_R > p_hX_R - \frac{\Delta}{\theta\phi}$. Rearranging terms, we obtain that $\Delta > \theta\phi(1-\phi)(p_h-p_m)X_R$ (*Contradicts* the first feasibility condition).

Therefore, either $X_S \leq qX_R$ or $(1-\theta)L+\theta(1-\phi)\min\left\{X_S, p_hX_R - \frac{\Delta}{\theta\phi}\right\}+\theta\phi p_hX_R \geq I+m_A+\theta m_g$.

(Sufficiency:) We will prove this by construction.

(i) Suppose $\Delta \leq \frac{\theta\phi(1-\phi)(p_h-p_m)}{q} \min\{X_S, qX_R\}$ and $X_S \leq qX_R$. Consider the contract $S_L = L$, $S_S = X_S$ and $S_R = \frac{X_S}{q}$ (therefore, $S_R \leq X_R$). It is easily verified that incentive compatibility constraint (15) holds. We only need to prove that the financing constraint (14) also holds. We will prove this by contradiction. Suppose (14) doesn't hold. Therefore, $(1-\theta)L+\theta(1-\phi)X_S+\theta\phi p_h\frac{X_S}{q} < I+m_A+\theta m_g$. Combining this with Assumption 3, we obtain that $I+m_P+\theta\phi X_S\left(\frac{p_h}{q}-1\right) < I+m_A+\theta m_g$. This simplifies to $\Delta > \frac{\theta\phi(1-\phi)(p_h-p_m)}{q}X_S$ (contradiction). Therefore, the financing constraint (14) holds.

(ii) Suppose $\Delta \leq \frac{\theta\phi(1-\phi)(p_h-p_m)}{q}\min\{X_S, qX_R\}$ and $(1-\theta)L + \theta(1-\phi)\min\left\{X_S, p_hX_R - \frac{\Delta}{\theta\phi}\right\} + \theta\phi p_hX_R \geq I + m_A + \theta m_g$. Consider the contract $S_L = L$, $S_S = \min\left\{X_S, p_hX_R - \frac{\Delta}{\theta\phi}\right\}$ and $S_R = X_R$. It can be easily shown that $qX_R \leq p_hX_R - \frac{\Delta}{\theta\phi}$ (because, otherwise, the first feasibility condition will get violated). We will focus on the case when $qX_R < X_S$ (because we have already addressed the $X_S \leq qX_R$ case in (i) above. Therefore, it follows that, for this contract, $qS_R \leq S_S$. Financing constraint (14) clearly holds from the feasibility condition above.

We will prove, by contradiction, that the incentive compatibility constraint (15) holds too. Suppose (15) doesn't hold. Therefore, $p_hX_R - \min\left\{X_S, p_hX_R - \frac{\Delta}{\theta\phi}\right\} < \frac{\Delta}{\theta\phi}$. If $X_S \leq p_hX_R - \frac{\Delta}{\theta\phi}$, the inequality simplifies to $p_hX_R - X_S < \frac{\Delta}{\theta\phi} \Rightarrow X_S > p_hX_R - \frac{\Delta}{\theta\phi}$ (contradiction). If $p_hX_R - \frac{\Delta}{\theta\phi} < X_S$, the inequality simplifies to $\frac{\Delta}{\theta\phi} < \frac{\Delta}{\theta\phi}$ (contradiction). Therefore, (15) holds. ■

Proof. (of Lemma-7):

(1) Consider the contract $S_L^* = L$, $S_S^* = S_R^* = \frac{I+m_P-(1-\theta)L}{\theta}$. Therefore, $\max\{qS_R^*, S_S^*\} = S_S^*$. The financing constraint (8) is now binding. Let (S_L', S_S', S_R') be another feasible contract that can implement outcome P1. Clearly $S_L' \leq L$(from limited liability)$= S_L^*$; therefore, the financing constraint (8) requires that $\max\{qS_R', S_S'\} \geq \max\{qS_R^*, S_S^*\}$. Assumptions L2 and L3 would then imply that (S_L', S_S', S_R') has a (weakly) higher liquidity cost than (S_L^*, S_S^*, S_R^*). Therefore, (S_L^*, S_S^*, S_R^*) is an optimal contract for implementing outcome P1.

(2) Consider the same contract as above in case of outcome P2. It must be the case that $\frac{I+m_P-(1-\theta)L}{\theta} > qX_R$. If $qX_R \geq \frac{I+m_P-(1-\theta)L}{\theta}$, it can be shown that outcome P1 pareto-dominates outcome P2 (refer to Lemma 8), and hence P2 will not occur in equilibrium. Consider any alternative feasible contract $\left(S_L', S_S', S_R'\right)$. $S_S' \geq qX_R$ (from renegotiation constraint (11)) and $S_R' \leq X_R$ (limited liability) imply that $\max\{qS_R', S_S'\} \geq qX_R$. The financing constraint (8) implies that $\max\{qS_R', S_S'\} \geq \frac{I+m_P-(1-\theta)L}{\theta}$. Combining the two inequalities, we obtain $\max\{qS_R', S_S'\} \geq \frac{I+m_P-(1-\theta)L}{\theta}$, which implies that $\max\{qS_R', S_S'\} \geq \max\{qS_R^*, S_S^*\}$. Also, $S_L' \leq L$ (limited liability). Then, Assumption L2 and L3 imply that contract (S_L^*, S_S^*, S_R^*) has a lower liquidity cost than $\left(S_L', S_S', S_R'\right)$. Therefore, (S_L^*, S_S^*, S_R^*) is an optimal contract as any other feasible contract would have a higher liquidity cost.

(3) We need to show that the optimal contract will satisfy $S_L^* = L$ and $S_S^* < X_S$. We will prove this by contradiction. Suppose $S_L^* < L$. Then, consider the contract (L, S_S^*, S_R^*). (L, S_S^*, S_R^*) is a feasible contract since increasing S_L does not affect either the incentive compatibility constraint (15) or the renegotiation constraint (17), and loosens up the financing constraint (14). Also, from Assumption L1 it follows that (L, S_S^*, S_R^*) has a lower liquidity cost than (S_L^*, S_S^*, S_R^*). This contradicts the fact that (S_L^*, S_S^*, S_R^*) is an optimal contract.

Therefore, we conclude that $S_L^* = L$. Now suppose that $S_S^* = X_S$.

The incentive compatibility constraint (15) requires that $\theta(1-\phi) S_S + \theta\phi p_h S_R - m_A - \theta m_g \geq \theta \max\{qS_R, S_S\} - m_P$. Rearranging terms, and substituting for S_S^*, it must be true that $\theta\phi p_h S_R^* \geq \Delta + \theta \max\{qS_R^*, X_S\} - \theta(1-\phi) X_S$. Since, $\max\{qS_R^*, X_S\} \geq X_S$, it must be true that $\theta\phi p_h S_R^* \geq \Delta + \theta\phi X_S$. Substituting this lower bound for $\theta\phi p_h S_R^*$ into the financing constraint (14), we obtain that $(1-\theta) L + \theta(1-\phi) X_S + \theta\phi p_h S_R^* \geq (1-\theta) L + \theta X_S + \Delta > I + m_A + \theta m_g$, where the last inequality follows from Assumption 3(a). In other words, the financing constraint (14) is not binding.

Case-(i) ($qS_R^* > X_S$): Now consider the alternative contract (S'_L, S'_S, S'_R) where $S'_L = L$, $S'_S = S_S^* - p_m z$ and $S'_R = S_R^* - z$, where $z > 0$ such that $q(S_R^* - z) > X_S - p_m z$ (this will ensure that incentive compatibility constraint (15) still holds), and the financing constraint (14) is satisfied. Therefore, (S'_L, S'_S, S'_R) is a feasible contract for implementing outcome A2. From Assumption L4, it also follows that (S'_L, S'_S, S'_R) has a lower liquidity cost than (S_L^*, S_S^*, S_R^*). *Contradicts* our assumption that (S_L^*, S_S^*, S_R^*) is an optimal contract.

Case-(ii) ($qS_R^* \leq X_S$): Now consider the alternative contract (S'_L, S'_S, S'_R) where $S'_L = L$, $S'_S = S_S^* - p_h z$ and $S'_R = S_R^* - z$, where $z > 0$ such that the financing constraint (14) is satisfied. It can be verified that the incentive compatibility constraint (15) still holds. Therefore, (S'_L, S'_S, S'_R) is a feasible contract for implementing outcome A2. From Assumption L4, it also follows that (S'_L, S'_S, S'_R) has a lower liquidity cost than (S_L^*, S_S^*, S_R^*). *Contradicts* our assumption that (S_L^*, S_S^*, S_R^*) is an optimal contract.

Therefore, it must be true that $S_S^* < X_S$. ■

Proof. (of Proposition 8): Suppose P1 is feasible. We need to show that $U_{M,P1} > U_{M,P2}$ and $U_{M,P1} > U_{M,A2}$.

(1) $U_{M,P1} > U_{M,P2}$: We need to show that $(1-\theta) L + \theta(C_1 + C_2 + qX_R) > (1-\theta) L + \theta(C_1 + X_S) \Rightarrow C_2 + qX_R > X_S$. Consider the following two cases:

(a) $qX_R \geq X_S$. This implies that $C_2 + qX_R > X_S$. Therefore, $U_{M,P1} > U_{M,P2}$.

(b) $qX_R < X_S$. Since P1 is feasible, Lemma (3) implies that $(1-\theta) L + \theta qX_R \geq m_P + I$. From, Assumption 2(b), we have that $(1-\theta) L + \theta(1-\phi)(X_S - C_S) + \theta\phi p_h X_R < m_P + I$. Combining, these two inequalities, we obtain that $\theta(1-\phi)(X_S - C_2) + \theta\phi p_h X_R < \theta qX_R$. Substituting for q and simplifying, we obtain $X_S - C_2 < p_m X_R$. But, $p_m X_R \leq qX_R$. Therefore, $X_S - C_2 < qX_R \Rightarrow C_2 + qX_R > X_S$. Therefore, $U_{M,P1} > U_{M,P2}$.

Therefore, we conclude that if P1 is feasible, the manager strictly prefers it to P2.

(2) $U_{M,P1} > U_{M,A2}$: We need to show that $(1-\theta) L + \theta(C_1 + C_2 + qX_R) - m_P - I > (1-\theta) L + \theta(1-\phi)(C_1 + X_S) + \theta\phi(C_1 + C_2 + p_h X_R) - m_A - \theta m_g - I$. Rearranging terms, this condition simplifies to $\theta(1-\phi)(C_2 + p_m X_R - X_S) > m_P - m_A - \theta m_g$. Notice that $m_P - m_A - \theta m_g < 0$. Therefore, it is sufficient

to show that $C_2 + p_m X_R - X_S \geq 0$. As shown in 1(b) above, this condition follows from Assumption 2(b) and the feasibility condition for P1. Therefore, we have shown that if P1 is feasible, then the manager strictly prefers it to A2. ■

Proof. (of Lemma 10) (Necessity): We will prove this by contradiction. Suppose $(1-\theta)\min\{L, X_S - C_2\} + \theta(1-\phi)(X_S - C_2) + \theta\phi p_h X_R < I + m_P$ and/ or $X_S - C_2 < 0$. If $i = m$, the manager chooses $a = a_S$ if and only if $C_2 + p_m(X_R - S_R) \leq X_S - S_S$. Similarly, if $i = h$, the manager chooses $a = a_R$ if and only if $C_2 + p_h(X_R - S_R) \geq X_S - S_S$. Combining the two inequalities, we require $p_m(X_R - S_R) \leq X_S - S_S - C_2 \leq p_h(X_R - S_R)$. From the first inequality, $S_S \leq X_S - C_2 - p_m(X_R - S_R) \leq X_S - C_2$. This proves the necessity of the condition $X_S - C_2 \geq 0$, because otherwise $S_S < 0$.

It must also be true that $S_L \leq S_S$, because otherwise the institution will liquidate the firm whenever the manager decides to choose a_S (thus signaling that $i = m$). Therefore, we require $S_L \leq \min\{L, X_S - C_2\}$. If this contract is to be feasible, it must satisfy the financing constraint $(1-\theta)S_L + \theta(1-\phi)S_S + \theta\phi p_h S_R \geq I + m_P$. Substituting the upper bounds for S_L, S_S and S_R, we obtain $(1-\theta)\min\{L, X_S - C_2\} + \theta(1-\phi)(X_S - C_2) + \theta\phi p_h X_R \geq I + m_P$. Contradiction. This proves the necessity of the second condition.

(Sufficiency): Suppose $X_S - C_2 \geq 0$ and $(1-\theta)\min\{L, X_S - C_2\} + \theta(1-\phi)(X_S - C_2) + \theta\phi p_h X_R \geq I + m_P$. Consider the contract with $S_L = \min\{L, X_S - C_2\}$, $S_S = X_S - C_2$ and $S_R = X_R$. It is easily verified that, in equilibrium, this contract separates subtypes $i = m$ and $i = h$. Also, since $m_P < m_A + \theta m_g$, the institution will monitor passively, thus satisfying the incentive compatibility constraint as well. This proves the sufficiency of the above condition. ■

Proof. (of Proposition 11) Given that outcome P3 is feasible. We have already shown that $U_{M,P3} > U_{M,A2}$. Also, $U_{M,P3} - U_{M,P2} = \theta\phi(p_h X_R + C_2 - X_S) > 0$ (by Assumption 1). Therefore, if P3 is feasible, the manager strictly prefers it to outcomes A2 and P2. That only leaves outcomes P1 and P3 to choose from.

(i) If $p_m X_R + C_2 - X_S < 0$, $U_{M,P3} > U_{M,P1}$. Therefore, P3 is now the most preferred outcome. Since it is also feasible, the manager chooses outcome P3.

(ii) If $p_m X_R + C_2 - X_S \geq 0$, $X_S - C_2 < p_m X_R$. Substituting this upper bound into Condition 2 of Lemma 10, and noting that $\min\{L, X_S - C_2\} \leq L$, we obtain $(1-\theta)L + \theta q X_R > I + m_P$. Therefore, outcome P1 is feasible. Also, $p_m X_R + C_2 - X_S \geq 0 \Rightarrow U_{M,P1} \geq U_{M,P3}$ and $U_{M,P1} > U_{M,P2}$. Therefore, P1 is also the most preferred outcome. Hence, the manager will choose outcome P1. ■

Proof. (of Assumptions L1-L4): Suppose that the institution's claims have expected future values A_1, A_2 or A_3 with probabilities p_1, p_2 or p_3, where $A_1 \leq A_2 \leq A_3$ and $p_1 + p_2 + p_3 = 1$.

Suppose that the institution has liquidity need such that value of \$1 now is \$(1+β); value of \$1 in the future is \$1. The institution issues equity in its claim. However, although it knows the value of the claim, outside institutions only know possible values $\{A_i\}$ and probabilities $\{p_i\}$.

Let ψ_i denote the share of equity on its claims that the institution issues when it knows the value of its claims to be A_i; let $p(\psi_i)$ denote the price at which the claim is issued. As shown in Winton (2002), the unique equilibrium that satisfies intuitive criterion etc. is as follows:

(1) If A_1, institution issues $\psi_1 = 1$ (institution sells entire claim), $p(\psi_1) = A_1$.
(2) If A_2, institution issues $\psi_2 = \frac{\beta A_1}{(1+\beta)A_2 - A_1}$, $p(\psi_2) = A_2$. This leaves type A_1 indifferent options (1) and (2), i.e., between issuing $\psi_1 = 1$ and getting value $(1+\beta)A_1$, or issuing ψ_2 and getting value $\psi_2(1+\beta)A_2 + (1-\psi_2)A_1$ (sell ψ_2 at A_2 now, retain $1-\psi_2$ worth A_1 for later).
(3) If A_3, institution issues $\psi_3 = \frac{\beta A_2}{(1+\beta)A_3 - A_2}\psi_2$, $p(\psi_3) = A_3$. This leaves type A_2 indifferent between options (2) and (3).

The institution gets expected value of $p_1(1+\beta)A_1 + p_2(1+\beta\psi_2)A_2 + p_3(1+\beta\psi_3)A_3 \equiv Y$. Since the institution gets utility $\$(1+\beta)$ for every \$1 it keeps on hand, its opportunity cost is $(1+\beta)^{-1}$ and so it values these expected cash flows at $(1+\beta)^{-1}Y$. It is willing to pay the entrepreneur $(1+\beta)^{-1}Y$.

By contrast, the entrepreneur makes expected payments $p_1A_1 + p_2A_2 + p_3A_3$. Thus, the net cost to the entrepreneur is:

$$\Lambda \equiv p_1A_1 + p_2A_2 + p_3A_3 - (1+\beta)^{-1}Y$$

Substituting for Y and simplifying, we obtain:

$$\Lambda = \frac{\beta}{1+\beta}\{(1-\psi_2)p_2A_2 + (1-\psi_3)p_3A_3\} \tag{28}$$

Application to the model: In our model, $p_1 = 1-\theta$, $p_2 = \theta(1-\phi)$, $p_3 = \theta\phi$ and $A_1 = S_L$.

Case-I (Passive monitoring): Now, $A_2 = A_3 = \max\{S_S, qS_R\}$. Therefore $\psi_2 = \psi_3 = \frac{\beta S_L}{(1+\beta)\max\{S_S,qS_R\} - S_L}$. Substituting in equation (28), we obtain:

$$\Lambda_{passive} = \beta\left(\frac{\max\{S_S, qS_R\} - S_L}{(1+\beta)\max\{S_S, qS_R\} - S_L}\right)\theta\max\{S_S, qS_R\} \tag{29}$$

Assumption L1 follows directly from equation (29).

Assumptions L2 and L3 follow by differentiation.

Case-II (Active Monitoring): Now $A_2 = S_S$ and $A_3 = p_hS_R$. Substituting in the expressions above, we obtain:

$$\psi_2 = \frac{\beta S_L}{(1+\beta) S_S - S_L}, \; \psi_3 = \frac{\beta S_S}{(1+\beta) p_h S_R - S_S}\psi_2$$

$$\Lambda_{active} = \beta\theta \left\{ \frac{(S_S - S_L)(1-\phi) S_S}{(1+\beta) S_S - S_L} + \left(\frac{p_h S_R - S_S \frac{1+\beta\psi_2}{1+\beta}}{(1+\beta) p_h S_R - S_S} \right) \phi p_h S_R \right\} \quad (30)$$

Again, L1 follows directly from equation (30). Since ψ_2 is increasing in S_L, differentiation shows that Λ_{active} is decreasing in S_L and increasing in S_R (Assumption L2).

To see A ssumption L4, first note that $\Lambda_{active} = \frac{\beta}{1+\beta}\{(1-\phi)\theta W_1 + \phi\theta W_2\}$, where $W_1 \equiv (1-\psi_2) S_S$ and $W_2 \equiv (1-\psi_3) p_h S_R$.

$$\frac{\partial W_1}{\partial S_S} = 1 - \psi_2 - S_S \frac{\partial \psi_2}{\partial S_S} = 1 + \psi_2 \frac{S_L}{(1+\beta) S_S - S_L} = 1 + \frac{\psi_2^2}{\beta} > 0$$

$$\begin{aligned}
\frac{\partial W_2}{\partial S_S} &= -p_h S_R \frac{\partial \psi_3}{\partial S_S} \\
&= -p_h S_R \left\{ \frac{\beta S_S}{(1+\beta) p_h S_R - S_S} \frac{\partial \psi_2}{\partial S_S} + \frac{\beta(1+\beta) p_h S_R}{[(1+\beta) p_h S_R - S_S]^2}\psi_2 \right\} \\
&= p_h S_R \frac{\beta(1+\beta)\psi_2}{(1+\beta) p_h S_R - S_S} \left\{ \frac{S_S}{(1+\beta) S_S - S_L} - \frac{p_h S_R}{(1+\beta) p_h S_R - S_S} \right\} \\
&= p_h S_R \frac{\beta(1+\beta)\psi_2}{[(1+\beta) p_h S_R - S_S]^2 [(1+\beta) S_S - S_L]} \left(p_h S_R S_L - S_S^2 \right)
\end{aligned}$$

The sign of $\frac{\partial W_2}{\partial S_S}$ depends on whether $p_h S_R S_L \geq S_S^2$ or $p_h S_R S_L \leq S_S^2$.

$$\frac{\partial \Lambda_{active}}{\partial (p_h S_R)} = \frac{\beta}{1+\beta}\theta\phi \frac{\partial W_2}{\partial (p_h S_R)}$$

$$\frac{\partial W_2}{\partial (p_h S_R)} = 1 - \psi_3 - p_h S_R \frac{\partial \psi_3}{\partial (p_h S_R)}, \; \frac{\partial \psi_3}{\partial (p_h S_R)} = -\frac{\beta(1+\beta) S_S}{[(1+\beta) p_h S_R - S_S]^2}\psi_2$$

Therefore,

$$\begin{aligned}
\frac{\partial W_2}{\partial (p_h S_R)} &= 1 - \psi_3 + p_h S_R \frac{\beta(1+\beta) S_S}{[(1+\beta) p_h S_R - S_S]^2}\psi_2 \\
&= 1 + \frac{S_S}{(1+\beta) p_h S_R - S_S}\psi_3 = 1 + \frac{\beta S_S^2}{[(1+\beta) p_h S_R - S_S]^2}\psi_2 > 0
\end{aligned}$$

Cases:

(1) If $p_h S_R S_L \geq S_S^2$, then $\frac{\partial \Lambda_{active}}{\partial S_S} > 0$. If we define z as in Assumption L4, then $\frac{\partial \Lambda}{\partial z} = \frac{\partial \Lambda}{\partial S_S}\frac{dS_S}{dz} + \frac{\partial \Lambda}{\partial (p_h S_R)}\frac{d(p_h S_R)}{dz} > 0$.

(2) If $p_h S_R S_L < S_S^2$, then $\frac{\partial \Lambda_{active}}{\partial S_S}$ ambiguous in sign.

$$\frac{\partial \Lambda}{\partial z} = \frac{\beta}{1+\beta}\left\{(1-\phi)\,\theta\frac{\partial W_1}{\partial S_S}\frac{dS_S}{dz} + \phi\theta\frac{\partial W_2}{\partial S_S}\frac{dS_S}{dz} + \phi\theta\frac{\partial W_2}{\partial (p_h S_R)}\frac{d(p_h S_R)}{dz}\right\}$$

Then $\frac{\partial W_2}{\partial (p_h S_R)} > 0 \Rightarrow$ if $\frac{d(p_h S_R)}{dz} \geq \frac{dS_S}{dz}$, then:

$$\frac{\partial \Lambda}{\partial z} \geq \frac{\beta}{1+\beta}\left\{(1-\phi)\,\theta\frac{\partial W_1}{\partial S_S} + \phi\theta\left(\frac{\partial W_2}{\partial S_S} + \frac{\partial W_2}{\partial (p_h S_R)}\right)\right\}\frac{dS_S}{dz}$$

Since $\frac{\partial W_1}{\partial S_S} > 0$, all we need to show is that $\frac{\partial W_2}{\partial S_S} + \frac{\partial W_2}{\partial (p_h S_R)} \geq 0$; then $\frac{\partial \Lambda}{\partial z} > 0$.

$$\begin{aligned}\frac{\partial W_2}{\partial S_S} + \frac{\partial W_2}{\partial (p_h S_R)} &= p_h S_R \frac{\beta(1+\beta)}{[(1+\beta)p_h S_R - S_S]^2[(1+\beta)S_S - S_L]}\left[p_h S_R S_L - S_S^2\right]\psi_2 \\ &+ 1 + \frac{\beta S_S^2}{[(1+\beta)p_h S_R - S_S]^2}\psi_2 \\ &= [(1+\beta)p_h S_R - S_S]^{-2}[(1+\beta)S_S - S_L]^{-1} K\end{aligned}$$

where,

$$\begin{aligned}K = &\left\{\beta(1+\beta)\left[(p_h S_R)^2 S_L - p_h S_R S_S^2\right] + \beta(1+\beta)S_S^3 - \beta S_L S_S^2\right\}\psi_2 \\ &+ [(1+\beta)p_h S_R - S_S]^2[(1+\beta)S_S - S_L]\end{aligned}$$

If the coefficient of ψ_2 is positive, then $K > 0$ and we are done. Otherwise, since $\psi_2 < 1$,

$$\begin{aligned}K > &\left\{\beta(1+\beta)\left[(p_h S_R)^2 S_L - p_h S_R S_S^2\right] + \beta(1+\beta)S_S^3 - \beta S_L S_S^2\right\} \\ &+ [(1+\beta)p_h S_R - S_S]^2[(1+\beta)S_S - S_L] \\ &= \beta(1+\beta)\left[(p_h S_R)^2 S_L - p_h S_R S_S^2 + S_S^3\right] - \beta S_L S_S^2 + (1+\beta)^3 (p_h S_R)^2 S_S \\ &- 2(1+\beta)^2 p_h S_R S_S^2 + (1+\beta)S_S^3 - (1+\beta)^2 (p_h S_R)^2 S_L + 2(1+\beta)p_h S_R S_S S_L - S_S^2 S_L \\ &= \left[\beta(1+\beta) - (1+\beta)^2\right](p_h S_R)^2 S_L - \left[\beta(1+\beta) + 2(1+\beta)^2\right](p_h S_R) S_S^2 \\ &+ [\beta(1+\beta) + (1+\beta)] S_S^3 + (1+\beta)^3 (p_h S_R)^2 S_S + 2(1+\beta)p_h S_R S_S S_L - (1+\beta)S_S^2 S_L\end{aligned}$$

Factoring out $(1+\beta)$, we have

$$\begin{aligned}H \equiv &-(p_h S_R)^2 S_L - [\beta + 2(1+\beta)](p_h S_R) S_S^2 + (1+\beta)S_S^3 \\ &+ (1+\beta)^2 (p_h S_R)^2 S_S + 2p_h S_R S_S S_L - S_S^2 S_L\end{aligned}$$

Therefore,

$$\frac{\partial H}{\partial \beta} = -3\left(p_h S_R\right) S_S^2 + S_S^3 + 2\left(1+\beta\right)\left(p_h S_R\right)^2 S_S$$

Two cases:

(1) $\frac{\partial H}{\partial \beta} \geq 0$ at $\beta = 0$. Since $\frac{\partial^2 H}{\partial \beta^2} = 2\left(p_h S_R\right)^2 S_S > 0$, it follows that $\frac{\partial H}{\partial \beta} > 0$ for all $\beta > 0$, and so H is smallest at $\beta = 0$.

At $\beta = 0$,

$$\begin{aligned} H &= -\left(p_h S_R\right)^2 S_L - 2\left(p_h S_R\right) S_S^2 + S_S^3 + \left(p_h S_R\right)^2 S_S + 2p_h S_R S_S S_L - S_S^2 S_L \\ &= \left[\left(p_h S_R\right)^2 - 2p_h S_R S_S + S_S^2\right]\left[S_S - S_L\right] \\ &= \left[p_h S_R - S_S\right]^2\left[S_S - S_L\right] \\ &\geq 0, \text{ since } p_h S_R > S_S \geq S_L \end{aligned}$$

Therefore, $H \geq 0$ for all $\beta \geq 0$, and so on. Therefore, $K = (1+\beta) H > 0 \Rightarrow \frac{\partial W_2}{\partial S_S} + \frac{\partial W_2}{\partial (p_h S_R)} > 0$.

(2) $\frac{\partial H}{\partial \beta} < 0$ at $\beta = 0 \Rightarrow -3\left(p_h S_R\right) S_S^2 + S_S^3 + 2\left(p_h S_R\right)^2 S_S < 0$. Factoring out S_S, we have $-3\left(p_h S_R\right) S_S + S_S^2 + 2\left(p_h S_R\right)^2 < 0$. But LHS$=\left(S_S - 2p_h S_R\right)\left(S_S - p_h S_R\right)$. Since $S_S \leq p_h S_R \leq p_h S_R$, LHS$\geq 0$. Contradiction.

So, $\frac{\partial H}{\partial \beta} \geq 0$ at $\beta = 0$, and proof holds. ■