IPP-QM-1: Basic quantum formalism

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MT24

The course

1. Basic quantum formalism
2. Density operators and entanglement
3. Decoherence
4. The measurement problem
5. Dynamical collapse theories
6. Bohmian mechanics
7. Everettian structure
8. Everettian probability
9. EPR and Bell's theorem
10. The Bell-CHSH inequalities and possible responses
11. Contextuality
12. The PBR theorem
13. Quantum logic
14. Pragmatism and QBism
15. Relational quantum mechanics
16. Wavefunction realism

Books etc.

- David Albert, *Quantum Mechanics and Experience*, Boston: Harvard University Press, 1994.
- Tim Maudlin, *Philosophy of Physics Volume II: Quantum Mechanics*, Princeton: Princeton University Press, 2019.
- Tim Maudlin, *Quantum Non-Locality and Relativity*, third edition, Oxford: Wiley-Blackwell, 2011.
- David Wallace, *The Emergent Multiverse*, Oxford: Oxford University Press, 2012.

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- David Wallace, *The Emergent Multiverse*, Oxford: Oxford University Press, 2012.

- John S. Bell, “Against Measurement”, in *Speakable and Unspeakable in Quantum Mechanics*, second edition, Cambridge: Cambridge University Press, 2004.
- David Wallace, “Philosophy of Quantum Mechanics”, in D. Rickles (ed.), *The Ashgate Companion to Contemporary Philosophy of Physics*, London: Routledge, 2008.

Formalism

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Highly recommended are:

- David Wallace, "The Formalism of Quantum Mechanics", 2005.
- David Wallace, "The Formalism of Quantum Mechanics II – Density Operators and Entanglement", 2005.
- Guido Bacciagaluppi, "Density Operators in Quantum Mechanics", 1998.
- Frederic P. Schuller, "Lectures on Quantum Theory", 2015.

(All on Canvas.)

Today

Quantum states

Hilbert spaces

More about Hilbert spaces

Linear operators

The physical significance of operators

Wavefunctions and infinite-dimensional Hilbert spaces

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  1. if the system is a one-dimensional spinless particle then *position* is maximally specific;
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- Write the states of definite m as $|m_1\rangle, |m_2\rangle, \ldots, |m_n\rangle$.
- $|m_i\rangle$ is a state such that, if we measure M when the system is in that state (assuming that we have an ideal measuring device), then we definitely get the result m_i.

Superpositions

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- Because probabilities sum to one, this implies that quantum states must be *normalised*,

$$\sum_{i=1}^{N} |\alpha_i|^2 = 1.$$

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- Elements of a Hilbert space needn't be normalised (only the physical ones are), and are called *vectors*.

Hilbert space inner product

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- If $|\psi\rangle = \sum_i \alpha_i |m_i\rangle$ and $|\phi\rangle = \sum_i \beta_i |m_i\rangle$ are vectors, then their inner product is defined as

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- Properties of the inner product:
  1. *Linear:* $\langle\phi| (\alpha |\psi_1\rangle + \beta |\psi_2\rangle) = \alpha \langle\phi|\psi_1\rangle + \beta \langle\phi|\psi_2\rangle$
  2. Satisfies: $\langle\phi|\psi\rangle = (\langle\psi|\phi\rangle)^*$.
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  3. *Positive:* If $|\psi\rangle \neq 0$ then $\langle\psi|\psi\rangle \in \mathbb{R}^+$.
- We can write the normalisation condition on states as: $|\psi\rangle$ is normalised iff $\langle\psi|\psi\rangle = 1$.

Mathematical approaches to Hilbert space

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- See e.g. the Schuller notes for more details.

Bases for Hilbert space

- A *basis* for a Hilbert space $\mathcal{H}$ is a set of vectors in $\mathcal{H}$ s.t.:
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- It is typical to insist that a basis be *orthonormal*, which means that it also satisfies:
  1. If $|\psi\rangle$ and $|\phi\rangle$ are (distinct) basis vectors, then $\langle\psi|\phi\rangle = 0$.
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- **Exercise:** Prove that any two bases for $\mathcal{H}$ have the same number of vectors.
- Let $|\psi\rangle \in \mathcal{H}$ and let $\{|i\rangle\}$ be any orthonormal basis. By definition, there will be complex numbers $\{\alpha_i\}$ s.t. $|\psi\rangle = \sum_i \alpha_i |i\rangle$. Because the basis is orthonormal, $\langle i|\psi\rangle = \alpha_i$; that is, we have the useful expansion

$$|\psi\rangle = \sum_i |i\rangle \langle i|\psi\rangle$$

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- If $|\psi\rangle$ is an arbitrary state, what's the probability of getting result k_i when we measure K?
- It's a *postulate* of 'standard', textbook quantum mechanics that the probability is $|\langle k_i|\psi\rangle|^2$.

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- States of that system are represented by normalised vectors in that Hilbert space.
- A maximally specific measurement is represented by a basis of the Hilbert space.
- If K is a maximally specific measurement and $|k_i\rangle$ has definite value k_i of K, then the probability is of getting k_i when we measure K on the state $|\psi\rangle$ is $|\langle k_i|\psi\rangle|^2$.

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- It's a smaller Hilbert space, entirely contained within the original one.
- Such spaces are called *subspaces*.

Global phase transformations

- The probability rule defined previously gives the same probabilities for all measurements if we carry out a *global phase transformation* of all states,

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- As such, it's standard to regard the physical system as unchanged by such transformations.
- So strictly, it isn't states in Hilbert space which represent physical systems—it's equivalence classes of states related by global phase, sometimes called *rays*.
- Practically, however, it's easier to just carry on working with vectors in Hilbert space.

On the reality of the global phase

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- For defences of the mainstream, see (Wallace 2022) or (Gao 2024); I won't question it further in these lectures.

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Linear operators

- A *linear operator* $\hat{A}$ is a map from Hilbert space to itself which has the linearity property:

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- Simple example: $|\phi\rangle \langle\chi|$, where $|\psi\rangle$ and $|\chi\rangle$ are arbitrary (normalised) vectors; this is defined as follows: $(|\phi\rangle \langle\chi|) |\psi\rangle = |\phi\rangle \times \langle\chi|\psi\rangle$.

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- E.g., acting on $|\psi\rangle = \alpha|\phi\rangle + \beta|\chi\rangle$ with $|\phi\rangle\langle\phi|$ yields $\alpha|\phi\rangle$ (where $|\phi\rangle$ and $|\psi\rangle$ are part of an orthonormal basis for the Hilbert space).

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- E.g., acting on $|\psi\rangle = \alpha|\phi\rangle + \beta|\chi\rangle$ with $|\phi\rangle\langle\phi|$ yields $\alpha|\phi\rangle$ (where $|\phi\rangle$ and $|\psi\rangle$ are part of an orthonormal basis for the Hilbert space).
- More general (i.e., higher-dimensional) projectors don't just project out one particular component of a vector: they project out all components in some particular subspace.

Resolutions of the identity

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- The limiting case of a projector is one which projects onto the entire Hilbert space—and so leaves the vector alone!
- The *identity operator* can be written in the same way as any other projector:

$$\hat{1} = \sum_i |n_i\rangle \langle n_i| \,,$$

where $\{|n_i\rangle\}$ is any basis. An expression like this for $\hat{1}$ is called a *resolution of the identity*.

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  - $\hat{A} = \hat{1}\hat{A}\hat{1} = \left(\sum_i |n_i\rangle \langle n_i| \right) \hat{A} \left(\sum_j |n_j\rangle \langle n_j| \right) = \sum_{i,j} |n_i\rangle \langle n_j| \langle n_i| \hat{A} |n_j\rangle$.

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- The later of these is called a *matrix representation* of $\hat{A}$, and we can write $A_{ij} := \langle n_i|\,\hat{A}\,|n_j\rangle$.
- This gives us the components A_{ij} of the matrix associated with the operator $\hat{A}$ in the $\{|n_i\rangle\}$ basis.

Eigenvalues and eigenvectors

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- An *eigensubspace* of an operator is the subspace defined by all those eigenvectors with some particular eigenvalue.

Eigenvalues and eigenvectors

- Recall that an *eigenvector* **v** of a matrix **M** is a column vector satisfying $\mathbf{Mv} = \lambda\mathbf{v}$, and the number λ is called the *eigenvalue* associated with **v**.
- This translates to Hilbert space in the obvious way: $|\psi\rangle$ is an eigenvector of an operator $\hat{M}$ iff for some $\lambda \in \mathbb{C}$, $\hat{M}|\psi\rangle = \lambda|\psi\rangle$.
- If $\hat{M}$ has more than one eigenvector with the same eigenvalue, it is called *degenerate*. The set of all eigenvalues of $\hat{M}$ is called the *spectrum* of $\hat{M}$.
- An *eigensubspace* of an operator is the subspace defined by all those eigenvectors with some particular eigenvalue.
- (So, non-degenerate operators have only one-dimensional eigensubspaces.)

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- (For further details on the spectral theorem, see e.g. the Schuller notes.)
- If $\hat{C}$ is non-degenerate, then there's a *unique* such basis.

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  (**Exercise:** Prove this!)
- If $\hat{A}$ and $\hat{B}$ are normal operators which commute, then there exists a normal operator $\hat{C}$ and functions f, g such that $\hat{A} = f(\hat{C})$ and $\hat{B} = g(\hat{C})$.

A zoo of operators

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- All self-adjoint operators are normal, but not *vice versa*.
- (E.g., *unitary* operators—for which $\hat{A}\hat{A}^\dagger = \hat{1}$—are normal but not self-adjoint.)

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More about Hilbert spaces

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- This operator has the following two properties:
  1. $\hat{M}$ is *self-adjoint*: $\hat{M}^\dagger = \hat{M}$. (**Exercise:** Show that an operator is self-adjoint iff its eigenvalues are all real.)
  2. $\hat{M}$ is non-degenerate. (This follows from the definition of a maximally specific measurement.)
- Conversely, the spectral theorem means that every non-degenerate self-adjoint operator determines a maximally specific measurement.

The orthodoxy on self-adjoint operators

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- However, for a recent and philosophically rich article arguing that other normal operators could serve just as well (despite having complex eigenvalues!), see (Roberts 2017).

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  - For example, a photon detector destroys a photon completely when measuring its location!

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- So if we measure $\hat{C}$ then we automatically measure $\hat{A}$ and $\hat{B}$, just by applying the appropriate functions to the outcome.
- We conclude that *commuting operators can be measured simultaneously*.
- Conversely, if two operators don't commute then it's at best unclear what it would mean to measure them simultaneously.

Unitary operators

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- Similarly, *unitary operators* can be defined either as (i) operators satisfying $\hat{A}^\dagger \hat{A} = \hat{1}$, or (ii) operators whose eigenvalues all have modulus 1 (so of the form $e^{i\theta}$).
- The significance of unitary operators is that they map physical states to physical states, for if a map is to do this then it must preserve normalisation: if $|\psi\rangle$ is a physical state, then $|\psi'\rangle = \hat{U}|\psi\rangle$ is a physical state only if

  $$1 = \langle\psi'|\psi'\rangle = \langle\psi|\hat{U}^\dagger \hat{U}|\psi\rangle\,,$$

  which is true for all states only if $\hat{U}^\dagger \hat{U} = \hat{1}$.

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1. *Time evolution:* It's a postulate of textbook quantum mechanics that states evolve linearly (in accordance with the Schrödinger equation, to be discussed later): so if $|\psi\rangle$ evolves into $|\psi'\rangle$ and $|\phi\rangle$ evolves into $|\phi'\rangle$, then $\alpha|\psi\rangle + \beta|\phi\rangle$ evolves into $\alpha|\psi'\rangle + \beta|\phi'\rangle$. This means that time evolution is described by unitary operators: for any times t, t_0, there must be a unitary operator $\hat{U}(t, t_0)$ which evolves states at time t_0 into states at time t:

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2. *Symmetry transformations:* Symmetries of quantum systems, such as rotations or translations, are described by (anti-)unitary operators. (See e.g. Weinberg 1995)

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which is the time-dependent Schrödinger equation.

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- (For more, see the seminal (von Neumann 1955).)

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- The *wavefunction* is defined as $\psi(x) := \langle x|\psi\rangle$.

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- So, the wavefunction can't straightforwardly be thought of as living on space at all.
- Rather, the wavefunction is a complex-valued field on *configuration space*, which is the space of all possible coordinates of all particles, with $3N$ dimensions for an N-particle system.

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    2. Hilbert spaces
    3. Operators and their connections with measurements
    4. Wavefunctions

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  3. Operators and their connections with measurements
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- Tomorrow, we'll look at *density operators* and *entanglement*.

References

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