IPP-QM-11: Contextuality

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MT24

The course

1. Basic quantum formalism
2. Density operators and entanglement
3. Decoherence
4. The measurement problem
5. Dynamical collapse theories
6. Bohmian mechanics
7. Everettian structure
8. Everettian probability
9. EPR and Bell's theorem
10. The Bell-CHSH inequalities and possible responses
11. Contextuality
12. The PBR theorem
13. Quantum logic
14. Pragmatism and QBism
15. Relational quantum mechanics
16. Wavefunction realism

Today

Contextuality

The Bell-Kochen-Specker theorem introduced

The Clifton-Stairs state-dependent proof

Proof of the Bell-Kochen-Specker theorem

The Klyachko-Can-Binicioglu-Shumovsky inequality

Coda: Gleason's theorem

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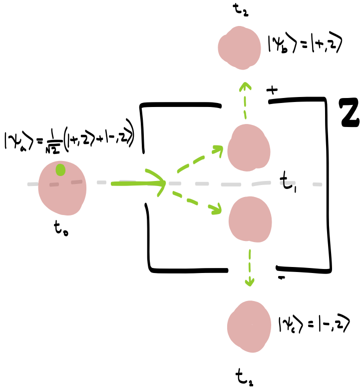
- Different experimental arrangements lead to *different* physics.
- So measurements are not *simply* revealing properties of the system being measured.

Contextuality reminder: Bohmian mechanics



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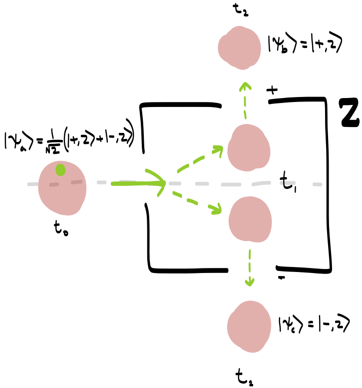

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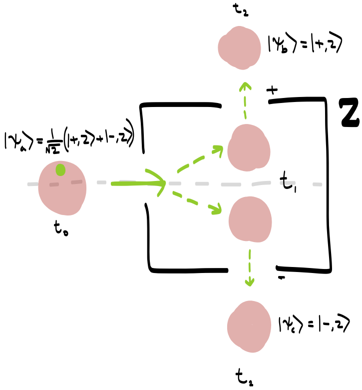

- The Bohm trajectories are deterministic. The outcomes of a measurement are fixed in advance.
- But the outcomes of e.g. Stern-Gerlach experiments also depend upon the context of *how* the measurement is performed.

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- The conjunction of 'outcome determinism' with 'measurement non-contextuality' used to be called 'non-contextuality' or 'non-contextual value definiteness.'
- Now it is sometimes called **traditional non-contextuality** or 'Kochen-Specker non-contextuality'.

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- They are provably not possible in any Hilbert space of larger dimension:
  - The Clifton-Stairs state-dependent proof.
  - The Bell-Kochen-Specker (BKS) state-independent proof.
  - The Klyachko-Can-Binicioglu-Shumovsky (KCBS) inequality.

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The Bell-Kochen-Specker theorem

- The BKS theorem tells us that any hidden variable theory which assigns values to all properties represented by projectors must be *contextual*.
- Specifically, whether or not a system is found, on measurement, to possess a given property must depend upon which *other* properties are measured simultaneously.
- As Wallace (2007, p. 51) writes, “Contextuality seems well-nigh inconsistent with the idea that systems determinately do or do not possess given properties and that measurements simply determine whether or not they do.”

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The Clifton-Stairs state-dependent proof

Before getting to the full BKS contextuality theorem, we'll first look at a simpler, 'state-dependent' proof due to Clifton (1993) and Stairs (1992).

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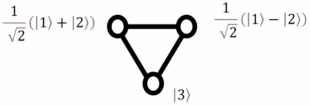

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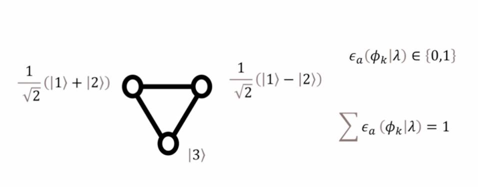

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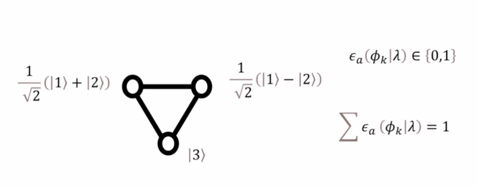

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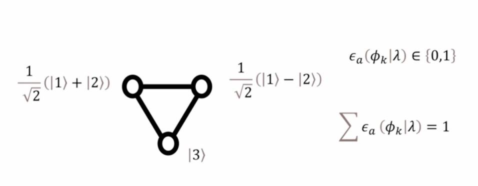
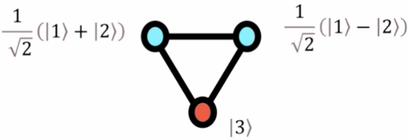


- Any triangle is a measurement scenario.
- We require that the measurement outcomes are 0 or 1, and that they sum to 1.
- So for any triple of ontic states, one of those states is going to be 1 and the other two 0.

The Clifton-Stairs proof



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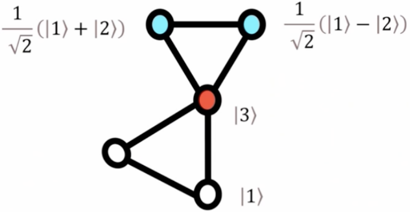

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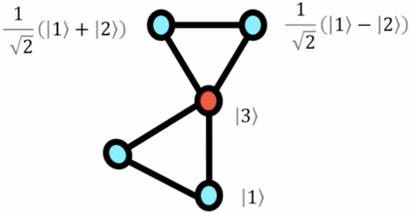
The Clifton-Stairs proof

So, naturally, the other circles are going to have to be blue:



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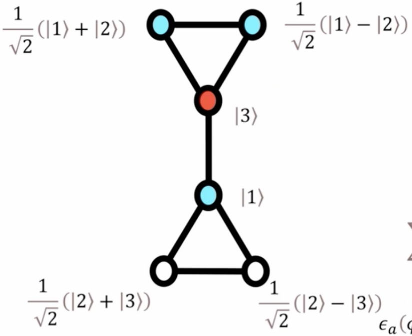
Now consider a scenario like this:



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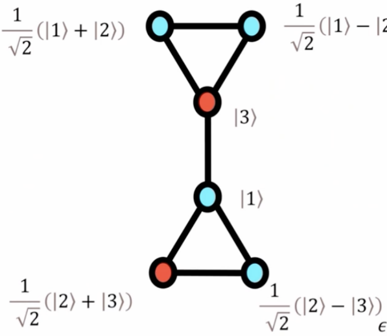
E.g.:



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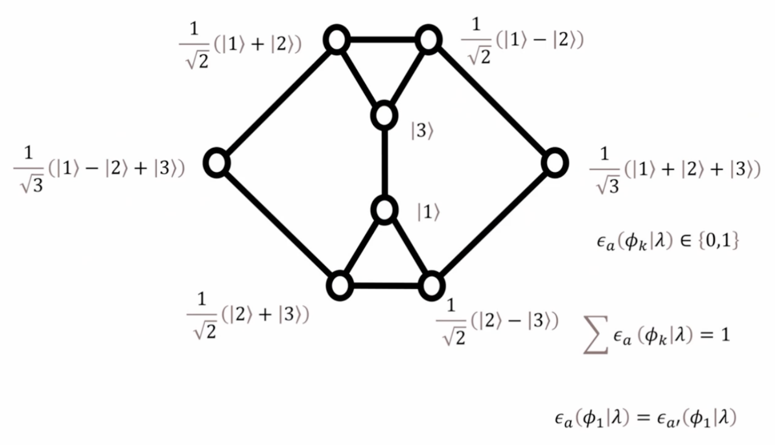

Notice that the far-left and far-right states are *not orthogonal to each other*.

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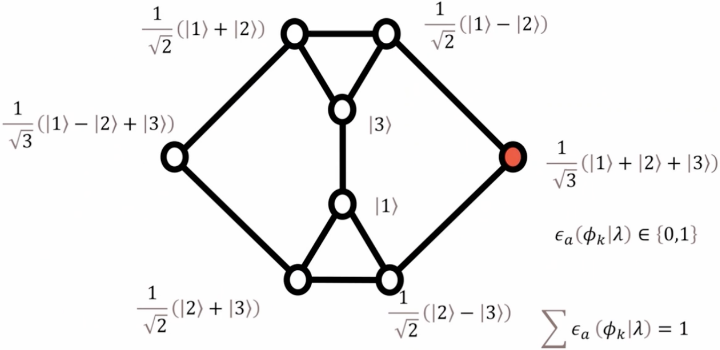

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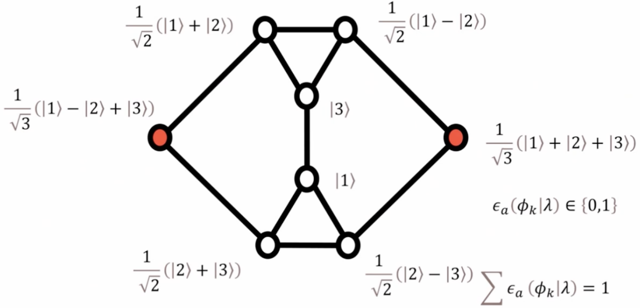
- In that case, we have:



$$\int \epsilon_a(\Psi|\lambda)\mu_\Psi(\lambda) = 1 \qquad |\Psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) \qquad \epsilon_a(\phi_1|\lambda) = \epsilon_{a'}(\phi_1|\lambda)$$

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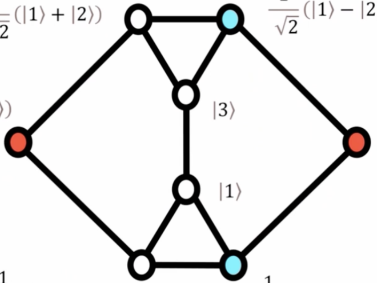

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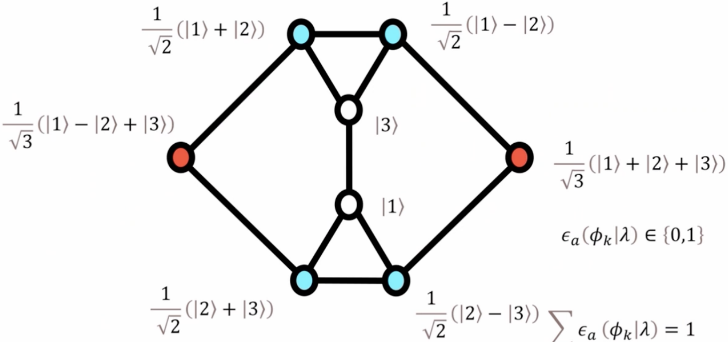

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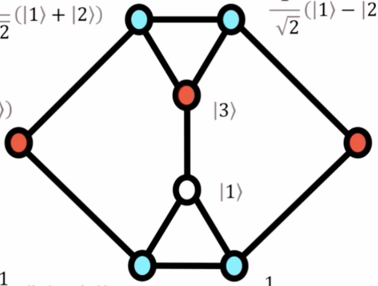

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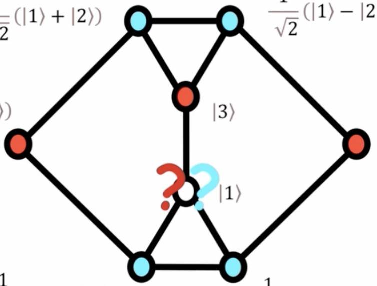

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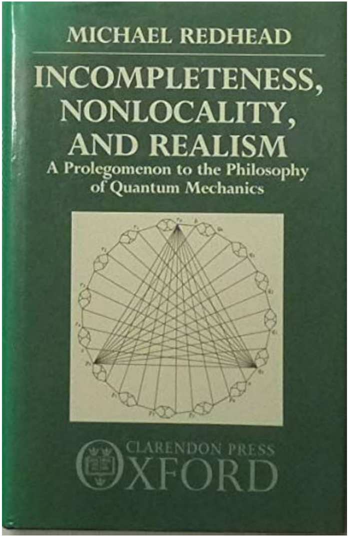

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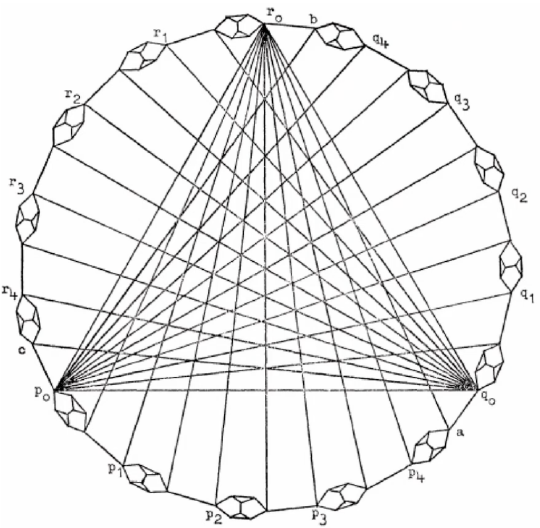

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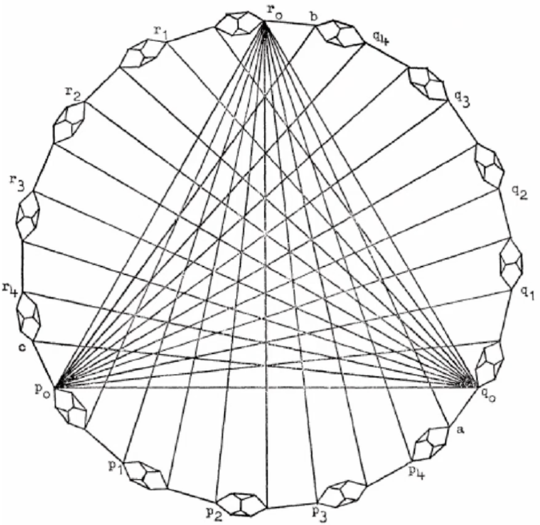

(Note: Stairs was working after Kochen & Specker, and noticed that one could use the mini-diagrams for a state-dependent proof.)

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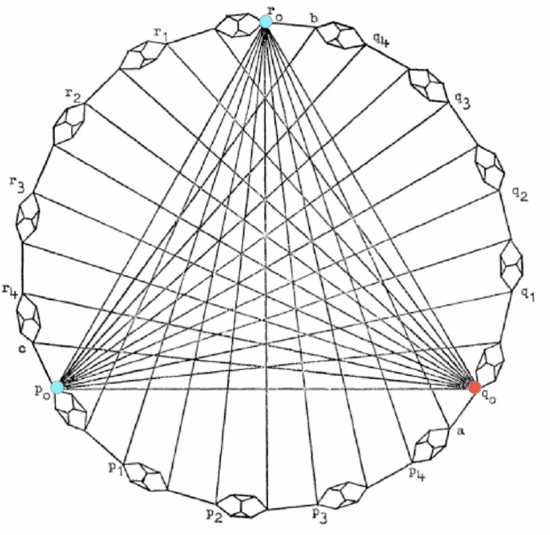

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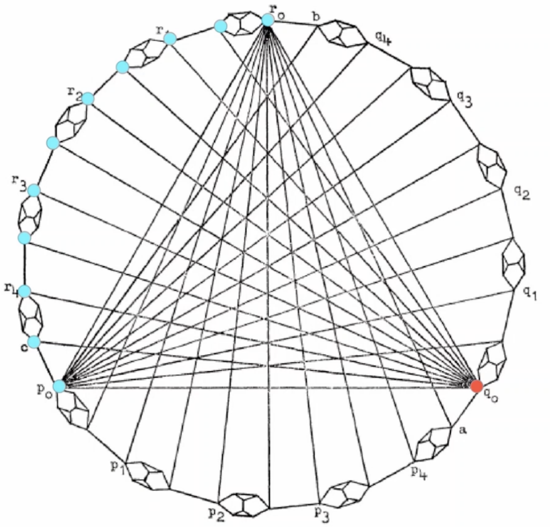

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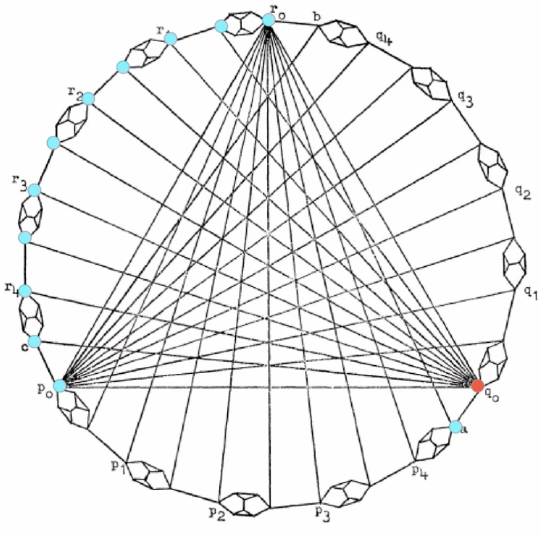

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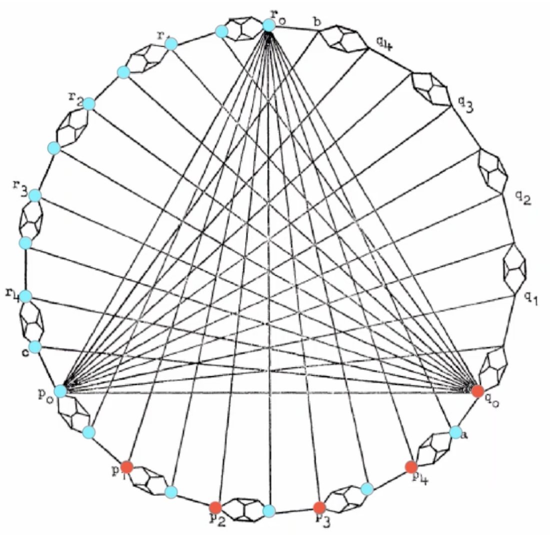

(At least one of the blobs for each of the pairs at the bottom has to be red given that we have a blue at the top.)

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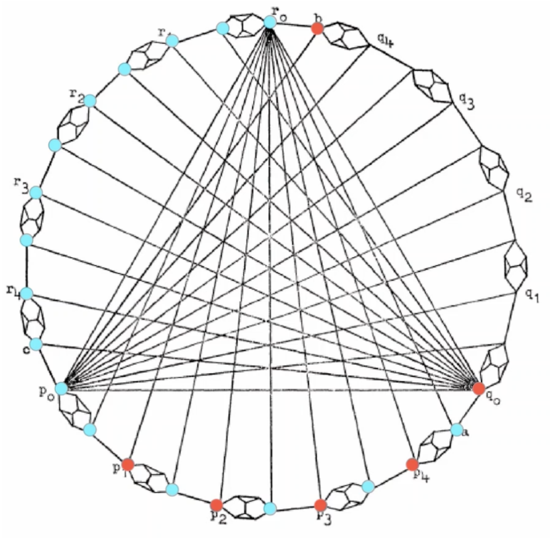

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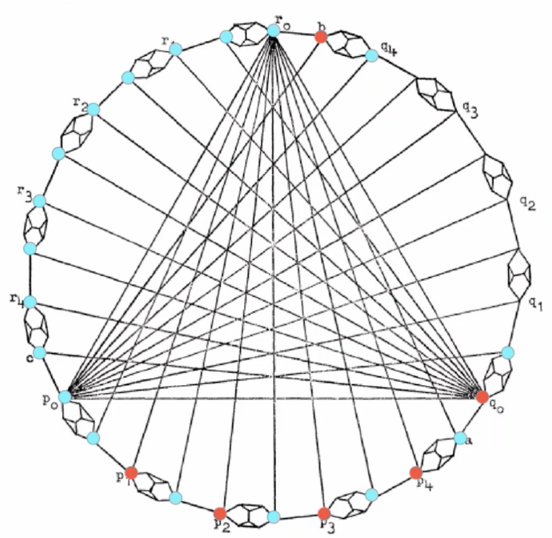
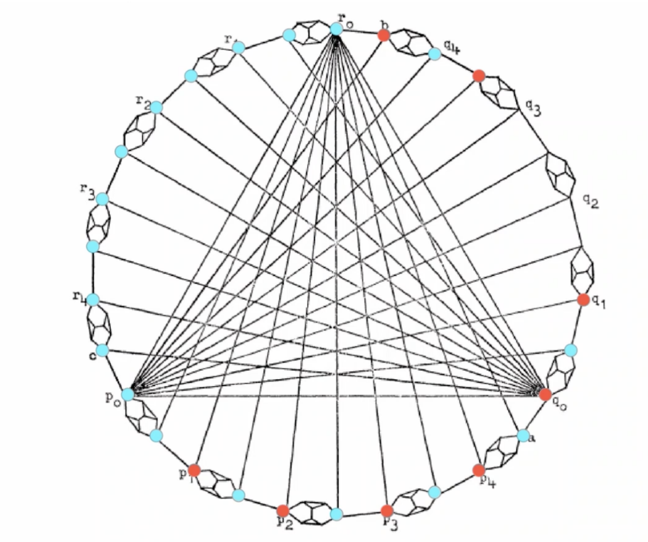


(This follows because we've seen from Clifton-Stairs that we can't have red on both sides of one of the little constructions.)

The BKS proof



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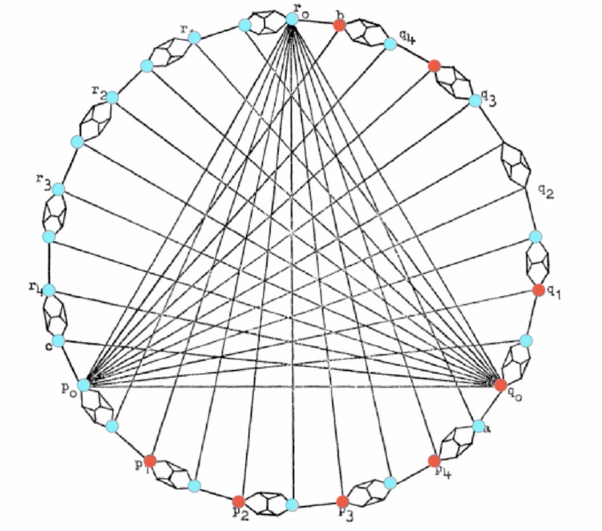

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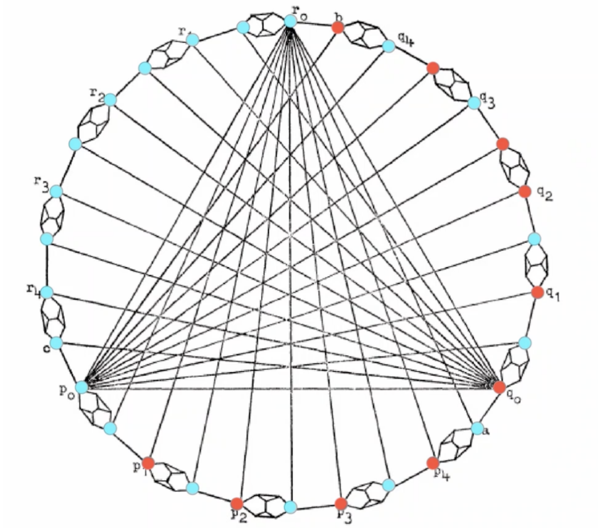

Contradiction, because we've already proven that if we've got one of the mini-diagrams then we can't have red on both sides!

On the BKS theorem

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- Obviously, the ingenuity is in designing this enormous set of states (117 projectors!).
- Bell (1966) actually did something similar to Kochen & Specker (1967) one year earlier than them, but nobody realised until afterwards (hence now the 'BKS theorem').

Later history of BKS-type results

- Since 1967, other sets of uncolourable directions have been discovered with fewer vectors.

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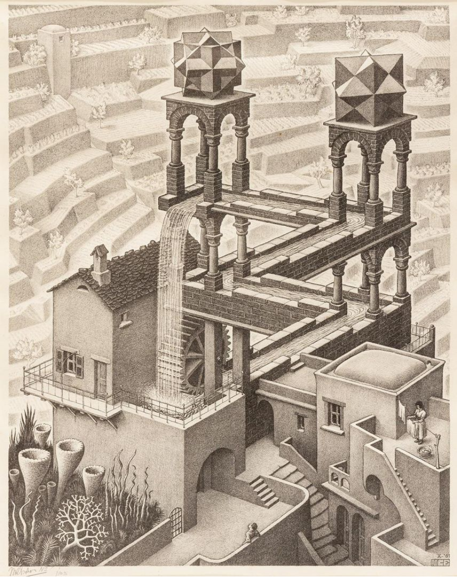
- E.g., Peres (1991) found a set of 33 with cubic symmetry.
- Penrose pointed out that Peres' set of 33 directions can be described as follows: take a cube and superimpose it with its 90-degree rotations about two perpendicular lines connecting its centre to the midpoints of an edge. Peres' directions point to the vertices and the centres of the faces and edges of the resulting set of three interpenetrating cubes. (Obviously...)

The Penrose cube



(Escher, *Waterfall*, 1961)

Today

Contextuality

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Coda: Gleason's theorem

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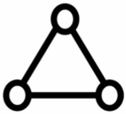
- I'll first present this, before discussing its significance.

The KCBS inequality



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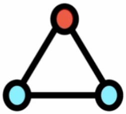

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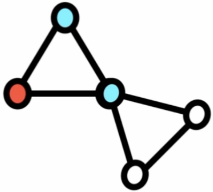

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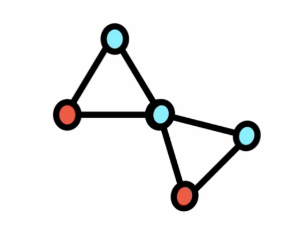

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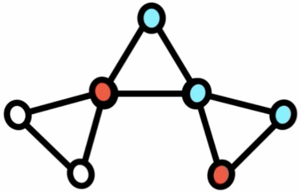

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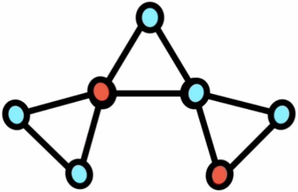

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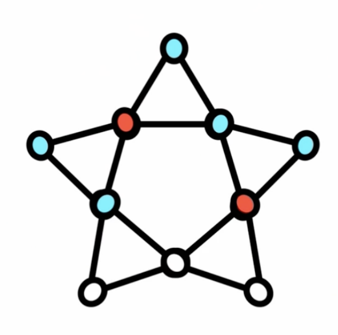

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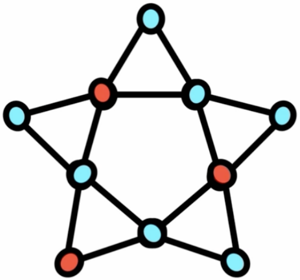

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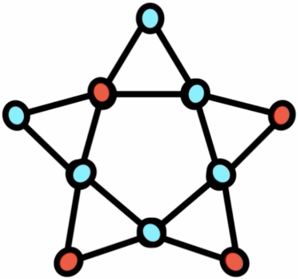

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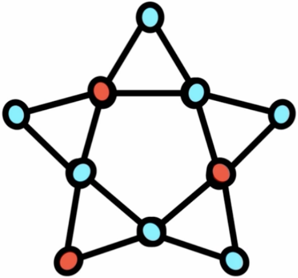

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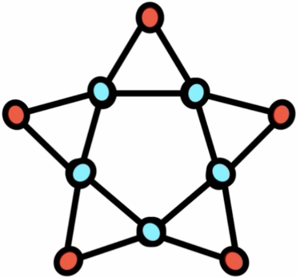

Main point: at least one of the outer points in the star must be red in a configuration like this.

The KCBS inequality

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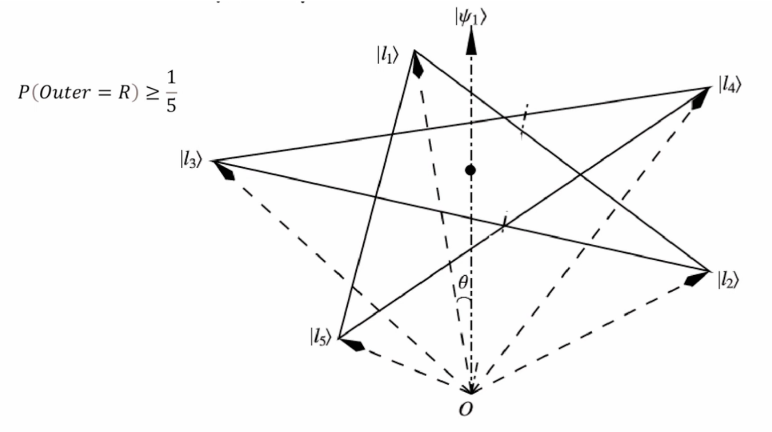
- From the five measurement bases (and measure it on that quantum state), $\Pr(\text{Outer} = R) \geq \frac{1}{5}$.

The KCBS inequality

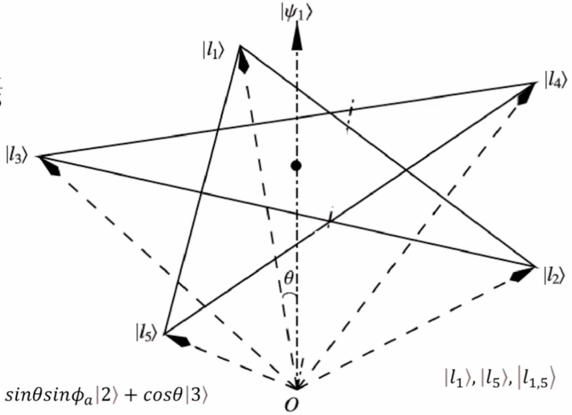
Now here comes quantum theory:



For any state you like, you can create this diagram around it.

The KCBS inequality

$$P(Outer = R) \geq \frac{1}{5}$$



$|l_1\rangle, |l_5\rangle, |l_{1,5}\rangle$

$$|l_a\rangle = sin\theta cos\phi_a|1\rangle + sin\theta sin\phi_a|2\rangle + cos\theta|3\rangle$$

$$|\langle 3|l_a\rangle|^2 = cos^2\theta = \frac{1}{\sqrt{5}}$$

$$cos\theta = \frac{1}{\sqrt[4]{5}} \qquad \phi_a = \frac{4}{5}\pi a$$

$$\left|\langle 3|l_{1,5}\rangle\right|^2 = 1 - \frac{2}{\sqrt{5}} = 0.1056\ldots < \frac{1}{5}$$

Experimental violation of the KCBS inequality

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- The KCBS inequality at least helps us to *quantify* quantum non-contextuality (see e.g. Cabello 2013).

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Gleason's theorem

Theorem (Gleason, 1957)

Let f be any function on projectors on a Hilbert space $\mathcal{H}$ of dimension $d > 2$ to the unit interval which is additive for any set of pairwise disjoint projectors on $\mathcal{H}$. Then there exists a unique density matrix $\hat{\rho}$ such that for any $\hat{P}$ on $\mathcal{H}$, $f(\hat{P}) = \mathrm{Tr}(\hat{\rho}\hat{P})$.

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- (The proof of Gleason's theorem is notoriously involved and I won't present it here.)
- One might think that Gleason's theorem counts as a derivaiton of the Born rule in unitary quantum mechanics—thereby (e.g.) solving (?) Everettians' worries about probability without the need to invoke decision theory etc.

Doubts about Gleason's theorem and quantum probabilities

However, as Saunders points out,

> *Gleason's theorem is a derivation of part of the Born rule, but of course it says nothing about 'measurements' or 'experiments'; nor, on reflection, is the premise of the theorem so clearly motivated.* (Saunders 2005, p. 213)

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Question: To what extent really is Gleason's theorem significant in the philosophy of quantum probabilities?

Gleason's theorem and the BKS theorem

- A direct consequence of Gleason's theorem is that when the Hilbert space $\mathcal{H}$ has dimension $d \geq 3$ then there does not exist a function f that assigns only the values 0 or 1 to projections $\hat{P} : \mathcal{H} \to \mathcal{H}$.

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- Therefore, both theorems rule out non-contextual hidden variable theories (though, being strictly stronger, Gleason's theorem does more).

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Next time: Ψ-ontic and Ψ-epistemic theories and the PBR theorem.

References I

- John S. Bell, “On the Problem of Hidden Variables in Quantum Mechanics”, Reviews of Modern Physics 38, pp. 447–52, 1966.
- Adán Cabello, “Simple Explanation of the Quantum Violation of a Fundamental Inequality”, Physical Review Letters 110, 060402, 2013.
- Rob Clifton, “Getting Contextual and Nonlocal Elements-of-Reality the Easy Way”, American Journal of Physics 61, pp. 443–47, 1993.
- A. M. Gleason, “Measures on the Closed Subspaces of a Hilbert Space”, Journal of Mathematics and Mechanics 6, pp. 885–93, 1957.
- Alexander A. Klyachko, M. Ali Can, Sinem Binicioglu, and Alexander S. Shumovsky, “Simple Test for Hidden Variables in Spin-1 Systems”, Physicsal Review Letters 101, 020403, 2008.
- S. Kochen and E. Specker, “The Problem of Hidden Variables in Quantum Mechanics”, Journal of Mathematics and Mechanics 17, pp. 59–87, 1967.

References II

- N. David Mermin, “Hidden Variables and the Two Theorems of John Bell”, Reviews of Modern Physics 65(3), 1993.
- A. Peres, “Two Simple Proofs of the Kochen-Specker Theorem”, Journal of Physics A: Mathematical and General 24(4), pp. L175–L178, 1991.
- Simon Saunders, “What is Probability?” in A. Elitzur, S. Dolev and N. Kolenda (eds.), *Quo Vadis Quantum Mechanics?*, Berlin: Springer, 2005.
- A. Stairs, “Value Definiteness and Contextualism: Cut and Paste with Hilbert Space”, PSA 1992 pp. 91–103, 1992.