IPP-SR-2: Galilean invariance

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The course

1. Newton's laws
2. Galilean invariance
3. The Michelson-Morley experiment
4. Einstein's 1905 derivation of the Lorentz transformations
5. Spacetime structure
6. General covariance
7. Relativity and conventionality of simultaneity
8. Frame-dependent effects
9. The twin paradox
10. Dynamical and geometrical approaches to relativity
11. Presentism and relativity
12. Acceleration and redshift

Today

The relativity principle

Active versus passive transformations

Galilean invariance

Newton on Galilean invariance

Poincaré invariance

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Wise words from Galileo

Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and in throwing something to your friend, you need to throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal space in every direction. ...

Wise words from Galileo (ctnd.)

... When you have observed all these things carefully, have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly ...

Wise words from Galileo (ctnd. ctnd.)

... The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in the water will swim toward the front of the bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals in keeping themselves in the air. (Galileo 1967, pp. 186-187.)

The relativity principle

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- We saw in the previous lecture that one can define the *inertial frames* as those frames in which the dynamical equations governing matter take their simplest form, and in which force-free bodies move with uniform velocity.
- The *relativity principle* states that the laws of physics take the same form (their simplest) in all inertial frames.

Questions

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1. In Newtonian mechanics, what are the coordinate transformations which take us from inertial frames to inertial frames?
2. Are these the *same* transformations which relate inertial frames in special relativity? (Spoiler: no.)

To answer the first question, we must investigate the *invariance properties* of the equations of Newtonian mechanics.

Today

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The transformations considered in this lecture can be understood either actively or passively.

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...the physics within the subsystem will be unchanged between the pre- and post-transformed cases.

This—an active boost applied to a subsystem, assuming the relativity principle and dynamical isolation—is what was going on in Galileo's example.

Today

Galilean transformations

A *Galilean transformation* is any coordinate transformation that can be expressed as the composition of a rigid spacetime translation, a rigid rotation, and a Galilean boost:

Spatial translation	$g_{\mathbf{a}}\left(\mathbf{a} \in \mathbb{R}^3\right)$:	$g_{\mathbf{a}}(t, \mathbf{x}) = (t, \mathbf{x} + \mathbf{a})$.
Time translation	$g_b\,(b \in \mathbb{R})$:	$g_b(t, \mathbf{x}) = (t + b, \mathbf{x})$.
Spatial rotation	$g_{\mathbf{R}}\,(\mathbf{R} \in SO(3))$:	$g_{\mathbf{R}}(t, \mathbf{x}) = (t, \mathbf{R}\mathbf{x})$.
Galilean boost	$g_{\mathbf{v}}\left(\mathbf{v} \in \mathbb{R}^3\right)$:	$g_{\mathbf{v}}(t, \mathbf{x}) = (t, \mathbf{x} - \mathbf{v}t)$.

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- For any such r and any time translation g_b, we can form the transformed structure $g_b r$:

$$\begin{aligned}(g_b r)(t) &= r(t-b)\\ &= Ae^{-k(t-b)}\\ &= \left(Ae^{+kb}\right)e^{-kt}.\end{aligned}$$

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- So we say that our equation is *time-translation invariant*.

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4. For general $g \in G$, identify the action of g on S.
5. Ask whether this action of G preserves the subset $D \subset S$ of solutions to Θ.

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- For the general solution $r(t) = Ae^{-kt}$, the transformed structure is given by

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- So our equation is not *Galilean boost invariant*.

Invariance

There are two (equivalent) ways of defining what it means for a given set of laws to be *invariant* under a given group of transformations:

1. Space-of-solutions approach.
2. Form-of-equations approach.

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- But asserting that our second equation holds for all t is equivalent to asserting that our first equation holds for all t.
- Thus, the original equation is time translation invariant.

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5. If the result is a set of equations equivalent to the original Θ, then Θ is G-invariant.

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- The ingredients of our equation transform as

$$g_v : \frac{d}{dt} \mapsto \frac{d}{dt};$$
$$g_v : k \mapsto k;$$
$$g_v : r(t) \mapsto r(t) - vt.$$

Toy example of non-invariance: form-invariance

- So the transformed equation is

$$\frac{d}{dt}\left(r\left(t\right) - vt\right) = -k\left(r\left(t\right) - vt\right).$$

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$$\frac{d}{dt}\left(r\left(t\right)-vt\right)=-k\left(r\left(t\right)-vt\right).$$

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- But this is equivalent to the original equation only if $-v=vkt$, which clearly cannot hold for all t.
- The *non*-equivalence of the untransformed and transformed equations means that the original equation is *not* boost-invariant.

Galilean boost invariance of Newtonian gravitation

- Newtonian gravity for two particles is given by (combining N2L and the law of gravitation):

$$\ddot{\mathbf{r}}_i = \frac{G_N m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_i - \mathbf{r}_{i+1}), \qquad i = 1, 2.$$

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- Let G be the group B_3 of *three*-dimensional boosts, $\left\{(g_{\mathbf{v}} : \mathbf{r} \mapsto \mathbf{r} - \mathbf{v}t) : \mathbf{v} \in \mathbb{R}^3\right\}$.

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- The quantities in our equation transform as

$$\begin{aligned} \mathbf{r}_i'(t) &:= (g_{\mathbf{v}} \mathbf{r}_i)(t) = \mathbf{r}_i(t) - \mathbf{v}t, \\ \ddot{\mathbf{r}}_i'(t) &:= (\ddot{g_{\mathbf{v}} \mathbf{r}_i})(t) = \ddot{\mathbf{r}}_i(t), \\ m_i' &:= g_{\mathbf{v}} m_i = m_i. \end{aligned}$$

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- The transformed equation is:

$$\ddot{\mathbf{r}}'_i = \frac{G_N m_1 m_2}{\left|\mathbf{r}'_1 - \mathbf{r}'_2\right|^3} \left(\mathbf{r}'_i - \mathbf{r}'_{i+1}\right), \qquad i = 1, 2.$$

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- Eliminating the primes, we have

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- **Exercise:** Generalise this to the *N*-body problem.
- **Exercise:** Show that Newtonian gravitation is invariant under Galilean boosts using the space-of-solutions approach.

Space-of-solutions versus form-invariance approaches

- The two approaches are equivalent (**Exercise:** prove this!), but often the form-invariance method is easier, because it doesn't involve having to figure out what the solutions of the equation under consideration *are*.

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- In each case above, we began with an *ansatz* about what the symmetry group of our equation. Figuring out the full symmetry group of a set of equations is highly non-trivial.
- While there is no general method for doing this, the task can be aided by formulating our theories in certain ways, using certain objects which have familiar symmetry properties. (See lecture 6.)

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Newton’s ‘Corollary V’

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  > *The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forward in a right line without any circular motion.* (Cajori 1934, p. 20)

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- This essentially *states* that the laws of physics are Galilean invariant.

Newton’s argument for Corollary V

For the differences of the motions tending towards the same parts, and the sums of those that tend toward contrary parts, are, at first (by supposition), in both cases the same; and it is from those sums and differences that the collisions and impulses do arise with which the bodies mutually impinge one upon another. Wherefore (by Law II) the effects of those collisions will be equal in both cases; and therefore the mutual motions of the bodies among themselves in the one case will remain equal to the mutual motions of the bodies among themselves in the other. A clear proof of which we have from experiment of a ship; where all motions happen after the same manner, whether the ship is at rest, or is carried uniformly forwards in a right line. (Cajori 1934, p. 20)

Two *non sequiturs* in Newton's argument

- It does not follow from the laws of motion alone that 'it is from those sums and differences that the collisions and impulses do arise with which the bodies mutually impinge upon one another'. This requires the *additional assumption* that forces depend only on (vectorial) differences of positions and/or velocities, not on absolute positions or absolute velocities. (Consider a particle affected by the force $\mathbf{F} = -k\mathbf{v}$.)

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- It does not follow that "the effects of those collisions will be equal" unless we further assume that the *mass* of a given body is independent of the body's absolute position and absolute velocity. (Consider particles whose masses are proportional to their absolute speeds.)

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(For more on this, see (Brown 2005, §3.2).)

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A *Poincaré transformation* is any coordinate transformation that can be expressed as the composition of a rigid spacetime translation, a rigid rotation, and a *Lorentz* boost:

Spacetime translation	$g_{a^\mu}\ \left(a^\mu \in \mathbb{R}^4\right)$:	$g_{a^\mu}\ (x^\nu) = x^\nu + a^\nu$.
Spatial rotation and Lorentz boost	$g_{\Lambda^\mu{}_\nu}\ \left(\Lambda^\mu{}_\nu \in SO(1,3)\right)$:	$g_{\Lambda^\mu{}_\nu}\ (x^\nu) = \Lambda^\nu{}_\sigma x^\sigma$.

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- Lorentz boosts in x direction ($\gamma := 1/\sqrt{1 - v^2/c^2}$):

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ x' &= \gamma\,(x - vt)\,. \end{aligned}$$

Galilean versus Lorentz transformations

- In both cases, we have a rigid translation, a rigid rotation, and a boost.
- But the boosts are *different* in the two cases.
- Galilean boosts in x direction:

$$
\begin{aligned}
t' &= t, \\
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\end{aligned}
$$

- Lorentz boosts in x direction ($\gamma := 1/\sqrt{1 - v^2/c^2}$):

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\begin{aligned}
t' &= \gamma \left(t - \frac{vx}{c^2} \right), \\
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\end{aligned}
$$

- **Exercise:** Show that we recover the Galilean boosts from the Lorentz boosts when $v \ll c$.

Maxwell's equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

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Question: What are the invariance properties of these equations?

Precipitating a crisis

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- One might say that it was the discovery of a set of dynamical laws which were Poincaré invariant, rather than Galilean invariant, which precipitated the crisis which led to the development of special relativity.
- We will see more of this next time.

Summary

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5. Argued that, with corrections, Newton was right to say that the dynamical equations of Newtonian mechanics are Galilean invariant.
6. Seen that Maxwell's equations of electromagnetism are Poincaré invariant.

References

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