EDL Exercises

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These are my problem sheets for the Prelims paper, *Elements of Deductive Logic* (EDL). Most are not original to me, but are based upon the sheets of Stephen Blamey, Adam Caulton (about 80% of everything below), Anthony Eagle, and James Studd. Please do contact me at the email address below if you have any questions or queries.

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1 Set theory and induction

1.1 Important concepts

These are the concepts introduced in this week's class and readings. Check that you understand them all.

- **Sets:** members and elements; extensionality; the empty set (or null set); singletons; intersection, union, relative complement; disjointness; subset and superset, proper subset and proper superset; power set; Russell's paradox.
- **Sequences:** the set-theoretic definition of an ordered pair; the criterion of identity for ordered pairs; sequences (or n-tuples); the Cartesian product.
- **Relations:** relations over a set; binary relations; domain and range; reflexive, irreflexive and non-reflexive; symmetric, asymmetric, anti-symmetric and non-symmetric; transitive, intransitive and non-transitive; the transitive closure (or ancestral) of a relation; equivalence relations and partitions; the identity relation; predicates determining a relation.
- **Functions:** argument and value; domain and range; total vs. partial functions on a set; criterion of identity for functions; one-to-one (or injective); into; onto (or surjective); one-to-one correspondences (or bijections); the inverse of a function; n-place functions (including $n = 0$); truth functions; enumeration; equinumerosity and cardinality; Cantor's theorem; countability, denumerability and uncountability.
- **Induction and recursion:** inductive definition of a set (base case, generating relation and closure condition); recursive definition of a function (base case and recurrence relation); proof by induction (base cases and inductive steps); the Weak Principle of Induction (or ordinary induction), the Strong Principle of Induction (or complete induction), the Least Number Principle.

1.2 Required exercises

1. Diagnose the following fallacious inductive "proofs":

   (a) *Claim:* All horses are the same colour. *Proof:* By induction on the number n of horses.

   - *Base case:* $n = 1$. Trivial: the single horse is the same colour as itself.
   - *Inductive hypothesis:* Assume the claim for all sets of n horses; now consider any set of $n + 1$ horses.
     Assume some enumeration of the $n + 1$ horses. By the inductive hypothesis, the first n horses are the same colour. By the inductive hypothesis again, the last n horses (horse 2 to horse $n+1$) are

the same colour. So then the first horse is the same colour as the middle $n-1$ horses, which are the same colour as the $(n+1)$th horse. So then all $n+1$ horses are the same colour.

By induction, all horses are the same colour. QED.

(b) *Claim:* π is rational. *Proof:* By induction on the number of decimal places n of π.

- *Base case:* $n = 0$. $x_0 := 3$ is rational.
- *Inductive hypothesis:* Assume that the decimal expansion of π truncated to the nth decimal place, x_n, is rational; now consider the expansion truncated to the $(n+1)$th decimal place, x_{n+1}. $x_{n+1} = x_n + \frac{m}{10^{n+1}}$ for some $m \in \{0, 1, \ldots, 9\}$. By the inductive hypothesis, x_n is rational. $\frac{m}{10^{n+1}}$ is clearly rational. The sum of any two rational numbers is rational. Therefore x_{n+1} is rational.

By induction, $\pi = x_\infty$ is rational. QED.

2. Prove the following:

   (a) $\subseteq$ is a partial order, i.e. it is reflexive, anti-symmetric and transitive.

   (b) $\subset$ is a strict order, i.e. it is irreflexive and transitive (and therefore asymmetric).

   (c) $\subseteq$ can be defined in terms of $\subset$, or *vice versa*.

3. Prove the following:

   (a) $A \subseteq B$ iff $A \cap B = A$, iff $A \cup B = B$.

   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. Provide an example for which $\wp(A) \cup \wp(B) \neq \wp(A \cup B)$.

5. (a) Prove that a function $f : A \to B$ has an inverse $f^{-1} : B \to A$ iff it is a bijection.

   (b) Give an example of a function that:

      i. is into but not onto;
      ii. is one-to-one but not a bijection;
      iii. is a bijection but has disjoint domain and range;
      iv. has no inverse;
      v. is its own inverse.

   (c) For each of the following sets, say whether or not it is denumerable:

      i. $\{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} \mid x + y = 12\}$;
      ii. $\mathbb{N} \times \mathbb{N}$;
      iii. the set of all finite sequences of natural numbers;
      iv. the set of all binary expansions of real numbers in the interval $[0, 1]$.

6. State and prove Cantor's theorem.

7. Let φ be an $\mathcal{L}_1$-sentence containing only zero or more instances of the connective $\neg$ and the sentence letter P. Show using induction on complexity of φ that either $\varphi \equiv P$ or $\varphi \equiv \neg P$.

8. Let φ be an $\mathcal{L}_1$-sentence containing only the connectives $\wedge$ and $\vee$. Show using induction on the complexity of φ that $\left|\varphi\right|_{\mathcal{A}} = \mathrm{F}$ whenever $|\Phi|_{\mathcal{A}} = \mathrm{F}$ for each sentence letter Φ in φ and that $\left|\varphi\right|_{\mathcal{A}} = \mathrm{T}$ whenever $|\Phi|_{\mathcal{A}} = \mathrm{T}$ for each sentence letter Φ in φ.

9. Prove the following claims about $\mathcal{L}_1$:

   (a) The number of occurrences of parentheses in a sentence is twice the number of occurrences of binary connectives.

   (b) Let $\mathcal{A}$ be a structure such that $|\alpha|_{\mathcal{A}} = \mathrm{T}$ for each sentence letter α. For any sentence φ with no occurrences of negation, $\left|\varphi\right|_{\mathcal{A}} = \mathrm{T}$.

   (c) If φ contains at most one occurrence of each sentence letter then there is a structure $\mathcal{A}$ such that $\left|\varphi\right|_{\mathcal{A}} = \mathrm{T}$ and a structure $\mathcal{B}$ such that $\left|\varphi\right|_{\mathcal{B}} = \mathrm{F}$.

   (d) If φ contains at most one occurrence of any sentence letter, then $\nvDash \phi$.

1.3 Optional exercises

1. Prove that, according to Kuratowski's definition, $\langle x, y\rangle = \langle u, v\rangle$ iff $x = u$ and $y = v$.

2. In axiomatic set theory, the *Axiom of Separation* (in fact an axiom schema) says that, for any $(n + 1)$-place formula φ, any items $y_1, \ldots, y_n$ and any set A, there is a set $B := \{x \in A \mid \varphi(x, y_1, \ldots, y_n)\}$. Show, using Russell's paradox, that the Axiom of Separation is inconsistent with the existence of the universal set Ω (the set of all sets).

3. Prove that the Weak Principle of Induction (WPI), the Strong Principle of Induction (SPI) and the Least Number Principle (LNP) are all equivalent. [*Hint:* only three claims need to be proven: (LNP) $\Rightarrow$ (SPI), (SPI) $\Rightarrow$ (WPI) and (WPI) $\Rightarrow$ (LNP); the first two claims are particularly easy.]

2 Truth functions, DNF/CNF, and substitution

2.1 Important concepts

These are the concepts introduced in this week's class and readings. Check that you understand them all.

- **Syntax:** alphabet (sentence letters, logical connectives, parentheses); inductive definition of $\mathcal{L}_1$-sentence ($\mathcal{L}_1$-formula); atomic vs. complex sentences; literals; the parenthesis matching theorem.
- **Semantics:** truth values; $\mathcal{L}_1$-structures; truth tables; recursive definition of truth in a $\mathcal{L}_1$-structure; proof by induction on complexity of sentences.
- **Entailment:** satisfaction; satisfiability (or semantic consistency); entailment (or semantic entailment); sequents (the meaning of '$\Sigma \vDash \varphi$'); tautologies; structural rules (permutation, contraction, weakening); logical equivalence; equivalence theorem; cut, transitivity, contraposition; the deduction theorem.
- **Substitution:** uniform substitution of sentence letters; substitution instances; the substitution theorem; logical equivalence of sentences with logically equivalent subsentences.
- **Truth functions and their expression:** truth functions; connectives (n-place, for any $n \in \mathbb{N}$); truth-functional connectives (the expression of a truth function); disjunctive normal form (DNF); conjunctive normal form (CNF); the DNF and CNF theorems.

2.2 Required exercises

1. For each of the following claims, either offer a proof or a counterexample.
   (a) If $\Gamma \vDash \neg\varphi$, then $\Gamma \nvDash \varphi$.
   (b) Either $\Gamma \vDash \varphi$ or $\Gamma \vDash \neg\varphi$.
   (c) If $\varphi \vDash \chi$ and $\psi \vDash \chi$, then $(\varphi \vee \psi) \vDash \chi$.
   (d) If $\varphi, \neg\psi \vDash$ and $\neg\varphi, \psi \vDash$, then $\vDash \varphi \leftrightarrow \psi$.
2. (a) How many n-place truth functions are there?
   (b) For each 2-place truth function, and using the sentence letters P and Q, offer the least complex $\mathcal{L}_1$-sentence which expresses it. (Recall that all sentence letters have complexity 0 and the complexity of any sentence is 1 more than the complexity of its most complex proper subsentence.)

(c) Let's call two truth functions *equivalent* iff they are expressed by logically equivalent $\mathcal{L}_1$-sentences. Further, let's call an n-place truth function *redundant* iff it is equivalent to some m-place truth function, where $m < n$.

How many non-redundant 0, 1, 2, and 3-place truth functions are there?

3. For each of the following $\mathcal{L}_1$-sentences, offer a logically equivalent sentence in disjunctive normal form. [*Hint:* first draw their truth-tables and consider how the DNF Theorem is proved.]

   (a) $(P \to (\neg Q \to \neg P))$

   (b) $((P \land \neg Q) \to (\neg P \land R)) \leftrightarrow ((R \to P) \to (P \to Q))$

   (c) $(P \leftrightarrow Q) \to (R \leftrightarrow \neg Q)$

   (d) $((P \leftrightarrow (\neg Q \land R)) \lor (R \to (P \leftrightarrow Q)))$

4. Prove, by induction, the generalised De Morgan equivalences,

$$\neg\left(\phi_1 \land \ldots \land \phi_n\right) \equiv \left(\neg\phi_1 \lor \ldots \lor \neg\phi_n\right),$$
$$\neg\left(\phi_1 \lor \ldots \lor \phi_n\right) \equiv \left(\neg\phi_1 \land \ldots \land \neg\phi_n\right).$$

5. The *CNF Theorem* states that

   Every truth function is expressed by an $\mathcal{L}_1$-sentence in conjunctive normal form.

   Prove this theorem in two different ways:

   (a) Using the DNF theorem and the generalised De Morgan equivalences.

   (b) Directly, in the style of Eagle's proof of the DNF theorem.

6. Crucial to proving the Substitution Theorem is the *Substitution Lemma*, which is as follows:

   Let φ and ψ be any $\mathcal{L}_1$-sentences, and let X be some sentence letter. For any $\mathcal{L}_1$-structure $\mathcal{A}$, define the $\mathcal{L}_1$-structure $\mathcal{A}_{\psi/X}$ as follows:

$$\mathcal{A}_{\psi/X}(Y) = \begin{cases} |\psi|_{\mathcal{A}} & \text{iff} \quad Y = X; \\ \mathcal{A}(Y) & \text{iff} \quad Y \neq X. \end{cases}$$

   Then $|\varphi|_{\mathcal{A}_{\psi/X}} = |\varphi[\psi/X]|_{\mathcal{A}}$.

   Prove this lemma.

2.3 Optional exercises

1. (a) Prove that if φ and ψ are nonempty sequences of characters of the alphabet of $\mathcal{L}_1$, such that $\varphi\psi$ (i.e., the sequence of characters consisting of the sequence φ followed immediately by the sequence ψ) is a sentence, then φ is not a sentence. [*Hint:* use the parentheses matching theorem and the fact that if $\varphi\psi$ is a sentence, then φ must have mismatched parentheses.]

   (b) Prove that if φ is a non-atomic $\mathcal{L}_1$-sentence, then there is exactly one formation clause from the inductive definition of '$\mathcal{L}_1$-sentence' that could have been applied to existing sentences to produce φ. [*Hint:* you need to show that, e.g., $(\varphi \vee \psi)$ cannot be the same sentence as $(\chi \wedge \xi)$.]

   (c) The *Unique Readability Theorem* states that

   > Every $\mathcal{L}_1$-sentence is *uniquely readable*; i.e. every $\mathcal{L}_1$-sentence can be produced from sentence letters in accordance with the formation clauses of $\mathcal{L}_1$ in exactly one way.

   Prove this theorem.

2. Take any $\mathcal{L}_1$-sentence φ whose only connectives are in $\{\neg, \wedge, \vee\}$. For any sentence letter X, we say that X is *positive in* φ iff X does not occur in the scope of a negation symbol. Prove that if X is positive in φ and $(\psi \to \xi)$ is a tautology, then $(\varphi[\psi/X] \to \varphi[\xi/X])$ is a tautology. (You may *not* assume the logical equivalence of sentences with logically equivalent subsentences.)

3 Duality, interpolation, and compactness

3.1 Important concepts

These are the concepts introduced in this week's class and readings. Check that you understand them all.

- **Expressive adequacy:** De Morgan equivalences; functional completeness; expressive adequacy; the Sheffer stroke ($\uparrow$, or NAND) and the Peirce arrow ($\downarrow$, or NOR).
- **Duality:** the duals of $\wedge$ and $\vee$; the dual of an arbitrary connective; self-dual connectives; self-dual sets of connectives; the recursive definition of the dual of an $\mathcal{L}_1^-$-sentence (where $\mathcal{L}_1^-$ is like $\mathcal{L}_1$, but only has the connectives $\neg, \wedge, \vee$); the dual of an $\mathcal{L}_1^-$-structure; the Duality Lemma; the Duality Theorem.
- **Interpolation:** an interpolant of a sequent; the Interpolation Theorem.
- **Compactness:** satisfiability vs. finite satisfiability; the Compactness Theorem for $\mathcal{L}_1$.
- **Decidability:** effective procedures; decidability; positive and negative decidability; the positive decidability of unsatisfiability in $\mathcal{L}_1$; the undecidability of unsatisfiability in $\mathcal{L}_1$.

3.2 Required exercises

1. For each of the following sets of connectives, either prove that it is expressively adequate or prove that it isn't.
   (a) $\{\neg, \wedge\}$
   (b) $\{\neg, \rightarrow\}$
   (c) $\{\rightarrow, \bot\}$, where $\bot$ is the 0-place connective which expresses the 0-place truth function $f_\bot = 0$.
   (d) $\{\rightarrow, \rightarrow^\star\}$, where $\rightarrow^\star$ is the connective dual to $\rightarrow$.
   (e) $\{\leftrightarrow, \leftrightarrow^\star\}$, where $\leftrightarrow^\star$ is the connective dual to $\leftrightarrow$.
   (f) $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$
2. Define every 2-place connective which is expressively adequate on its own (by giving its truth table) and show that it is expressively adequate.
3. For any n-place connective $\mathfrak{c}$, its dual $\mathfrak{c}^\star$ is defined as the connective which expresses the truth function $f_{\mathfrak{c}^\star}$, where

$$f_{\mathfrak{c}^\star}(t_1, \ldots, t_n) := \left(f_{\mathfrak{c}}(t_1^\star, \ldots, t_n^\star)\right)^\star \ ,$$

where each $t_i \in \{1, 0\}$ (i.e. true or false), $0^\star = 1$ and $1^\star = 0$, and $f_\mathfrak{c}$ is the truth function expressed by $\mathfrak{c}$.

Use this fact to find all of the self-dual 2-place connectives.

4. Consider any propositional logic $\mathcal{L}_\mathfrak{C}$, whose connectives are given by the set $\mathfrak{C}$, which is self-dual (i.e. $\mathfrak{C}^\star := \{\mathfrak{c}^\star \mid \mathfrak{c} \in \mathfrak{C}\} = \mathfrak{C}$) and expressively adequate. The truth rules for $\mathcal{L}_\mathfrak{C}$ are as follows: for all $\mathfrak{c} \in \mathfrak{C}$, all $\mathcal{L}_\mathfrak{C}$-structures $\mathcal{A}$, and all $\mathcal{L}_\mathfrak{C}$-sentences $\varphi_1, \ldots, \varphi_n$ (where $\mathfrak{c}$ is an n-place connective),

$$\left|\mathfrak{c}(\varphi_1, \ldots, \varphi_n)\right|_\mathcal{A} := f_\mathfrak{c}\left(|\varphi_1|_\mathcal{A}, \ldots, |\varphi_n|_\mathcal{A}\right) ,$$

where $f_\mathfrak{c}$ is the truth function expressed by $\mathfrak{c}$. Also recall the definition of $f_{\mathfrak{c}^\star}$ in Q3.

   (a) Offer a sensible recursive definition for the dual $\varphi^\star$ of any $\mathcal{L}_\mathfrak{C}$-sentence φ.

   (b) Let $\mathfrak{n}(\varphi)$ be some $\mathcal{L}_\mathfrak{C}$-sentence formed from the $\mathcal{L}_\mathfrak{C}$-sentence φ such that $|\mathfrak{n}(\varphi)|_\mathcal{A} = 1$ iff $|\varphi|_\mathcal{A} = 0$, for all $\mathcal{L}_\mathfrak{C}$-structures $\mathcal{A}$. (Since $\mathfrak{C}$ is expressively adequate, we are guaranteed that $\mathfrak{n}(\varphi)$ exists for each φ.)

   Let $\tilde{\varphi}$ be the $\mathcal{L}_\mathfrak{C}$-sentence constructed from the $\mathcal{L}_\mathfrak{C}$-sentence φ by substituting each of its sentence letters with its corresponding "negation"; i.e.

$$\tilde{\varphi} = \varphi[\mathfrak{n}(P)/P][\mathfrak{n}(Q)/Q][\mathfrak{n}(R)/R][\mathfrak{n}(P_1)/P_1]\cdots .$$

   Prove the Duality Lemma for $\mathcal{L}_\mathfrak{C}$, which states that $\varphi^\star \equiv \mathfrak{n}(\tilde{\varphi})$, for all $\mathcal{L}_\mathfrak{C}$-sentences φ.

   (c) Prove the Duality Theorem for $\mathcal{L}_\mathfrak{C}$, which states that if $\varphi \vDash \psi$, then $\psi^\star \vDash \varphi^\star$, for all $\mathcal{L}_\mathfrak{C}$-sentences φ and ψ.

   (d) For any $\mathcal{L}_\mathfrak{C}$-structure $\mathcal{A}$, define its dual $\mathcal{A}^\star$ so that, for all sentence letters X,

$$\mathcal{A}^\star(X) := (\mathcal{A}(X))^\star .$$

   Prove that $|\varphi|_\mathcal{A} = (|\varphi^\star|_{\mathcal{A}^\star})^\star$, for all $\mathcal{L}_\mathfrak{C}$-sentences φ.

5. Determine interpolants for the following sequents. In each case give the simplest interpolant (i.e. the interpolant of least complexity).

   (a) $((Q \vee P) \rightarrow R) \vDash ((P_1 \wedge \neg R) \rightarrow \neg Q)$

   (b) $(\neg(P \vee Q) \wedge (P \leftrightarrow R)) \vDash ((R \rightarrow P) \vee (\neg P_1 \wedge R))$

   (c) $((Q_2 \leftrightarrow Q) \wedge \neg((R \rightarrow \neg P_1) \vee \neg(P \rightarrow Q))) \vDash (R_1 \rightarrow (P_2 \rightarrow (\neg P \vee Q)))$

6. Prove Craig's interpolation theorem. [The proof can be found in Eagle—but write it in your own words, and make sure you understand every step!]

7. (a) Let Γ be a possibly infinite set of sentences of $\mathcal{L}_1$ such that $\Gamma \vDash$. Show that there is a finite disjunction δ, each disjunct of which is the negation of a sentence in Γ, and such that $\vDash \delta$. (You may assume the Compactness Theorem.)

   (b) Consider the following relation holding between sets of sentences:

   > Where Γ and Δ are any sets of $\mathcal{L}_1$-sentences, $\Gamma \vDash_\infty \Delta$ iff every $\mathcal{L}_1$-structure which satisfies *every* sentence in Γ is also one which satisfies *at least one* sentence in Δ.

   Show that if $\Gamma \vDash_\infty \Delta$, then there is a finite conjunction of $\mathcal{L}_1$-sentences in Γ,

   $$\Phi = (\varphi_1 \wedge \ldots \wedge \varphi_m) ,$$

   and a finite disjunction of $\mathcal{L}_1$-sentences in Δ,

   $$\Psi = (\psi_1 \vee \ldots \vee \psi_n) ,$$

   such that $\Phi \vDash \Psi$.

3.3 Optional exercises

1. (a) Formulate a notion of satisfiability and compactness for natural languages, such as English.

   (b) Is English compact? Justify your answer.

   (c) What, if anything, does you answer to (b) say about the expressive capabilities of $\mathcal{L}_1$?

4 Natural deduction for $\mathcal{L}_1$

4.1 Important concepts

These are the concepts introduced in this week's class and readings. Check that you understand them all.

- **Proofs:** provability (or syntactic entailment); syntactic sequents (the meaning of '$\Sigma \vdash \varphi$'); assumptions (premises) and conclusion; theorems.
- **Natural Deduction:** introduction and elimination rules for *ND*; discharging assumptions; the Deduction Theorem for *ND*; alternative natural deduction rules: double negation elimination, *ex falso quodlibet*, *tertium non datur* (non-constructive dilemma).
- **Soundness:** the statement of the Soundness Theorem for *ND* and $\mathcal{L}_1$; the proof of the Soundness Theorem for *ND* and $\mathcal{L}_1$ (by induction on the complexity of proofs).
- **Completeness:** the statement of the Completeness Theorem for *ND* and $\mathcal{L}_1$; (syntactic) consistency; maximal consistency; deductive closure; negation completeness, conditional completeness, etc.; the Satisfiability Theorem (satisfiability of maximal consistent sets); the proof of the Completeness Theorem for *ND* and $\mathcal{L}_1$ (by the construction of a maximally consistent set); compactness via soundness, completeness and the finitude of proofs.
- **Axioms:** Hilbert-style axiomatic proof systems; axioms and axiom schemata; soundness and completeness for axiomatic systems and $\mathcal{L}_1$.

4.2 Required exercises

1. The compactness of $\mathcal{L}_1$ has a 'quick' proof, which goes via soundness and completeness. Outline this proof.
2. Give proofs of the soundness and completeness of *ND* for $\mathcal{L}_1$ (and make sure you understand them!).
3. In this question we consider modifications to *ND*, the natural deduction system for $\mathcal{L}_1$.
   (a) Let's replace ¬-Elim with *ex falso quodlibet:*

   $$\frac{\varphi \qquad \neg\varphi}{\psi}\,\text{EFQ}$$

   Is the resulting system equivalent to *ND*? Justify your answer.

   (b) In addition, let's replace ¬-Intro with *tertium non datur*:

$$\dfrac{\begin{matrix}[\varphi]^n & & [\neg\varphi]^n \\ \vdots & & \vdots \\ \psi & & \psi\end{matrix}}{\psi}\ \text{TND}, n$$

Is the resulting system equivalent to *ND*? Justify your answer.

4. Consider the language $\mathcal{L}_{\to,\bot}$, whose only connectives are material implication $\to$ and $\bot$, the 0-place connective which expresses the 0-place truth function $f_\bot = 0$. Also consider the language $\mathcal{L}_{\neg,\lor}$, whose only connectives are $\neg$ and $\lor$.

   (a) Devise a translation scheme between sentences of $\mathcal{L}_{\to,\bot}$ and $\mathcal{L}_{\neg,\lor}$.

   (b) A sound and complete natural deduction system $ND_{\to,\bot}$ for $\mathcal{L}_{\to,\bot}$ needs only three rules. What are they?

   (c) $ND_{\neg,\lor}$ is the (sound and complete) natural deduction system for $\mathcal{L}_{\neg,\lor}$ (i.e., it is the fragment of *ND* which governs the connectives $\neg$ and $\lor$).

   Show that $ND_{\to,\bot}$ and $ND_{\neg,\lor}$ are equivalent. (This involves providing several proof schemata: some to show that if $\Gamma \vdash_{\to,\bot} \varphi$, then $\Gamma^* \vdash_{\neg,\lor} \varphi^*$, where φ^* is the $\mathcal{L}_{\neg,\lor}$-translation of the $\mathcal{L}_{\to,\bot}$-sentence φ, etc.; and some to show that if $\Gamma \vdash_{\neg,\lor} \varphi$, then $\Gamma^\dagger \vdash_{\to,\bot} \varphi^\dagger$, where $\varphi^\dagger$ is the $\mathcal{L}_{\to,\bot}$-translation of the $\mathcal{L}_{\neg,\lor}$-sentence φ, etc.)

5. (a) A mystery connective $\oplus$ has the following introduction and elimination rules:

   $$\dfrac{\varphi}{\varphi \oplus \psi}\ \oplus\text{I} \qquad\qquad \dfrac{\varphi \oplus \psi}{\psi}\ \oplus\text{E}$$

   Are these rules sound? Justify your answer.

   (b) A mystery connective $\ominus$ has the following introduction and elimination rules:

   $$\dfrac{\varphi}{\varphi \ominus \psi}\ \ominus\text{I} \qquad\qquad \dfrac{\varphi \ominus \psi}{\varphi}\ \ominus\text{E}$$

   Suggest truth rules for $\ominus$ for which these rules are sound.

   (c) Using your answers to Parts (a) and (b), and your knowledge of the rules of *ND*, suggest a rule of thumb that ensures that the introduction and elimination rules governing a connective are sound.

4.3 Optional exercises

1. (a) Check, by means of truth tables, that $(((P \to Q) \to P) \to P)$ is a tautology.
   (b) Give a natural deduction proof of $(((P \to Q) \to P) \to P)$.
   (c) Every valid argument involving only sentences containing $\wedge$ as their only connective can be proved valid using just the rules $\wedge$-Intro and $\wedge$-Elim. In that sense, $\wedge$ is completely characterised by its introduction and elimination rules. What do your results in Parts (a) and (b) show about $\to$ in this connection?

2. Devise introduction and elimination rules for a natural deduction system for the language $\mathcal{L}_{\uparrow}$, which has the Sheffer Stroke as its only connective. Prove that your system is sound and complete. You may prove these directly, or else use the fact that $ND_{\neg,\to}$, which is the fragment of ND governing $\neg$ and $\to$, is sound and complete for the language $\mathcal{L}_{\neg,\to}$.

3. Consider the language $\mathcal{L}_{\neg,\to}$, whose only connectives are $\neg$ and $\to$. We define the natural deduction system $ND_{\neg,\to}$ to be the fragment of ND governing $\neg$ and $\to$.

   We also define the natural deduction system $ND_{Ł}$ as follows. Its only rules are: $\to$-Elim (a.k.a. *Modus Ponens*); and that any instance of Łukasiewicz's three axiom schemata (see Parts (e)–(g) in Q1) may be discharged, which we could represent as follows:

$$[\varphi \to (\psi \to \varphi)]^{Ł1} \qquad [(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))]^{Ł2} \qquad [(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)]^{Ł3}$$
$$\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$$

   (a) Show that for any $\mathcal{L}_{\neg,\to}$-sentence φ, $\vdash_{Ł} \varphi \to \varphi$. ('$\vdash_{Ł}$' refers to proofs in $ND_{Ł}$). [*Hint:* Look at Eagle, p. 61 for inspiration.]
   (b) Show that for any $\mathcal{L}_{\neg,\to}$-sentence φ, $\vdash_{Ł} \varphi \to \alpha$, where α is any instance of one of Łukasiewicz's axiom schemata. [*Hint:* Ł1 will come in useful.]
   (c) Show that for any $\mathcal{L}_{\neg,\to}$-sentences φ and γ, where $\gamma \in \Gamma$, $\Gamma \vdash_{Ł} \varphi \to \gamma$.
   (d) Using your results in Parts (a)–(c), prove the *Deduction Theorem* for $ND_{Ł}$: i.e., $\Gamma, \varphi \vdash_{Ł} \psi$ iff $\Gamma \vdash_{Ł} \varphi \to \psi$. [*Hint:* for the left-to-right direction, Ł2 will come in useful.]
   (It follows that $\to$-Intro is a derived rule of $ND_{Ł}$.)
   (e) Show that $ND_{Ł}$ obeys *Cut*; i.e. if $\Gamma \vdash_{Ł} \varphi$ and $\Delta, \varphi \vdash_{Ł} \psi$, then $\Gamma, \Delta \vdash_{Ł} \psi$.
   (f) Show that *Ex Falso Quodlibet* (EFQ) is a derived rule of $ND_{Ł}$; i.e. $\varphi, \neg\varphi \vdash_{Ł} \psi$.
   (g) Show that *Double Negation Elimination* (DNE) is a derived rule of $ND_{Ł}$; i.e. $\neg\neg\varphi \vdash_{Ł} \varphi$. [*Hint:* Start with the fact that $\varphi, \neg\varphi \vdash_{Ł} \neg(\varphi \to \varphi)$, due to EFQ. You will also need your result in Part (a).]

(h) Show that $\varphi \to \neg\psi \vdash_{Ł} \psi \to \neg\varphi$.

(i) Show that $\varphi \to \neg\varphi \vdash_{Ł} \neg\varphi$.

(j) Show that $\neg$-Intro is a derived rule of $ND_{Ł}$; i.e. if $\Gamma, \varphi \vdash_{Ł} \psi$ and $\Delta, \varphi \vdash_{Ł} \neg\psi$, then $\Gamma, \Delta \vdash_{Ł} \neg\varphi$. [*Hint:* Use your result in Part (i).]

(k) Show that $\neg$-Elim is a derived rule of $ND_{Ł}$; i.e. if $\Gamma, \neg\varphi \vdash_{Ł} \psi$ and $\Delta, \neg\varphi \vdash_{Ł} \neg\psi$, then $\Gamma, \Delta \vdash_{Ł} \varphi$. [*Hint:* Use your result in Part (j) and DNE.]

(l) Using your results in Parts (d)–(k) and Q1 Parts (e)–(g), show that $ND_{\neg,\to}$ and $ND_{Ł}$ are equivalent; i.e., $\Gamma \vdash_{\neg,\to} \varphi$, iff $\Gamma \vdash_{Ł} \varphi$.

(m) Offer an argument that $ND_{Ł}$ is equivalent to Łukasiewicz's "Hilbert-style" axiomatic proof system, presented in Eagle, Section 4.5, and therefore (by Part (l)) that Łukasiewicz's proof system is equivalent to $ND_{\neg,\to}$.

5 Meta-theory for $\mathcal{L}_2$

5.1 Important concepts

These are the concepts introduced in this week's class and readings. Check that you understand them all.

- **Syntax of $\mathcal{L}_2$:** terms (constants and variables); universal and existential quantifiers; atomic formulae; the inductive definition of a (well-formed) formula; free vs. bound variables; open vs. closed formulae (sentences).
- **Semantics of $\mathcal{L}_2$:** $\mathcal{L}_2$-structures; domain; interpretation; semantic values; extensions; variable assignments over structures; the satisfaction of a formula by a variable assignment over a structure; the truth of a sentence in a structure.
- **Simple metatheorems of $\mathcal{L}_2$:** interdefinability of quantifiers; Substitution of Co-designating Terms; satisfaction of sentences in every variable assignment; introduction and elimination of the existential and universal quantifiers; Satisfaction of Sentences; Substitution of Equivalent Formulae.
- **Alternative semantics for $\mathcal{L}_2$:** Semantics for sentences only; substitutional quantification; truth defined for open formulae.

5.2 Required exercises

1. Briefly explain the difference between satisfaction and truth in $\mathcal{L}_2$.
2. Which of the following equivalence claims are true? For any invalid sequents, give a counterexample.
   (a) $\forall x(Px \wedge Qx) \equiv (\forall xPx \wedge \forall xQx)$
   (b) $\forall x(Px \vee Qx) \equiv (\forall xPx \vee \forall xQx)$
   (c) $\exists x(Px \wedge Qx) \equiv (\exists xPx \wedge \exists xQx)$
   (d) $\exists x(Px \vee Qx) \equiv (\exists xP^1x \vee \exists xQx)$
   (e) $\forall x\forall y(Px \to Qy) \equiv (\exists xPx \to \forall yQy)$
   (f) $\forall x\exists yRxy \equiv \exists y\forall xRxy$
   (g) $\forall x(Px \to \exists yRxy) \equiv \neg\exists x\forall y(Px \wedge \neg Rxy)$
3. Any binary relation R on a set S is *dense* iff, for any $a, b \in S$ such that $\langle a, b\rangle \in R$, there is some $c \in S$ such that both $\langle a, c\rangle \in R$ and $\langle c, b\rangle \in R$.
   (a) Show that if R is reflexive on S, then it is dense on S.
   (b) R is *idempotent* on S iff R is both dense and transitive on S. Give an example of: (i) a non-empty idempotent relation; and (ii) a non-empty, non-idempotent, transitive relation.

(c) R is a *strict dense order* on S iff it is both irreflexive and idempotent on S. Suppose there is a non-empty strict dense order on S. What is the minimum number of objects in S?

(d) Using the dictionary:

$$R^2 \ : \ \langle \ldots_1, \ldots_2 \rangle \in R$$

Write down an $\mathcal{L}_2$-sentence σ that is true in a structure iff the relation R is a strict dense order on that structure's domain.

(e) With σ the sentence above, show that

$$\nvDash (\sigma \wedge \exists x \exists y R^2 xy) \rightarrow \exists x \forall y \neg R^2 xy$$

[*Hint:* $<$ is a strict dense order on $\mathbb{Q}$, the set of rational numbers.]

(f) What do you think the prospects are for constructing an effective procedure which (correctly) answers whether $\vDash \exists x \forall y \varphi(x, y)$, where $\varphi(x, y)$ is an arbitrary $\mathcal{L}_2$-formula with only variables x and y free? What are the consequences for the decidability of validity in $\mathcal{L}_2$? [I'm not asking for a rigorous proof! An educated guess will do.]

4. (a) Show that $\mathcal{L}_2$ obeys $\exists$-Introduction and $\forall$-Elimination:
Let φ be any $\mathcal{L}_2$-formula in which only the variable v occurs free, and let τ be any constant. Then:

($\exists$-Intro) $\varphi[\tau/v] \vDash \exists v \varphi$.

($\forall$-Elim) $\forall v \varphi \vDash \varphi[\tau/v]$.

(b) Show that $\mathcal{L}_2$ obeys $\exists$-Elimination and $\forall$-Introduction:
Let Γ be any set of $\mathcal{L}_2$-sentences, let φ be any $\mathcal{L}_2$-formula in which at most the variable v occurs free, and let τ be any constant not occurring in Γ or φ. Then:

($\exists$-Elim) If $\Gamma, \varphi[\tau/v] \vDash \psi$, then $\Gamma, \exists v \varphi \vDash \psi$, where τ does not occur in ψ.

($\forall$-Intro) If $\Gamma \vDash \varphi[\tau/v]$, then $\Gamma \vDash \forall v \varphi$.

(c) Prove that, for any $\mathcal{L}_2$-formula φ, all $\mathcal{L}_2$-structures $\mathcal{A}$ and all variable assignments α over $\mathcal{A}$, $|\exists v \varphi|^\alpha_\mathcal{A} = |\neg \forall v \neg \varphi|^\alpha_\mathcal{A}$.

(d) Use your result to explain briefly the similarities between: (i) $\exists$-Introduction and $\forall$-Elimination; and (ii) $\exists$-Elimination and $\forall$-Introduction.

5. Suppose ψ and χ are any formulae, in which possibly (but not necessarily) $v_1, \ldots, v_n$ occur free. Let φ be any sentence and let $\Phi\tau_1 \cdots \tau_m$ be any atomic formula whose only free variables are among the $v_1, \ldots, v_n$. Using induction on the complexity of φ, establish the result that $\forall v_1 \cdots \forall v_n(\psi \leftrightarrow \chi) \vDash \varphi[\psi/\Phi\tau_1 \cdots \tau_m] \leftrightarrow \varphi[\chi/\Phi\tau_1 \cdots \tau_m]$.

6. (a) Which of the following are equivalence relations on the specified sets? Explain your answers.

i. The relation of *studying the same subjects as* on the set of Oxford undergraduates.
ii. The relation $\{\langle d, e\rangle : d \text{ is logically equivalent to } e\}$ on the set of $\mathcal{L}_1$-sentences.
iii. The relation $\{\langle d, e\rangle : d \text{ is logically equivalent to } e\}$ on the set of $\mathcal{L}_2$-formulae.
iv. The relation $\{\langle d, e\rangle : \text{if } \{d\} \text{ is consistent, then } \{d\}\cup\{e\} \text{ is consistent}\}$ on the set of $\mathcal{L}_1$-sentences.

(b) P is an *apartness* relation on $D_{\mathcal{A}}$ iff, for some equivalence relation R on $D_{\mathcal{A}}$, $\forall x \forall y(P^2xy \leftrightarrow \neg R^2xy)$ is true in some structure $\mathcal{A}$, where $I_{\mathcal{A}}(P^2) = P$ and $I_{\mathcal{A}}(R^2) = R$. Which minimal set of sentences from the following list characterises P as an apartness relation on $D_{\mathcal{A}}$ in structures $\mathcal{A}$ such that $I_{\mathcal{A}}(P^2) = P$? Justify your answer.

(A1) $\forall x \forall y(P^2xy \to \neg P^2yx)$
(A2) $\forall x \forall y(P^2xy \lor P^2yx)$
(A3) $\forall x \forall y(P^2xy \to P^2yx)$
(A4) $\forall x \forall y \forall z(P^2xy \to (P^2zx \lor P^2zy))$
(A5) $\forall x \forall y \forall z((P^2yx \land P^2zx) \to P^2yz)$
(A6) $\forall x \neg P^2xx$.

5.3 Optional exercises

1. We define a *finite universe* $\mathfrak{A}$ to be any ordered sequence $\mathfrak{A} = \langle F, \mathfrak{C}, \mathfrak{P}_0, \mathfrak{P}_1, \ldots, \mathfrak{P}_n\rangle$, where $n \in \mathbb{N}$ and:

   - F is a finite set, with cardinality N;
   - $\mathfrak{C}$ is a finite set of constants of $\mathcal{L}_2$, with cardinality c;
   - $\mathfrak{P}_0$ is a finite set of 0-place predicate letters of $\mathcal{L}_2$, with cardinality p_0;
   - $\mathfrak{P}_1$ is a finite set of 1-place predicate letters of $\mathcal{L}_2$, with cardinality p_1;

   $\vdots$

   - $\mathfrak{P}_n$ is a finite set of n-place predicate letters of $\mathcal{L}_2$, with cardinality p_n.

   Any $\mathcal{L}_2$-structure $\mathcal{A} = \langle D_{\mathcal{A}}, I_{\mathcal{A}}\rangle$ will be called $\mathfrak{A}$-*compatible* iff:

   - $F \subseteq D_{\mathcal{A}}$;
   - for all $\tau \in \mathfrak{C}$, $I_{\mathcal{A}}(\tau) \in F$;
   - for all $\Phi^1 \in \mathfrak{P}_1$, $I_{\mathcal{A}}(\Phi^1) \subseteq F$;

   $\vdots$

   - for all $\Phi^n \in \mathfrak{P}_n$, $I_{\mathcal{A}}(\Phi^n) \subseteq F^n$;

Any two $\mathfrak{A}$-*compatible* $\mathcal{L}_2$-structures will be called $\mathfrak{A}$-*equivalent* iff they agree on all their semantic values for the members of $\mathfrak{C} \cup \mathfrak{P}_0 \cup \mathfrak{P}_1 \cup \ldots \cup \mathfrak{P}_n$.

(a) Give an argument that $\mathfrak{A}$-equivalence is an equivalence relation on the set $\mathfrak{S}_{\mathfrak{A}}$ of all $\mathfrak{A}$-compatible $\mathcal{L}_2$-structures.

(b) Show that the number of $\mathfrak{A}$-equivalence classes of $\mathfrak{S}_{\mathfrak{A}}$ is

$$N^c \times 2^{p_0} \times 2^{p_1 N} \times \ldots \times 2^{p_n N^n} .$$

(c) Let $i : \mathfrak{C} \to F$ be some function from the constants in $\mathfrak{C}$ into the finite domain F. Consider the subset $\mathfrak{S}_{(\mathfrak{A},i)} \subseteq \mathfrak{S}_{\mathfrak{A}}$ of $\mathfrak{A}$-compatible $\mathcal{L}_2$-structures $\mathcal{A} = \langle D_{\mathcal{A}}, I_{\mathcal{A}} \rangle$ such that $I_{\mathcal{A}}(\tau) = i(\tau)$ for all $\tau \in \mathfrak{C}$.

Give an argument that the $\mathfrak{A}$-equivalence classes of $\mathfrak{S}_{(\mathfrak{A},i)}$ are in a one-to-one correspondence with rows of the truth table for some number M of sentence letters. What is the value of M?

(d) Show that it is decidable whether $\vDash \varphi$, where φ is any quantifier-free $\mathcal{L}_2$-sentence.

6 Natural deduction for $\mathcal{L}_2$

6.1 Important concepts

These are the topics and concepts introduced in this week's class and readings. Check that you understand them all. (Topics in parentheses are not crucial to the course.)

- **Natural deduction (ND_2) proofs in $\mathcal{L}_2$:** introduction and elimination rules for $\forall$ and $\exists$, and their conditions for application; uniqueness of $\exists$; the cut rule.
- **Soundness of ND_2 in $\mathcal{L}_2$:** steps in inductive proof of soundness governing $\forall$ and $\exists$.
- **Completeness of ND_2 in $\mathcal{L}_2$:** existential and universal completeness; witnesses; compactness via soundness, completeness and finitude of proofs.
- **Decidability:** prenex normal form (PNF); positive decidability of validity; negative undecidability of validity; (the halting problem; decidability of validity of quantifier-free sentences, $\exists$-sentences, $\forall\exists$-sentences and monadic sentences).

6.2 Required exercises

1. Show the following:
   (a) $\forall x Px \vdash \exists x Px$
   (b) $\exists x \forall y Rxy \vdash \forall y \exists x Rxy$
   (c) $\neg \exists x Px \vdash \forall x \neg Px$
   (d) $\exists x \neg Px \vdash \neg \forall x Px$
   (e) $\exists x Px \to Q \vdash \forall x (Px \to Q)$
   (f) $\forall x (Px \to Q) \vdash \exists x Px \to Q$
   (g) $\exists x (Px \to Q) \vdash \forall x Px \to Q$
   (h) $\forall x Px \to Q \vdash \exists x (Px \to Q)$
2. (a) Give a natural deduction proof that $Pa, \forall x(Px \to Qx) \vDash \exists x Qx$.
   (b) Give a natural deduction proof that $\exists x Qx, \forall x(Qx \to Rx) \vDash \exists x Rx$.
   (c) Give a natural deduction proof that $Pa, \forall x(Px \to Qx), \forall x(Qx \to Rx) \vDash \exists x Rx$.
   (d) Modify your proofs for Parts (b) and (c) so that your proof for Part (a) and your modified proof for Part (b) are both subproofs of your modified proof for Part (c).

(e) The *Cut Theorem* for any proof theory is: If $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$.

The proof of the Cut Theorem for *ND* (natural deduction for propositional logic $\mathcal{L}_1$) is straightforward: one can simply append any proof $\Gamma \vdash \varphi$ to any proof $\Delta, \varphi \vdash \psi$ to yield a proof $\Gamma, \Delta \vdash \psi$. However, the proof of the Cut Theorem for ND_2 is more delicate. Under what precise conditions would a simple appending of proofs *not* suffice? What, if anything, is the remedy?

(f) Prove the Cut Theorem for ND_2.

3. Prove the soundness of ND_2 for $\mathcal{L}_2$.

4. Explain why the inference rule

$$\frac{\exists v \varphi}{\varphi[\tau/v]}\ \exists\mathrm{E}?$$

is unsound with respect to the semantics of $\mathcal{L}_2$.

5. Prove, using natural deduction, the following claims.

(a) Any asymmetric relation is irreflexive.

(b) Any transitive, irreflexive relation is asymmetric.

6. Say that the binary formula φ, with variables x and y free, *expresses* the relation $\mathcal{R}$ in the structure $\mathcal{A}$ iff:

$|\varphi|^{\alpha}_{\mathcal{A}} = 1$ for all v.a.s α over $\mathcal{A}$ such that $\alpha(x) = a$ and $\alpha(y) = b$ iff $\langle a, b\rangle \in \mathcal{R}$.

Let $\mathcal{G}$ be an $\mathcal{L}_2$-structure in which the atomic binary formula Pxy expresses the relation $\mathcal{P} := \{\langle a, b\rangle \mid a \text{ is a parent of } b\}$ on the set of people.

(a) Write down a binary formula, containing the binary predicate letter P, which expresses the relation $\mathcal{P}_2 := \{\langle a, b\rangle \mid a \text{ is a grandparent of } b\}$ in $\mathcal{G}$.

(b) Write down a binary formula, containing the binary predicate letter P, which expresses the relation $\mathcal{P}_3 := \{\langle a, b\rangle \mid a \text{ is a great-grandparent of } b\}$ in $\mathcal{G}$.

Given any relation $\mathcal{R}$ on some set S, the *transitive closure* of $\mathcal{R}$, a.k.a. the *ancestral of* $\mathcal{R}$, which we will denote $\mathcal{R}^*$, is the smallest relation $\mathcal{Q}$ on S such that: (i) $\mathcal{R} \subseteq \mathcal{Q}$; and (ii) $\mathcal{Q}$ is transitive. ("Smallest" can be understood here as follows: $\mathcal{Q}$ is the relation satisfying the two conditions which is a subset of any other such relation.)

(c) Give a succinct description of the relation $\mathcal{P}^*$, where $\mathcal{P} := \{\langle a, b\rangle \mid a \text{ is a parent of } b\}$.

(d) Let $\mathcal{P}$ be the relation $\{\langle a, b\rangle \mid b = a+1\}$ on the set $\mathbb{N}$ of natural numbers. Give a succinct description of the relation $\mathcal{P}^*$.

(e) Is there a systematic way in $\mathcal{L}_2$ to define a binary formula, containing the binary predicate letter P, which expresses in $\mathcal{G}$ the ancestral of the relation that Pxy expresses in $\mathcal{G}$?

6.3 Optional exercises

1. A relation R on the set S is called *Euclidean* iff, for all $a, b, c \in S$: if $\langle a, b\rangle \in R$ and $\langle a, c\rangle \in R$, then $\langle b, c\rangle \in R$.

   R is called *serial* iff, for every $a \in S$, there is some $b \in S$ such that $\langle a, b\rangle \in R$.

   R is called *inverse serial* iff, for every $a \in S$, there is some $b \in S$ such that $\langle b, a\rangle \in R$.

   (a) Prove that a relation that is Euclidean and serial on S need not be an equivalence relation on S.

   (b) Prove, using natural deduction, that a relation is inverse serial and Euclidean on S iff it is an equivalence relation on S.

2. Prove the completeness of ND_2 for $\mathcal{L}_2$. [Warning: hard! Have a look at the references in Eagle if you get stuck.]

7 Meta-theory and natural deduction for $\mathcal{L}_=$

7.1 Important concepts

These are the topics and concepts introduced in this week's class and readings. Check that you understand them all. (Topics in parentheses are not crucial to the course.)

- **Syntax of $\mathcal{L}_=$:** the equality sign; (well-formed) formulae of $\mathcal{L}_=$.
- **Semantics of $\mathcal{L}_=$:** the identity relation; Theoremhood of Identity; Substitution of Co-Designating Constants; Identity is an Equivalence Relation; Leibniz' Law (the "indiscernibility of identicals").
- **Natural deduction (*$ND_=$*) proofs in $\mathcal{L}_=$:** introduction and elimination rules for =; dispensability of =-Elim-l/=-Elim-r; steps in inductive proof of soundness governing =.
- **Numerical quantification and definite descriptions:** expressing 'at least n', 'at most n' and 'exactly n' for all $n \in \mathbb{N}$; Russell's account of expressing 'the φ such that ψ'; (the notation $\imath v : \varphi$ and $\psi(\imath v : \varphi)$; the notation $\mathrm{I}v(\varphi, \psi)$; Strawson's account of definite descriptions and presupposition failure); scope ambiguities in sentences containing definite descriptions (e.g. $\mathrm{I}v(\varphi, \neg\psi)$ vs. $\neg\mathrm{I}v(\varphi, \psi)$).
- **Compactness and cardinality:** the indefinability of finitude in $\mathcal{L}_=$; (the Löwenheim-Skolem Theorem; Overspill).

7.2 Required exercises

1. Give proofs in $ND_=$ for the following claims:
   (a) $\exists x(x = a \wedge Px) \equiv Pa$
   (b) $\forall x(x = a \rightarrow Px) \equiv Pa$
   (c) $\models \exists x(x = a \wedge \forall y(y = a \rightarrow y = x))$
   (d) Identity is an equivalence relation on the domain of any structure.
2. Give an argument that we can do without constants altogether in $\mathcal{L}_=$ (although we *cannot* do without them in the proof theory $ND_=$), so long as we always adopt a suitable set of assumptions. [*Hint:* consider 1(a)–(c) above.]
3. Consider the following natural deduction rules for a new quantifier symbol, $\mho$:

$$\dfrac{\begin{matrix}[\varphi[\tau/v]]\\ \vdots \\ \kappa = \tau\end{matrix}}{\text{℩}v\varphi}\ \text{℩I}$$

Provided τ doesn't occur in any undischarged assumption other than $\varphi[\tau/v]$ in the proof of $\kappa = \tau$, neither κ nor τ appears in φ, and κ and τ are distinct terms.

$$\dfrac{\varphi[\tau/v] \qquad \varphi[\kappa/v] \qquad \text{℩}v\varphi}{\tau = \kappa}\ \text{℩E}$$

(a) Show that, if we add these two rules to $ND_=$, then $\text{℩}xPx$ is interderivable with $\exists x \forall y(Py \to y = x)$.

(b) What, heuristically, does $\text{℩}v\varphi$ mean? Suggest a semantics for $\text{℩}v\varphi$ such that the rules above are sound.

4. (a) Write down a sentence in $\mathcal{L}_=$ which expresses, given a suitable dictionary, the following:

    If there is exactly one natural satellite of Earth and exactly one UK Foreign Secretary, and no UK Foreign Secretary is a natural satellite of Earth, then there are exactly two things which are either a natural satellite of Earth or a UK Foreign Secretary.

   (b) Give a proof in $ND_=$ that this sentence is a tautology.

   (c) Let $\exists!_n v\varphi$ express the claim, 'There are exactly n things such that φ'. Offer a recursive definition of $\exists!_n v\varphi$ in terms of formulae of $\mathcal{L}_=$. [*Hint:* There are exactly $n + 1$ things with the property P iff there is something x such that: (i) x has the property P; and (ii) there are exactly n things distinct from x which have the property P.]

   (d) Offer an inductive proof that $\exists!_n v\varphi$ is satisfied by a variable assignment α over an $\mathcal{L}_=$-structure $\mathcal{A}$ iff there are exactly n variable assignments β over $\mathcal{A}$, which differ from α at most in what they assign to v, and which satisfy φ.

   (e) Using the "numerical quantifiers" $\exists!_n$, suggest an axiom schema that best expresses the arithmetical proposition, '$7 + 5 = 12$', such that every instance is an $\mathcal{L}_=$-tautology.

   (f) Does the collection of all axiom schemata of this kind (i.e. corresponding to each arithmetical proposition, '$\overline{m} + \overline{n} = \overline{m+n}$', where e.g. $\overline{m}$ denotes the numeral corresponding to the natural number m), each instance of which is a tautology of $\mathcal{L}_=$, constitute a reduction of arithmetic to logic? Justify your answer.

5. (a) Specify a set Δ_∞ of $\mathcal{L}_=$-sentences such that Δ_∞ is satisfied by a structure $\mathcal{A}$ iff $\mathcal{A}$ has an infinite domain. *Show* that Δ_∞ has this property.

(b) For each $n \in \mathbb{N}$, let M_n be an $\mathcal{L}_=$-sentence expressing that there are at most n objects. Give an informal argument for the truth or falsity of the following claims:

   i. For any finite subset $\Delta \subset \Delta_\infty$, there is some n such that $\Delta \cup \{M_n\}$ is satisfiable.

   ii. There is some n such that, for any finite subset $\Delta \subset \Delta_\infty$, $\Delta \cup \{M_n\}$ is satisfiable.

   iii. $\Delta_\infty \cup \{M_n\}$ is unsatisfiable, for any n.

(c) Is there a set Δ_f of $\mathcal{L}_=$-sentences such that Δ_f is satisfied by a structure $\mathcal{A}$ iff $\mathcal{A}$ has a finite domain? Justify your answer.

(d) Give an argument that, if the answer to the question above is No, then: (i) there is no sentence σ_f which is satisfied by a structure $\mathcal{A}$ iff $\mathcal{A}$ has a finite domain; and (ii) there is no sentence σ_∞ which is satisfied by a structure $\mathcal{A}$ iff $\mathcal{A}$ has an infinite domain.

7.3 Optional exercises

1. (*From the 2016 Prelims paper. Numbers in square brackets indicate the marks available for the corresponding question.*)

   (a) Formalize the following as a valid argument in $\mathcal{L}_=$. Comment on any points of interest in your formalization. Provide a natural deduction proof demonstrating the argument's validity.

   > If neo-Lockean views about personal identity are correct, then, if I undergo personal fission, no one who exists after the experiment will be identical to me. The reason is that, if I undergo personal fission, there will be exactly two people who exist after the experiment who are psychologically continuous with me. And, if the neo-Lockeans are right, someone is identical to me only if they're psychologically continuous with me. However, on the neo-Lockean view, for any two people who are psychologically continuous with me, one of them is identical to me if and only if the other one is too. [17]

   (b) Using the dictionary "P : ... is a potato", translate the following sentences of $\mathcal{L}_=$ into idiomatic English.

   i. $\exists x \exists y(\neg x = y \wedge \forall z((Pz \wedge \neg \exists x_1(Px_1 \wedge \neg z = x_1)) \leftrightarrow (x = z \vee y = z)))$

   ii. $\exists x \exists y(\neg x = y \wedge \forall z((Pz \wedge \exists x_1 \forall y_1((Py_1 \wedge \neg z = y_1) \leftrightarrow y_1 = x_1)) \leftrightarrow (x = z \vee y = z)))$

   For each sentence, either provide an informal argument to show that it is inconsistent, or provide a model. [8]

2. (a) Propose sound natural deduction rules for the "exactly one" quantifier $\exists!$, which is governed by the following satisfaction rules:

$|\exists! v\varphi|^{\alpha}_{\mathcal{A}} = 1$ iff $|\varphi|^{\beta}_{\mathcal{A}} = 1$ for *exactly one* variable assignment β over $\mathcal{A}$ differing from α in *at most* in its assignment to v.

(b) Show that, if we add these rules to $ND_{=}$, then $\exists! xPx$ is inter-derivable with $\exists x \forall y(Py \leftrightarrow y = x)$.

8 Logic and natural language

8.1 Important concepts

The following concepts constitute a mild excursion into the philosophy of logic and language. While, strictly speaking, nothing here is examinable, a cursory appreciation of the issues are likely to help with the rest of the course.

- **Conditionals:** indicative vs. subjunctive conditionals; the material conditional as an adequate formalisation of the natural language indicative conditional; modus ponens and conditional proof.
- **Gricean ideas:** Gricean maxims; Gricean implicature: conventional (detachable) vs. conversational (cancellable).
- **Entailment:** the paradoxes of entailment (explosion and LEM); disjunctive syllogism and its discontents (relevance logic, dialethic logic, inconsistent belief).
- **Designation:** direct reference (the meaning of a constant is the object it denotes); informative identities; empty names; transparent vs. opaque contexts; indexicals; non-count nouns; descriptions.
- **Properties and relations:** extension vs. intension; sparse vs. abundant properties.

8.2 Required exercises

1. (a) Explain the difference between natural language indicative and counterfactual conditionals.
   (b) Is $\rightarrow$ an adequate formalisation of the natural language indicative conditional? Is it an adequate formalisation of the natural language counterfactual conditional?
   (c) Suppose we introduce a new logical symbol, $\Box\!\!\rightarrow$, which is intended to represent the counterfactual conditional. Construct a suitable semantics for $\Box\!\!\rightarrow$. [Hard without help! Have a think—then, once you get stuck, look at Sider's *Logic for Philosophy*.]
2. (a) Consider this rule of inference in English: if φ is a truth of logic, then 'Necessarily, φ' is too. Is this rule intuitively correct? Is the corresponding sequent '$\varphi \vDash_{\text{English}}$ Necessarily φ' correct?
   (b) Is it possible to formalise claims of necessity and possibility in $\mathcal{L}_=$? If so, how? If not, why not? In the latter case, what would need to be added to $\mathcal{L}_=$ such that it would be able to formalise adequately such claims?
3. Explain why maximal consistent sets are sometimes regarded as being good stand-ins for possible worlds.

4. What kinds of natural language phenomena could a three-place connective formalise? Construct a suitable semantics for your favourite three-place connective.

5. What is wrong with the following argument that reflexivity is a consequence of symmetry and transitivity?

   > If $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$ since we assume R is symmetric. If both $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$, then since R is transitive, $\langle x, x\rangle \in R$—so R is reflexive.

6. We might normally expect 'is similar to' to be a symmetric relation: after all, if there is a respect in which *a* is similar to *b*, then *b* must be similar to *a* in that very same respect. But many people seem to judge that similarity is *not* symmetric:

   > When people are asked to make comparisons between a highly familiar object and a less familiar one, their responses reveal a systematic asymmetry: The unfamiliar object is judged as more similar to the familiar one than vice versa. For example, people who know more about the USA than about Mexico judge Mexico to be more similar to the USA than the USA is to Mexico. (Kunda 1999, p. 520)

   Can you provide a rationale behind these psychological results? Do they indicate that people are systematically mistaken about the meaning of the relational predicate 'is similar to', or do they indicate that our theory of similarity in terms of matching respects of similarity is incorrect?