Discrete General Covariance

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Outline

1. What is special about GR?
   - General Covariance? (20 min)
   - Diffeomorphism Invariance?
   - Background Independence?
2. Review of Nyquist Shannon Sampling Theory (10 min)
   - Bandlimited Functions
   - Uniform Sampling
   - Non-uniform Sampling
3. Discrete General Covariance (20 min)

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I will take questions after each part. Please save major questions for then.

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How do these three concepts differ and how are they related to each other?

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C) Background independence is what makes GR special.

(Continuous) General Covariance

Example 1) 2D Klein Gordon Equation:

$$\partial_t^2 \phi(t, x, y) = (\partial_x^2 + \partial_y^2 - M^2)\, \phi(t, x, y) \tag{1}$$

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This formulation is not generally covariant. For instance, written in terms of the coordinates ($t' = t$, $x' = x + \frac{1}{2} a\, t^2$, $y' = y$) we have,

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To arbitrary coordinates x'^μ

$$(\eta^{\sigma\rho}\frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\rho}\partial_\mu\partial_\nu - M^2)\,\phi + \eta^{\sigma\rho}\frac{\partial^2 x'^\mu}{\partial x^\sigma\,\partial x^\rho}\,\partial_\mu\phi = 0. \tag{4}$$

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Consider the space of kinematically possible models (KPMs) given by:

$$\text{KPMs:} \quad \langle \mathcal{M}, \eta^{\mathsf{ab}}, \phi \rangle \tag{5}$$

where $\mathcal{M}$ is a differentiable (2+1)-manifold, η^{ab} is a fixed metric field with signature $(-1, 1, 1)$ and $\phi : \mathcal{M} \to \mathbb{R}$ is a scalar field.

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Consider the dynamically possible models (DPMs) picked out by

$$\text{DPMs:} \quad (\eta^{\mathsf{ab}} \nabla_{\mathsf{a}} \nabla_{\mathsf{b}} - M^2)\, \phi = 0 \tag{6}$$

where ∇_{a} is the unique derivative compatible with the metric, i.e. with $\nabla_{\mathsf{c}}\, \eta^{\mathsf{ab}} = 0$.

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We now have the Klein Gordon equation in a generally covariant form:

$$\text{SR1} \qquad \begin{aligned} &\text{KPMs:} \quad \langle \mathcal{M}, \eta^{\mathsf{ab}}, \phi \rangle \quad \text{with } \eta^{\mathsf{ab}} \text{ fixed,} \\ &\text{DPMs:} \quad (\eta^{\mathsf{ab}} \nabla_{\mathsf{a}} \nabla_{\mathsf{b}} + M^2)\, \phi = 0. \end{aligned} \tag{7}$$

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Given a generic diffeomorphism $d \in \mathrm{Diff}(\mathcal{M})$ and a solution $\langle \mathcal{M}, \eta^{\mathrm{ab}}, \phi \rangle$,

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$$\text{Continuum Heat} \qquad \text{KPMs:} \quad \langle \mathcal{M}, t_{\mathsf{ab}}, h^{\mathsf{ab}}, \nabla_{\mathsf{a}}, T^{\mathsf{a}}, \psi \rangle \tag{8}$$

h^{ab} and t_{ab} are space and time metrics with signatures $(0, 1, 1)$ and $(1, 0, 0)$ respectively. ∇_{a} is a derivative operator which is compatible with these metrics and flat (i.e., with $R^{\mathsf{a}}{}_{\mathsf{bcd}} = 0$). T^{a} is a constant unit time-like vector field which picks out a standardized way of moving forward in time (i.e, translation generated by $T^{\mathsf{a}} \nabla_{\mathsf{a}}$).

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The DPMs are picked out by:

$$\text{Continuum Heat} \qquad \text{DPMs:} \quad T^{\text{a}} \, \nabla_{\text{a}} \psi = \alpha \, h^{\text{bc}} \nabla_{\text{b}} \nabla_{\text{c}} \psi \tag{9}$$

(Continuous) General Covariance: Newtonian Gravity

Repeating this process for Newtonian Gravity we have

$$\text{Newtonian Gravity} \qquad \text{KPMs:} \quad \langle \mathcal{M}, t_{ab}, h^{ab}, \nabla_a, \varphi, \Phi \rangle \tag{10}$$
$$\text{DPMs:} \quad h^{bc}\nabla_b\nabla_c\varphi = 4\pi G\,\rho$$
$$u^a\nabla_a u^b = -h^{bc}\nabla_c\varphi$$

where φ is the gravitational potential and Φ is a stand in for the matter content of the theory (ρ is calculated from Φ somehow). u^a is the 4-velocity of a test particle (normalized and time-like).

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Note there is no time-like vector field T^a assumed here.
This theory is has the Galilean symmetry group.

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In a coordinate-independent framing, there are no passive symmetry transformations. The symmetry of a theory is just the subset of the diffeomorphisms which map solutions to solutions.

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This is not right. We can reformulate special relativity to be diffeomorphism invariant as

$$\begin{aligned} \text{SR2} \qquad \text{KPMs:} \quad & \langle \mathcal{M}, g^{ab}, \phi \rangle, \\ \text{DPMs:} \quad & (g^{ab}\nabla_a \nabla_b - M^2)\,\phi = 0 \\ & R^a{}_{bcd} = 0. \end{aligned} \tag{11}$$

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Note g^{ab} is not a fixed field, it is dynamical.

Given a generic diffeomorphism $d \in \text{Diff}(\mathcal{M})$ and a solution $\langle \mathcal{M}, g^{ab}, \phi \rangle$ we do in fact have that $\langle \mathcal{M}, d^* g^{ab}, d^* \phi \rangle$ is a solution.

Compare SR2 with GR

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with GR,

$$\begin{aligned} \text{GR} \qquad \text{KPMs:} \quad & \langle \mathcal{M}, g^{ab}, \phi \rangle, & (13) \\ \text{DPMs:} \quad & (g^{ab}\nabla_a \nabla_b - M^2)\, \phi = 0 & (14) \\ & G_{ab} = 8\pi\, T_{ab}. \end{aligned}$$

SR2 has background structure whereas GR does not.

Background Independence

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Let's move on to Part 2 of the presentation. Questions before we do?

Discrete Background Independence?

Can we extend these notions to discrete-space (e.g., lattice) theories?

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Do lattices always break continuous symmetries?
Translations, rotations, Galilean boosts, Lorentzian boosts, etc.

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Do lattices always break continuous symmetries?
Translations, rotations, Galilean boosts, Lorentzian boosts, etc.

To answer this question it would be very helpful to have a notion of discrete general covariance.

Discrete General Covariance

Inspired by the work of Achim Kempf,[1][2] I suggest the following analogy:

Coordinate Systems	$\leftrightarrow$	Lattice Structure
Changing Coordinates	$\leftrightarrow$	Nyquist-Shannon Resampling
Gen. Covariant Formulation	$\leftrightarrow$	Bandlimited Formulation

[1] Achim Kempf, New J. of Physics, Volume 12, November 2010. arXiv:1010.4354
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Before jumping into this, we need to review Sampling Theory.

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Part 2: Review of Nyquist Shannon Sampling Theory

A bandlimited function is one whose Fourier transform has compact support. That is, a function $f_{\mathrm{B}}(x)$ is bandlimited with bandwidth K iff $\mathcal{F}_k[f_{\mathrm{B}}(x)]$ has support only for wavenumbers $|k| \leq K$.

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The Nyquist Shannon Sampling Theorem tells us that we can **exactly** reconstruct any bandlimited function knowing only its values at a sufficiently dense set of sample points.

How does that work?

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Suppose we know $f_n = f_{\rm B}(x_n)$ at the regularly spaced sample points $x_n = n\,a$ and that $f_{\rm B}$ is bandlimited with bandwidth K.

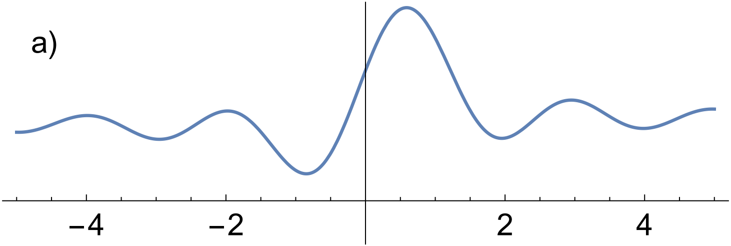
The following reconstruction,

$$f_{\rm B}(z) =^? \sum_{n=-\infty}^{\infty} S_n(z/a)\; f_n; \quad S(y) = \frac{\sin(\pi y)}{\pi y}, \quad S_n(y) = S(y-n). \quad (15)$$

is **exact** when our sample points are sufficiently dense
(here meaning $a \leq a^* = \pi/K$).

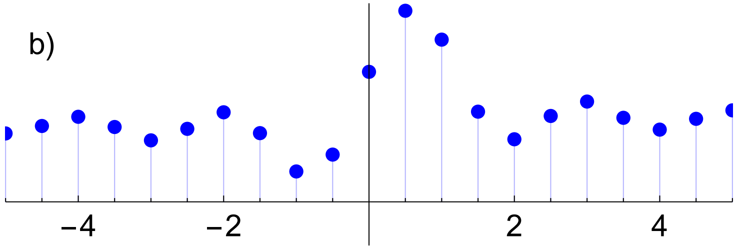
Exact Example

Consider that $f_{\mathrm{B}}(x) = 1 + S(x - 1/2) + x\,S(x/2)^2$ has a bandwidth of $K = \pi$ and so a critical sample spacing of $a^* = 1$



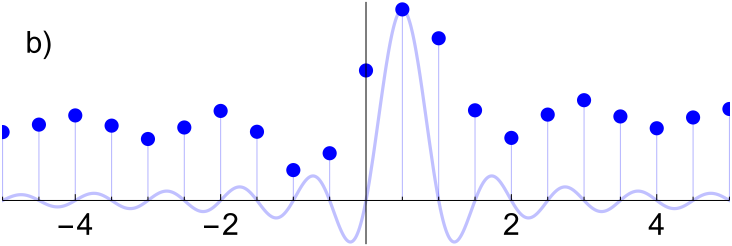
Exact Example

We can recover $f_B(x)$ exactly knowing only its values at $x_n = n\,a$ with $a = 1/2 < a^* = 1$



Exact Example

To recover $f_B(x)$ we associate each x_n with a shifted and rescaled sinc function as



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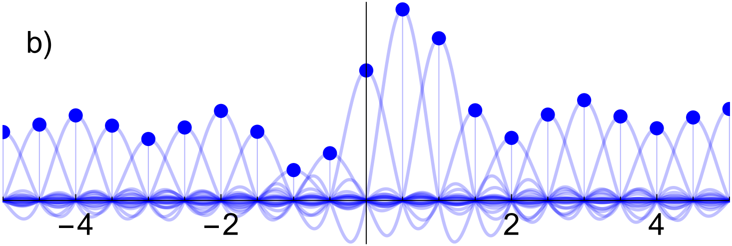
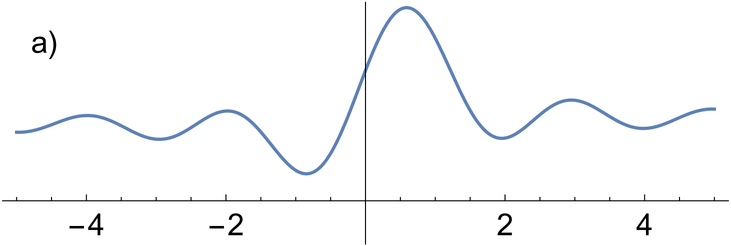


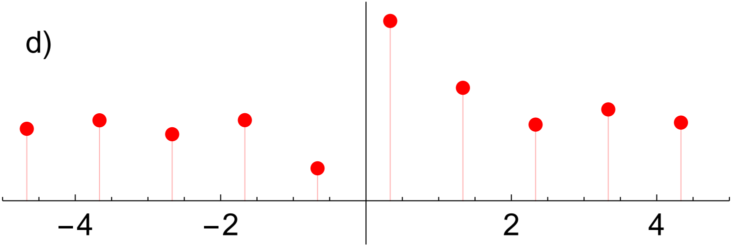
Exact Example

Adding together all of these sinc functions gives back $f_{\mathrm{B}}(x)$ with no approximation



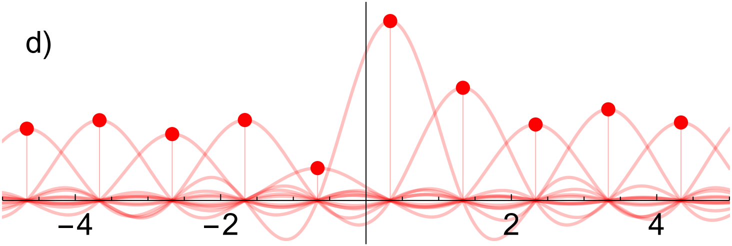
Exact Example

We oversampled in the previous example. We can recover $f_{\mathrm{B}}(x)$ exactly knowing only its values at $x_n = n\,a + 1/3$ with $a = a^* = 1$



Exact Example

Just as before we recover $f_{\mathrm{B}}(x)$ by associating each x_n with a shifted and rescaled sinc function as



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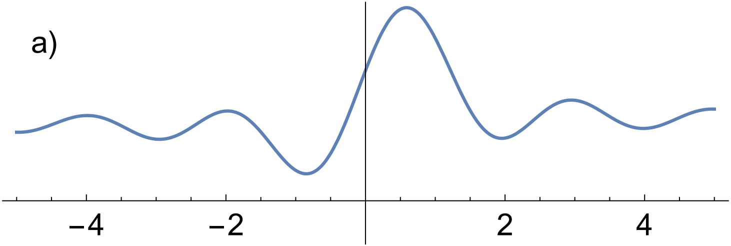

Non-uniform Sampling

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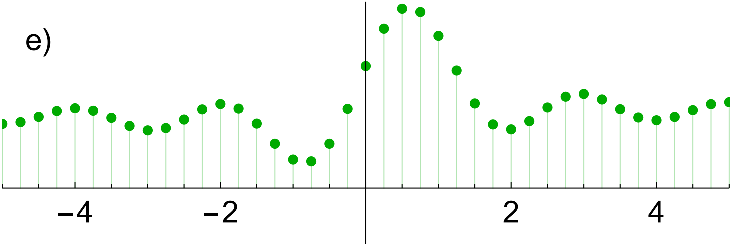
Let's see how this works.

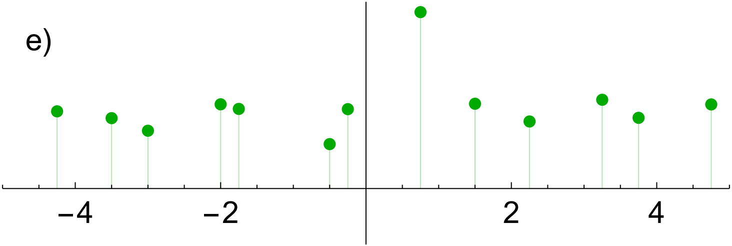
Exact Example

Consider the following oversampling of $f_{\mathrm{B}}(x)$ with $a = 1/4 < a^* = 1$. We do not need all of these sample points to reconstruct (we need approximately one quarter of them).



Exact Example

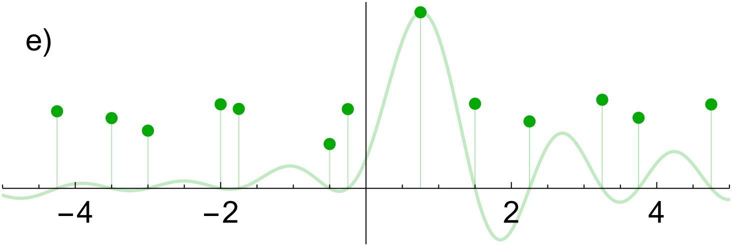
The samples which we drop do not need to be selected uniformly. The following non-uniform sampling works,



Exact Example

The reconstruction function for each sample point is now more complicated. But ultimately,

$$f_{\rm B}(z) = \sum_{m=-\infty}^{\infty} G_m(z; \{x_n\})\, f_{\rm B}(x_m) \tag{16}$$



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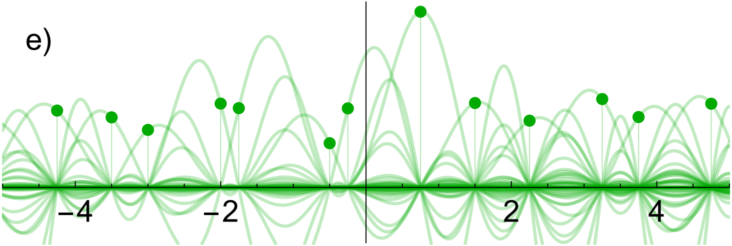

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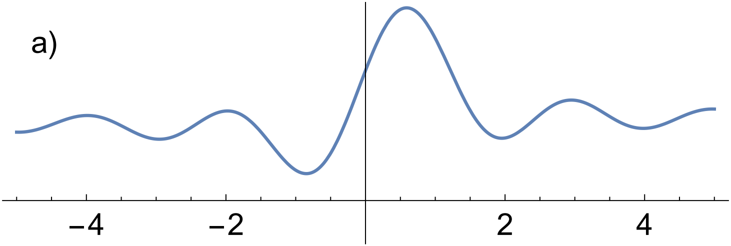

Higher Dimensions

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Consider $f_{\mathrm{B}}(x,y) = J_1(\pi\, r)/(\pi\, r)$ where J_1 is the first Bessel function and $r = \sqrt{x^2+y^2}$. This function is bandlimited with $\sqrt{k_x^2+k_y^2} \leq K = \pi$.

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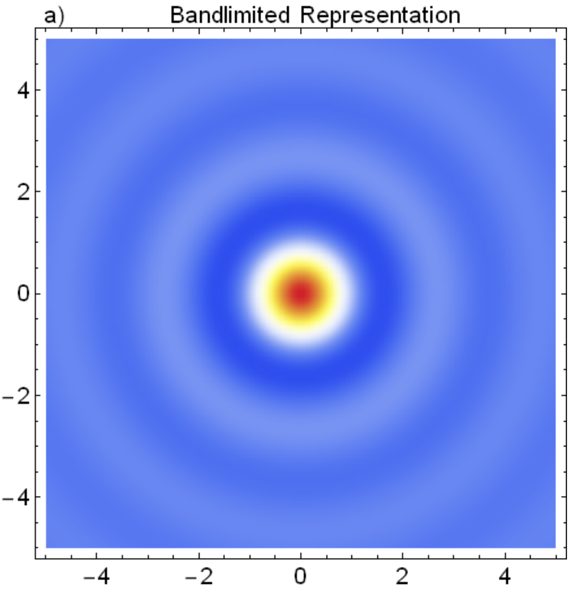
The following figures are all equivalent representations of $f_{\mathrm{B}}(x, y)$

Higher Dimensions



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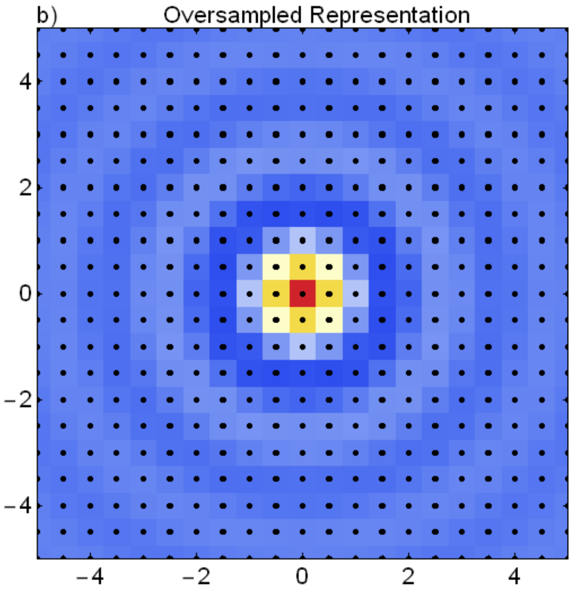

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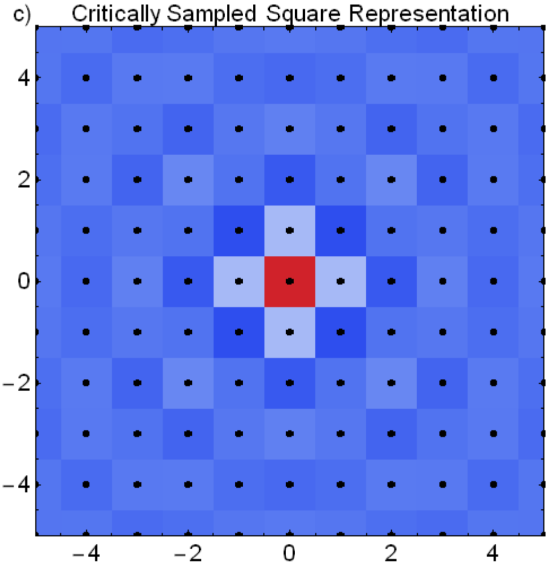

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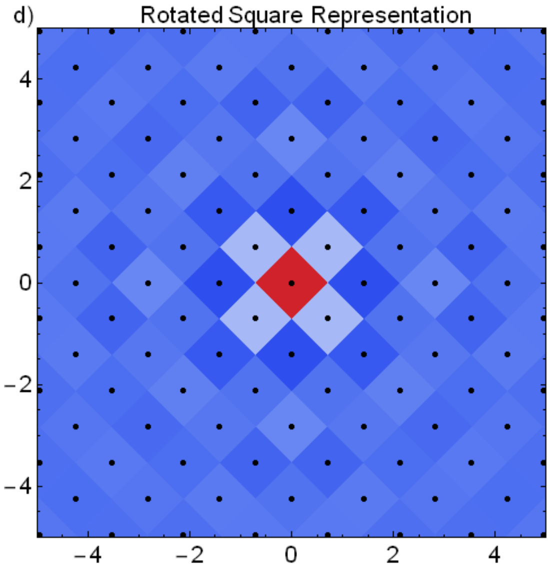

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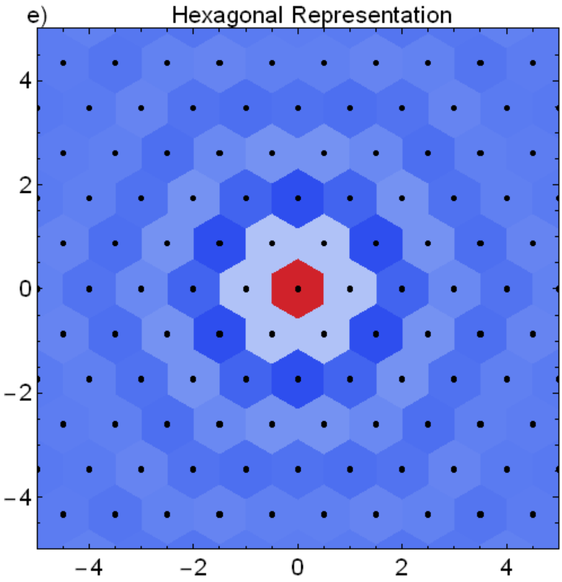

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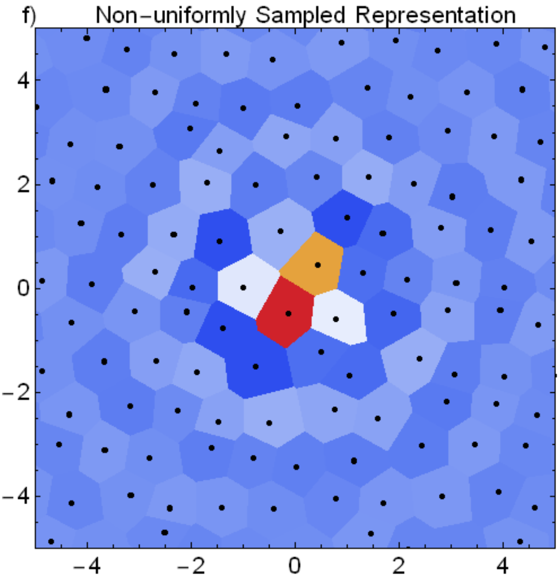

Let's Review

What is remarkable about bandlimited functions is that they have a finite density of degrees of freedom, but these degrees of freedom have no fixed definite location[3].

[3] Achim Kempf, New J. of Physics, Volume 12, November 2010. arXiv:1010.4354

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Moreover, we have near total freedom in how to pick our sample points.

Questions before we move on to Part 3?

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Part 3: Discrete General Covariance

So far we have started with a bandlimited function and induced discrete lattice representations from it.

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Next, we will do some physics by adding dynamics. We will start from a lattice-formulation and from it find a bandlimited formulation.

Recall the proposed analogy:

Coordinate Systems	$\leftrightarrow$	Lattice Structure
Changing Coordinates	$\leftrightarrow$	Nyquist-Shannon Resampling
Gen. Covariant Formulation	$\leftrightarrow$	Bandlimited Formulation

Example: 1D Nearest-Neighbor Heat Equation

Consider the 1D nearest-neighbor heat equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_n(t) = \alpha\,\frac{\psi_{n+1}(t) - 2\psi_n(t) + \psi_{n-1}(t)}{a^2} \tag{19}$$

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or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}(t) = \frac{\alpha}{a^2}\,\Delta^2_{(1)}\boldsymbol{\psi}(t) \tag{20}$$

where $\Delta^2_{(1)}$ is the nearest neighbor approximation to the second derivative and $\boldsymbol{\psi}(t) = (\dots, \psi_{-1}(t), \psi_0(t), \psi_1(t), \dots)$.

Example: 1D Nearest-Neighbor Heat Equation

At each time we can take these discrete values $\psi_n(t)$ and imagine them as samples which are drawn from a bandlimited function ψ_B as,

$$\psi_n(t) = \psi_\text{B}(t, x_n), \qquad x_n = n\,a. \tag{21}$$

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We can then use these samples to reconstruct $\psi_\mathrm{B}(t, x)$ as

$$\psi_\mathrm{B}(t, x) = \sum_{n=-\infty}^{\infty} S_n(x/a)\; \psi_n(t). \tag{22}$$

Adding Dynamics

In addition to moving the state-of-the-world at each time into the bandlimited setting we can also move the dynamics,

$$\frac{\partial}{\partial t}\psi_{\mathrm{B}}(t,x) = \sum_n S_n(x)\ \frac{\mathrm{d}}{\mathrm{d}t}\psi_n(t) \tag{23}$$

$$= \dots$$

$$= \frac{\alpha}{a^2}\,\frac{\cosh(a\,\partial_x)-1}{1/2}\ \psi_{\mathrm{B}}(t,x)$$

The complicated cosh term is the continuum analog of $\Delta^2_{(1)}$. Note $\exp(a\,\partial_x)f(x) = f(x+a)$.

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$$\frac{\cosh(a\,\partial_x) - 1}{a^2/2} = \partial_x^2 + \frac{a^2}{12}\partial_x^4 + O(a^4) \tag{24}$$

Three Different Dynamics

H1:
$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}(t) = \frac{\alpha}{a^2}\,\Delta^2_{(1)}\,\boldsymbol{\psi}(t) \tag{25}$$
$$\partial_t \psi_{\mathrm{B}}(t,x) = \frac{\alpha}{a^2}\,\frac{\cosh(a\,\partial_x) - 1}{1/2}\;\psi_{\mathrm{B}}(t,x)$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\alpha}{a^2}\,\Delta^2_{(2)}\,\psi(t) \tag{26}$$
$$\partial_t\psi_{\mathrm{B}}(t,x) = \frac{\alpha}{a^2}\,\frac{-\cosh(2a\,\partial_x)+16\cosh(a\,\partial_x)-15}{6}\,\psi_{\mathrm{B}}(t,x)$$

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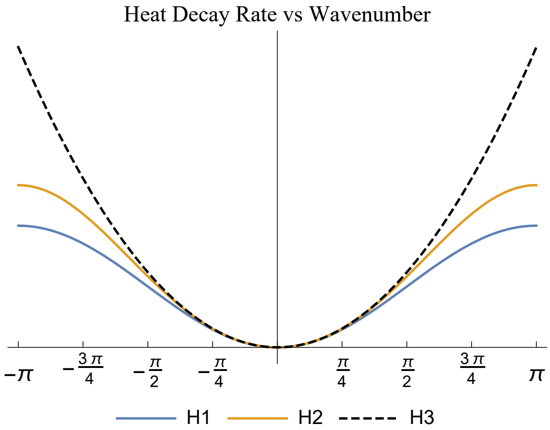
H3: $$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\alpha}{a^2}\,D^2_{\mathrm{B}}\,\psi(t) \tag{27}$$
$$\partial_t\psi_{\mathrm{B}}(t,x) = \alpha\,\partial^2_x\psi_{\mathrm{B}}(t,x)$$

where $D^2_{\mathrm{B}} = \lim_{n\to\infty}\Delta^2_{(n)}$ is the infinite-range derivative approximation.

In each of these cases the eigensolutions are planewaves, with $|k| \leq K$, which decay exponentially at some rate.



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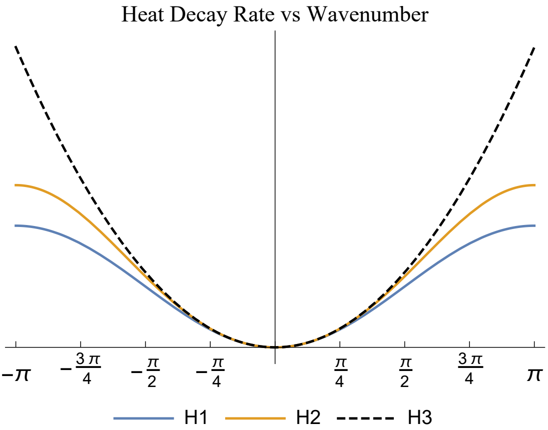
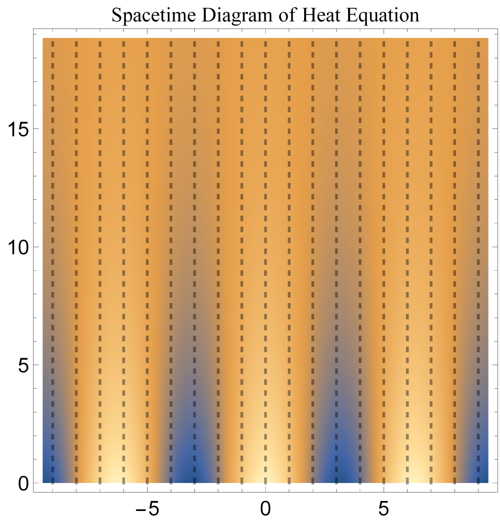


Lesson 1: There is no reason that a lattice theory needs to have different dynamics than the continuum theory (at least not below the bandwidth, K).

Dynamic Resampling



We can plot the discrete values $\psi_n(t)$ and the bandlimited function $\psi_{\mathrm{B}}(t, x)$ in a spacetime diagram.

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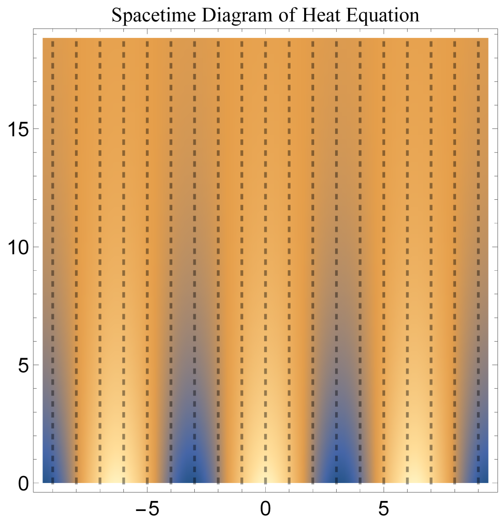
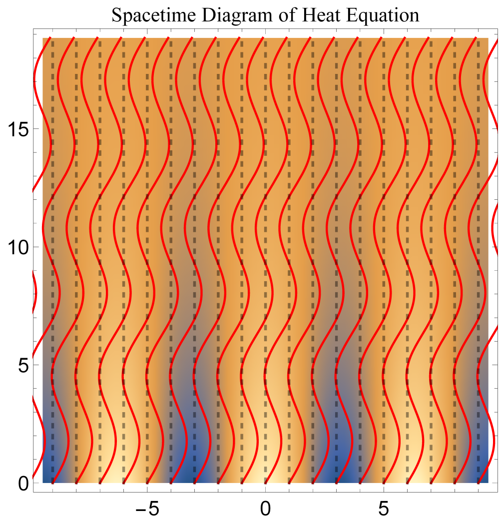


We can plot the discrete values $\psi_n(t)$ and the bandlimited function $\psi_{\text{B}}(t, x)$ in a spacetime diagram.

We can then pick new sample point at each time.

Resampling and Symmetry



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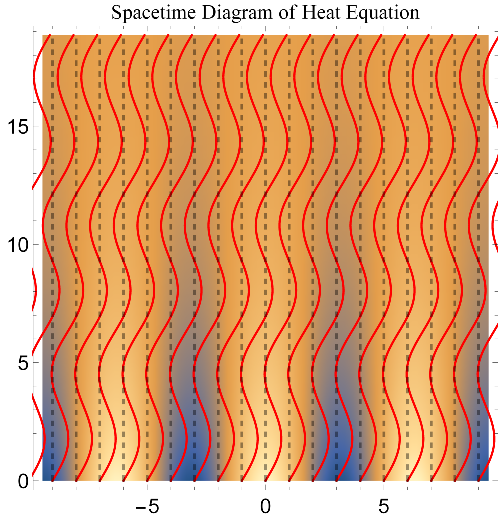
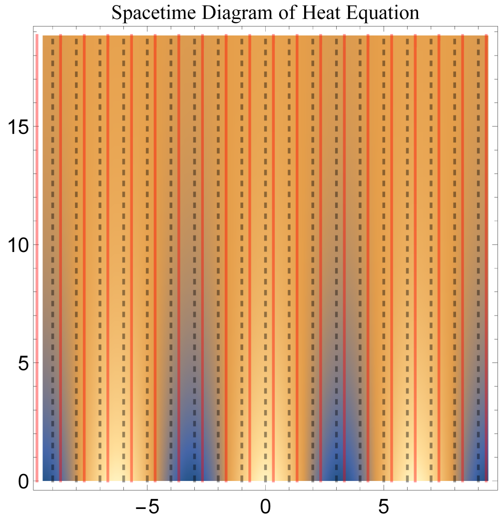


We can pick new sample point at each time.

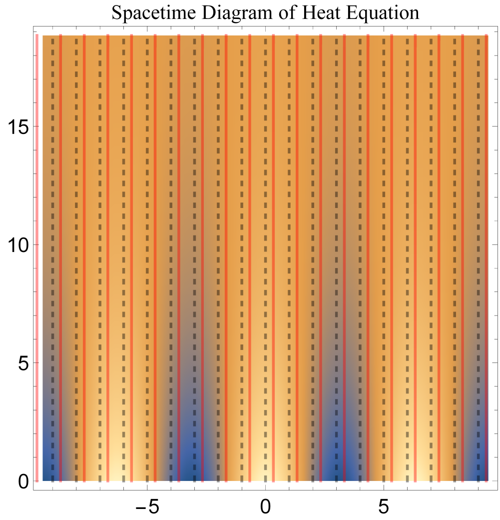
These new sample values will not obey the same equation that the old ones did. But they are completely sufficient to represent the dynamics.

Resampling and Symmetry



What about this resampling? Do the shifted red sample values obey the same equations as the original dashed sample points?

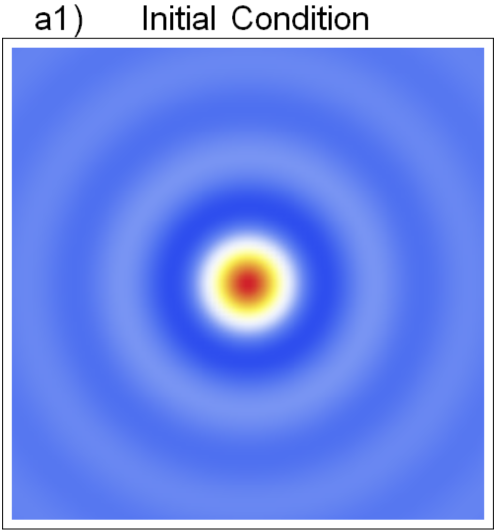
Resampling and Symmetry



What about this resampling? Do the shifted red sample values obey the same equations as the original dashed sample points?

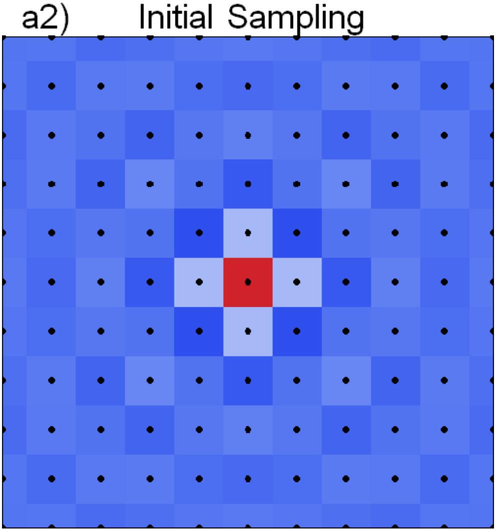
Indeed they do. Lesson 2: There is no reason that a lattice theory can't have a continuous symmetry.

Another Example: 2D Heat Equation



Consider this initial condition for the 2D Heat Equation.

Another Example: 2D Heat Equation



We can sample this initial condition and then evolve it via one of our discrete dynamical equations.

Nearest Neighbor Der.:
H4 : via $\Delta^2_{(1),x} + \Delta^2_{(1),y}$

Bandlimited Derivative:
H5 : via $D^2_{\mathrm{B,x}} + D^2_{\mathrm{B,y}}$.

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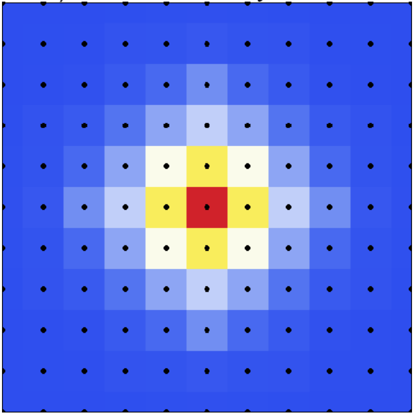

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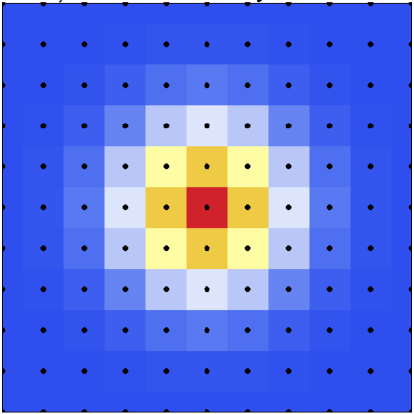

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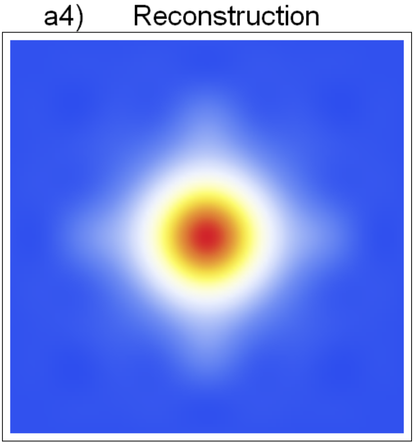

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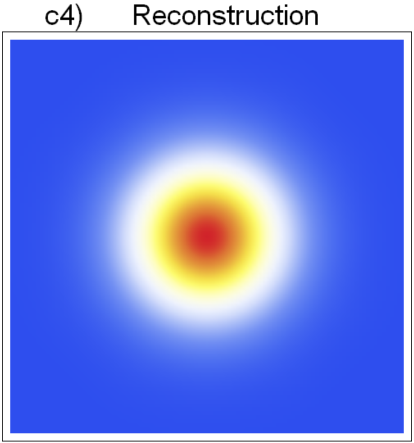

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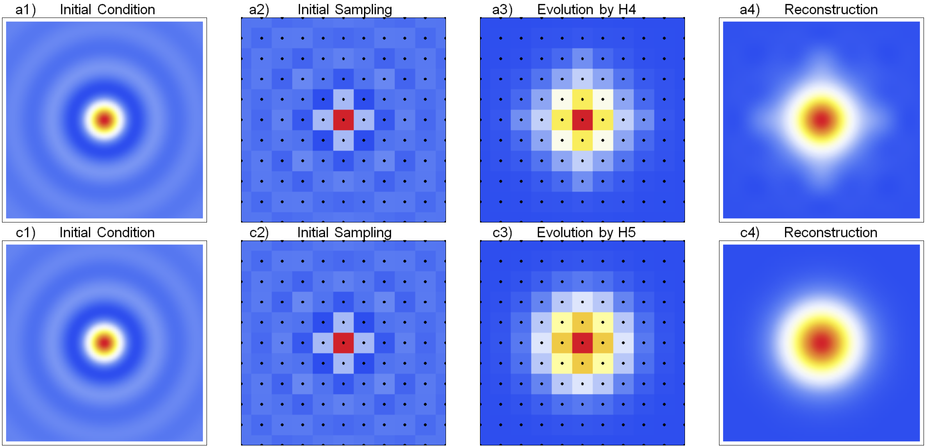

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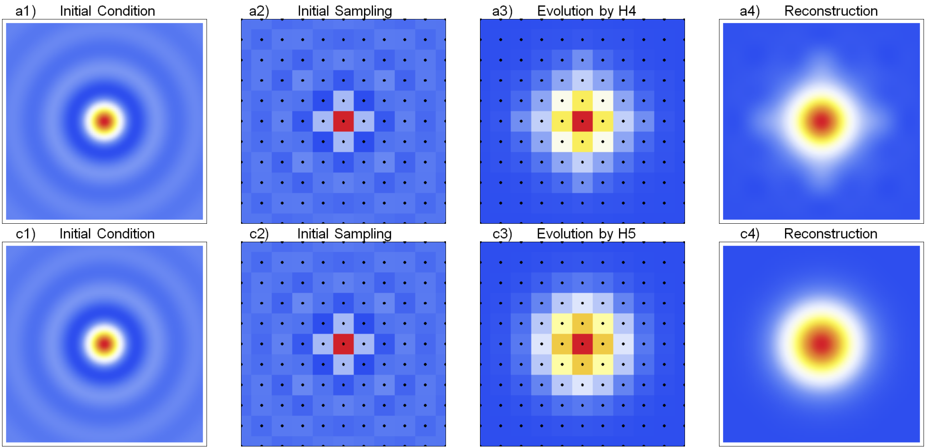
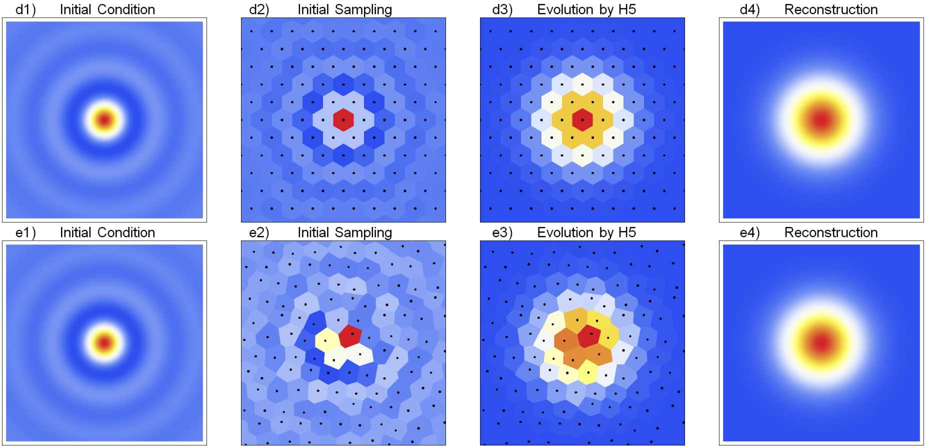


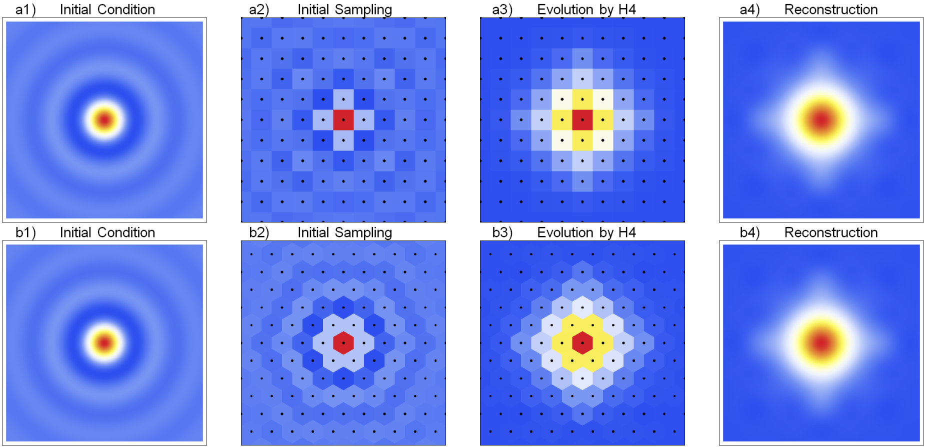
<u>Lesson 2:</u> We can have rotation invariant dynamics described in terms of a square lattice.

Another Example: 2D Heat Equation



Lesson 2: We can have rotation invariant dynamics described in terms of any lattice.

Another Example: 2D Heat Equation



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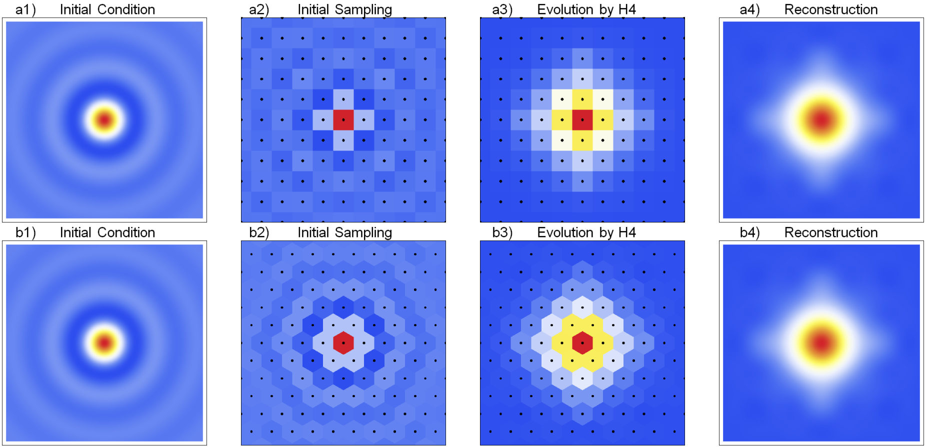

Lesson 3: The 4-fold symmetry of the H4 dynamics has nothing to do with the dynamics being represented in terms of a square lattice.

The discrete 2D heat equation on a square lattice,

$$\text{H5:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\alpha}{a^2}\left(D_{\mathrm{B,x}}^2 + D_{\mathrm{B,y}}^2\right)\psi(t), \tag{28}$$

Bandlimited and Generally Covariant Heat Equation

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We can further reformulate this in a generally covariant way as,

$$\text{H5} \qquad \text{KPMs:} \quad \langle \mathcal{M}, t_{\mathrm{ab}}, h^{\mathrm{ab}}, \nabla_{\mathrm{a}}, T^{\mathrm{a}}, \psi_{\mathrm{B}} \rangle \tag{30}$$

$$\text{DPMs:} \quad T^{\mathrm{a}}\,\nabla_{\mathrm{a}}\psi_{\mathrm{B}} = \alpha\, h^{\mathrm{bc}}\nabla_{\mathrm{b}}\nabla_{\mathrm{c}}\psi_{\mathrm{B}}$$

Compare this with the generally covariant continuum heat equation,

$$\begin{array}{lll} \text{H5} & \text{KPMs:} & \langle \mathcal{M}, t_{\mathrm{ab}}, h^{\mathrm{ab}}, \nabla_{\mathrm{a}}, T^{\mathrm{a}}, \psi_{\mathrm{B}} \rangle \\ & \text{DPMs:} & T^{\mathrm{a}}\, \nabla_{\mathrm{a}} \psi_{\mathrm{B}} = \alpha\, h^{\mathrm{bc}} \nabla_{\mathrm{b}} \nabla_{\mathrm{c}} \psi_{\mathrm{B}} \end{array} \tag{31}$$

$$\begin{array}{lll} \text{Continuum Heat} & \text{KPMs:} & \langle \mathcal{M}, t_{\mathrm{ab}}, h^{\mathrm{ab}}, \nabla_{\mathrm{a}}, T^{\mathrm{a}}, \psi \rangle \\ & \text{DPMs:} & T^{\mathrm{a}}\, \nabla_{\mathrm{a}} \psi = \alpha\, h^{\mathrm{bc}} \nabla_{\mathrm{b}} \nabla_{\mathrm{c}} \psi \end{array} \tag{32}$$

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The only difference is that ψ_{B} is bandlimited whereas ψ is unrestricted.

Bandlimited and Generally Covariant Heat Equation

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H5	KPMs:	$\langle \mathcal{M}, t_{\mathsf{ab}}, h^{\mathsf{ab}}, \nabla_{\mathsf{a}}, T^{\mathsf{a}}, \psi_{\mathsf{B}} \rangle$	(31)
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The only difference is that ψ_{B} is bandlimited whereas ψ is unrestricted.

Since these dynamics preserve bandlimits, this ultimately amounts to a restriction of the initial condition.

Bandlimited and Gen. Covariant Klein Gordon Equation

The discrete 2D Klein Gordon equation on a square lattice,

$$\text{Discrete KG:} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi(t) = \left(\frac{1}{a^2}D_{\mathrm{B,x}}^2 + \frac{1}{a^2}D_{\mathrm{B,y}}^2 - M^2\right)\phi(t), \tag{33}$$

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Bandlimited and Gen. Covariant Klein Gordon Equation

Compare this with continuum Klein Gordon dynamics,

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Bandlimited and Gen. Covariant Klein Gordon Equation

Compare this with continuum Klein Gordon dynamics,

$$\text{Discrete KG} \qquad \begin{aligned} &\text{KPMs:} \quad \langle \mathcal{M}, \eta^{\mathsf{ab}}, \phi_{\mathsf{B}} \rangle \\ &\text{DPMs:} \quad (\eta^{\mathsf{ab}} \nabla_{\mathsf{a}} \nabla_{\mathsf{b}} - M^2)\phi_{\mathsf{B}} = 0 \end{aligned} \tag{36}$$

$$\text{Continuum KG} \qquad \begin{aligned} &\text{KPMs:} \quad \langle \mathcal{M}, \eta^{\mathsf{ab}}, \phi \rangle \\ &\text{DPMs:} \quad (\eta^{\mathsf{ab}} \nabla_{\mathsf{a}} \nabla_{\mathsf{b}} - M^2)\phi = 0 \end{aligned} \tag{37}$$

The only difference is that ϕ_{B} is bandlimited whereas ϕ is unrestricted.

Since these dynamics preserve bandlimits, this ultimately amounts to a restriction of the initial condition.

The discrete 2D Klein Gordon equation on a square lattice,

$$\text{Discrete KG:} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi(t) = \left(\frac{1}{a^2}D_{\mathrm{B,x}}^2 + \frac{1}{a^2}D_{\mathrm{B,y}}^2 - M^2\right)\phi(t), \qquad (38)$$

has the full* Poincare symmetry group.

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has the full* Poincare symmetry group.

*with one slight exception. The value of the bandwidth K depends on which flat space-like hypersurface you compute it on.

Conclusion

I have argued for the following analogy:

Coordinate Systems	$\leftrightarrow$	Lattice Structure
Changing Coordinates	$\leftrightarrow$	Nyquist-Shannon Resampling
Gen. Covariant Formulation	$\leftrightarrow$	Bandlimited Formulation

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Note: Once a “lattice” theory has been given a bandlimited reformulation it can then be given a generally covariant reformulation as well.

Conclusion

We have seen that the lattice structure underlying a “lattice” theory has the same level of physical import as coordinates do, i.e., none at all.

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(Ask me about how this changes for non-linear theories.)

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In particular, the lattice does not need to cause modified heat decay rates or modified dispersion relations.

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C2) The lattice does not restrict in any way which symmetries our theory can have.

The symmetry that our dynamics is completely independent of the symmetries of any given lattice structure. We can have:

4-fold rotation symmetric dynamics on a hexagonal lattice.
Continuous rotation symmetric dynamics on a irregular lattice.
Poincare-invariant dynamics on a square lattice.

Questions this Raises for Me

Q1) What would it be like if the world really had an certain lattice structure underlying it? Given the above, could this ever be established experimentally?

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Q2) What is local in the lattice formulation (nearest neighbor, $\Delta^2_{(1)}$) is non-local in terms of the bandlimited formulation ($\cosh(a\,\partial_x)$).

Likewise, What is local in terms of the bandlimited formulation (∂_x) is non-local in terms of the lattice formulation (infinite range, D_{B}).

If we care about locality, which of these notions should we prefer?

Questions this Raises for Me

Q2) Partial Answer: If we care about maximizing symmetry in our future theories (necessary to minimize background structure) then the bandlimited locality seems to be preferred.

[4] Achim Kempf, New J. of Physics, Volume 12, November 2010. arXiv:1010.4354

[5] Achim Kempf, Phys. Rev. Lett., Vol 92, Issue 22, June 2004. arXiv:gr-qc/0310035

Questions this Raises for Me

Q2) Partial Answer: If we care about maximizing symmetry in our future theories (necessary to minimize background structure) then the bandlimited locality seems to be preferred.

Q3) What possibilities are there for a bandlimited theory of gravity[4][5]? E.g., Bandlimited Newton Cartan. What about a bandlimited background independent theory? E.g., Bandlimited GR.

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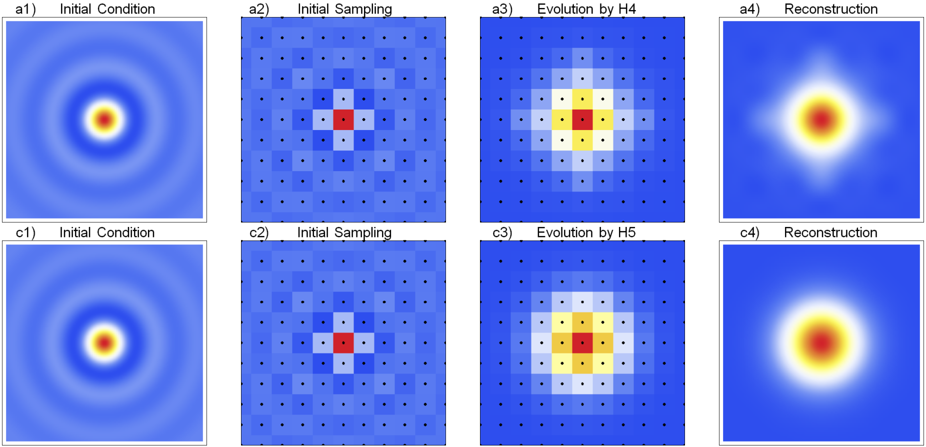
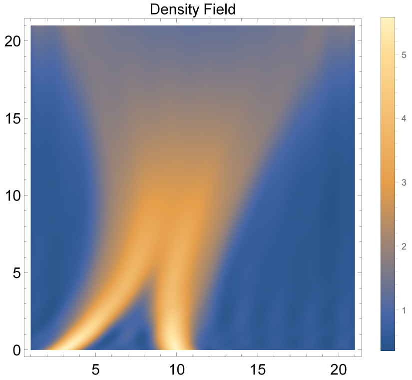
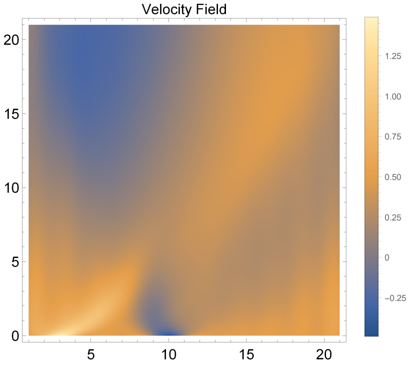
Thanks for your attention



Bandlimited Self-grav Navier Stokes

Both the heat equation and the Klein Gordon equation were linear. This stuff works for non-linear dynamics too (with a bit of work). Here is some bandlimited self-gravitating Navier Stokes dynamics.





Self-gravitating fluid

Consider this model of a self-gravitating fluid,

$$\text{KPMs:} \quad \langle \mathcal{M}, t_{\mathsf{ab}}, h^{\mathsf{ab}}, \nabla_{\mathsf{a}}, \varphi, \rho, u^{\mathsf{a}} \rangle \tag{39}$$

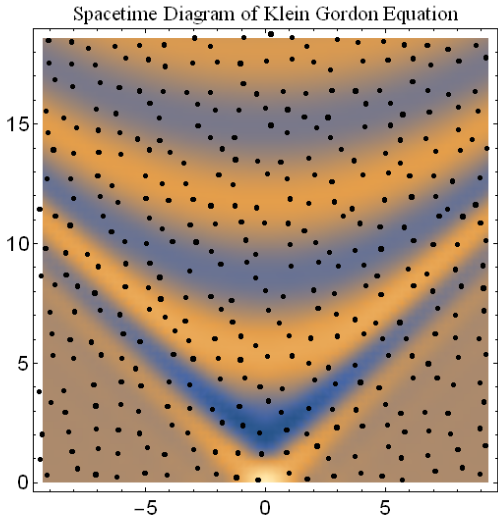
where φ is the grav. potential, ρ is the density, u^{a} is the time-like velocity

$$\begin{aligned} \text{DPMs:} \quad h^{\mathsf{ab}}\nabla_{\mathsf{a}}\nabla_{\mathsf{b}}\varphi &= 4\pi G\,\rho \\ \mathcal{B}_K[u^{\mathsf{a}}\nabla_{\mathsf{a}}u^{\mathsf{b}}] &= \nu\, h^{\mathsf{cd}}\nabla_{\mathsf{c}}\nabla_{\mathsf{d}}u^{\mathsf{b}} - \beta\, h^{\mathsf{bd}}\nabla_{\mathsf{d}}\rho - h^{\mathsf{bd}}\nabla_{\mathsf{d}}\varphi \\ \mathcal{B}_K[\nabla_{\mathsf{a}}(\rho\, u^{\mathsf{a}})] &= 0 \end{aligned} \tag{40}$$

ν is the viscosity and pressure is $p = \beta\rho^2/2$.

$\mathcal{B}_K$ applies a bandlimit with bandwidth K. Something like this is needed because products of bandlimited function can have up to the sum of their bandwidths.

Bandlimited in Time too



If the initial condition $\phi(0, x)$ of the Klein Gordon equation is bandlimited in space, then the full solution $\phi(t, x)$ is bandlimited in time.

As such we can describe it in both space and time via some sufficiently dense sample points.

Does this have anything to do with causal sets? I don't know.