A Discrete Analog of General Covariance

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Slides available at: http://users.ox.ac.uk/∼pemb6003/talks.html

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January 22, 2021

Outline

1. Review: What is special about GR? (20 min)
   - General Covariance?
   - Diffeomorphism Invariance?
   - Background Independence?
2. Review: Nyquist Shannon Sampling Theory (15 min)
   - Bandlimited Functions
   - Uniform Sampling
   - Non-uniform Sampling
3. Discrete General Covariance (25 min)
   - Put the above two together

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I will take questions after each part. Please save major questions for then.

Part 1: Some Philosophy of GR

What is special about GR? (as opposed to merely SR theories)[1]

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How do these three concepts differ and how are they related to each other?

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Spoiler Part 1: Its background independence.

What is special about GR?

A) General covariance is not a special property of GR.

- Any theory can be made generally covariant (Kretschmann, 1917)[2].

- However, general covariance is important because it exposes background structure, and clarifies many questions about symmetry.

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C) Background independence is what makes GR special.

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(Continuous) General Covariance

Example 1) Klein Gordon Equation:

$$\partial_t^2 \phi(t,x,y,z) = (\partial_x^2 + \partial_y^2 + \partial_z^2 - M^2)\,\phi(t,x,y,z) \tag{1}$$

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This formulation is not generally covariant. For instance, written in terms of the coordinates $(t' = t,\ x' = x + \frac{1}{2} a\, t^2,\ y' = y,\ z' = z)$ we have,

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to arbitrary coordinates x'^μ

$$(\eta^{\sigma\rho}\frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\rho}\partial_\mu\partial_\nu - M^2)\,\phi + \eta^{\sigma\rho}\frac{\partial^2 x'^\mu}{\partial x^\sigma\,\partial x^\rho}\,\partial_\mu\phi = 0. \tag{4}$$

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Consider the space of kinematically possible models (KPMs) given by:

$$\text{KPMs:} \quad \langle \mathcal{M}, \eta^{\mathsf{ab}}, \phi \rangle \tag{5}$$

where $\mathcal{M}$ is a differentiable (3+1)-manifold, η^{ab} is a fixed metric field with signature $(-1, 1, 1, 1)$ and $\phi : \mathcal{M} \to \mathbb{R}$ is a scalar field.

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Consider the dynamically possible models (DPMs) picked out by

$$\text{DPMs:} \quad (\eta^{\mathsf{ab}} \nabla_{\mathsf{a}} \nabla_{\mathsf{b}} - M^2)\, \phi = 0 \tag{6}$$

where ∇_{a} is the unique derivative compatible with the metric, i.e. with $\nabla_{\mathsf{c}}\, \eta^{\mathsf{ab}} = 0$.

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We now have the Klein Gordon equation in a generally covariant form:

$$\begin{aligned} \text{SR1} \qquad \text{KPMs:} \quad & \langle \mathcal{M}, \eta^{\mathsf{ab}}, \phi \rangle \quad \text{with } \eta^{\mathsf{ab}} \text{ fixed,} \\ \text{DPMs:} \quad & (\eta^{\mathsf{ab}} \nabla_{\mathsf{a}} \nabla_{\mathsf{b}} - M^2)\, \phi = 0. \end{aligned} \tag{7}$$

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Given a generic diffeomorphism $d \in \text{Diff}(\mathcal{M})$ and a solution $\langle \mathcal{M}, \eta^{ab}, \phi \rangle$,

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These are only solutions if d is in the Poincare group.

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$$\text{Continuum Heat} \qquad \text{KPMs:} \quad \langle \mathcal{M}, h^{\mathrm{ab}}, t_{\mathrm{ab}}, \nabla_{\mathrm{a}}, T^{\mathrm{a}}, \psi \rangle \tag{8}$$

Many details:

- Fixed space and time metrics: h^{ab} and t_{ab}. With signatures $(0,1,1)$ and $(1,0,0)$ respectively. They are orthogonal: $h^{\mathrm{ab}} t_{\mathrm{bc}} = 0$.
- Fixed covariant derivative operator: ∇_{a}. It is compatible with the metrics, $\nabla_{\mathrm{a}} h^{\mathrm{bc}} = 0$ and $\nabla_{\mathrm{a}} t_{\mathrm{bc}} = 0$, and flat, $R^{\mathrm{a}}{}_{\mathrm{bcd}} = 0$.
- Fixed vector field: T^{a}. It is constant, $\nabla_{\mathrm{a}} T^{\mathrm{b}} = 0$, time-like, $t_{\mathrm{ab}} T^{\mathrm{a}} T^{\mathrm{b}} > 0$, and normalized, $t_{\mathrm{ab}} T^{\mathrm{a}} T^{\mathrm{b}} = 1$.

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Note that T^{a} picks our a canonical way of moving forward in time (i.e, translation generated by $T^{\mathrm{a}}\nabla_{\mathrm{a}}$).

(Continuous) General Covariance: Heat Equation

In total we have the heat equation $\partial_t \psi = \alpha \, (\partial_x^2 + \partial_y^2) \, \psi$ reformulated as,

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The DPMs are picked out by:

$$\text{Continuum Heat} \qquad \text{DPMs:} \quad T^{\mathsf{a}}\,\nabla_{\mathsf{a}}\psi = \alpha\,h^{\mathsf{bc}}\nabla_{\mathsf{b}}\nabla_{\mathsf{c}}\psi \tag{10}$$

(Continuous) General Covariance: Newtonian Gravity

Repeating this process for Newtonian Gravity we have

$$\text{Newtonian Gravity} \qquad \text{KPMs:} \quad \langle \mathcal{M}, t_{\mathrm{ab}}, h^{\mathrm{ab}}, \nabla_{\mathrm{a}}, \varphi, \Phi \rangle \tag{11}$$

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$$\text{DPMs:} \quad h^{\mathrm{bc}} \nabla_{\mathrm{b}} \nabla_{\mathrm{c}} \varphi = 4\pi G\, \rho$$

$$u^{\mathrm{a}} \nabla_{\mathrm{a}} u^{\mathrm{b}} = -h^{\mathrm{bc}} \nabla_{\mathrm{c}} \varphi$$

where φ is the gravitational potential and Φ is a stand in for the matter content of the theory (ρ is calculated from Φ somehow). u^{a} is the 4-velocity of a test particle (normalized and time-like).

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where φ is the gravitational potential and Φ is a stand in for the matter content of the theory (ρ is calculated from Φ somehow). u^{a} is the 4-velocity of a test particle (normalized and time-like).

Note there is no time-like vector field T^{a} assumed here.
This theory is has the Galilean symmetry group.

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Since any theory can be represented in terms of any coordinates (or in terms of no coordinates at all) it is now obvious that coordinates play no role in symmetry. In a coordinate-independent framing, there are no passive symmetry transformations.

Diffeomorphism invariance?

What about diffeomorphism invariance? Maybe this sets GR-like theories apart from SR-like theories.

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Unfortunately, this is not right. We can reformulate special relativity to be diffeomorphism invariant as[3]

$$\begin{aligned} \text{SR2} \qquad \text{KPMs:} \quad & \langle \mathcal{M}, g^{ab}, \phi \rangle, \\ \text{DPMs:} \quad & \left(g^{ab} \nabla_a \nabla_b - M^2 \right) \phi = 0 \\ & R^a{}_{bcd} = 0. \end{aligned} \tag{12}$$

Note g^{ab} is not a fixed field, it is dynamical.

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Given a generic diffeomorphism $d \in \mathrm{Diff}(\mathcal{M})$ and a solution $\langle \mathcal{M}, g^{ab}, \phi \rangle$ we do in fact have that $\langle \mathcal{M}, d^* g^{ab}, d^* \phi \rangle$ is a solution.

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Compare SR2 with GR

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with GR,

$$\text{GR} \qquad \text{KPMs:} \quad \langle \mathcal{M}, g^{\mathrm{ab}}, \phi \rangle, \tag{14}$$
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$$G_{\mathrm{ab}} = 8\pi\, T_{\mathrm{ab}}.$$

Intuitively SR2 has background structure whereas GR does not.

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For the rest of the talk, all that is important is that we understand how general covariance supports our understanding of diffeomorhism invariance and background structure:

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Let's move on to Part 2 of the presentation. Questions before we do?

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To answer this question it would be very helpful to have a notion of discrete general covariance.

Discrete General Covariance

Inspired by the work of Achim Kempf,[4][5] I suggest the following analogy:

Coordinate Systems	$\leftrightarrow$	Lattice Structure
Changing Coordinates	$\leftrightarrow$	Nyquist-Shannon Resampling
Gen. Covariant Formulation	$\leftrightarrow$	Bandlimited Formulation

[4] Achim Kempf, "Spacetime could be simultaneously continuous and discrete, in the same way that information can be" New J. of Physics (2010). arXiv:1010.4354

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Before jumping into this, we need to review Sampling Theory.

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   - Uniform Sampling
   - Non-uniform Sampling
3. Discrete General Covariance (25 min)
   - Put the above two together

A bandlimited function is one whose Fourier transform has compact support. That is, a function $f_{\mathrm{B}}(x)$ is bandlimited with bandwidth K iff $\mathcal{F}_k[f_{\mathrm{B}}(x)]$ has support only for wavenumbers $|k| \leq K$.

Part 2: Review of Nyquist Shannon Sampling Theory

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The Nyquist Shannon Sampling Theorem tells us that we can **exactly** reconstruct any bandlimited function knowing only its values at a sufficiently dense set of sample points.

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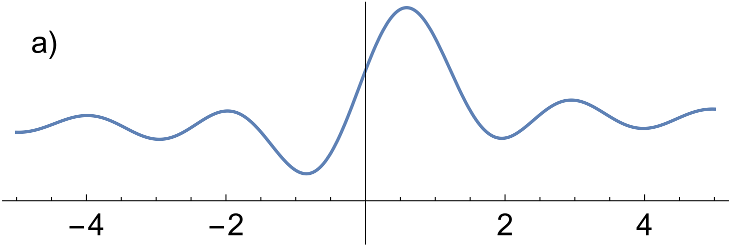
The following reconstruction,

$$f_{\mathrm{B}}(z) =^{?} \sum_{n=-\infty}^{\infty} S_n(z/a)\ f_n; \quad S(y) = \frac{\sin(\pi y)}{\pi y}, \quad S_n(y) = S(y-n). \quad (15)$$

is **exact** when our sample points are sufficiently dense
(here meaning $a \leq a^* = \pi/K$).

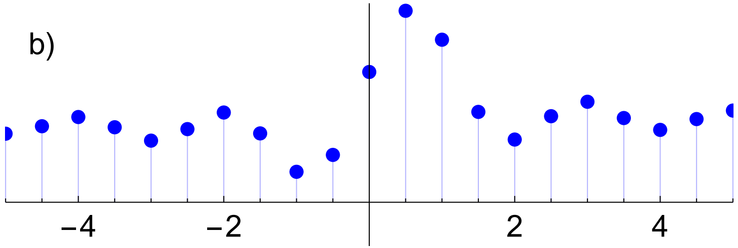
Exact Example

Consider that $f_B(x) = 1 + S(x - 1/2) + x\,S(x/2)^2$ has a bandwidth of $K = \pi$ and so a critical sample spacing of $a^* = 1$



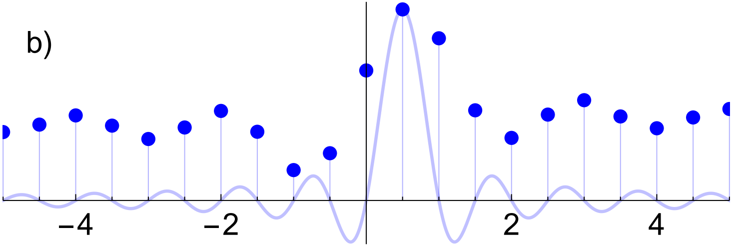
Exact Example

We can recover $f_{\mathrm{B}}(x)$ exactly knowing only its values at $x_n = n\,a$ with $a = 1/2 < a^* = 1$



Exact Example

To recover $f_B(x)$ we associate each x_n with a shifted and rescaled sinc function as



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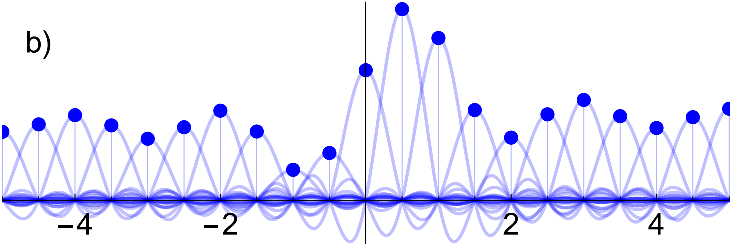
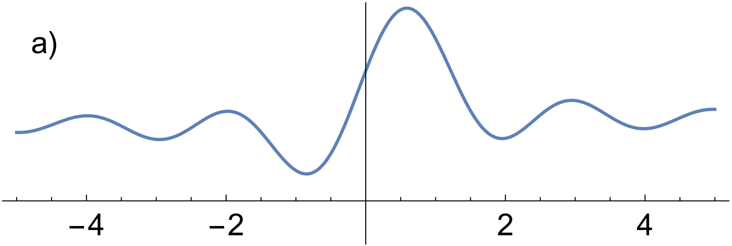


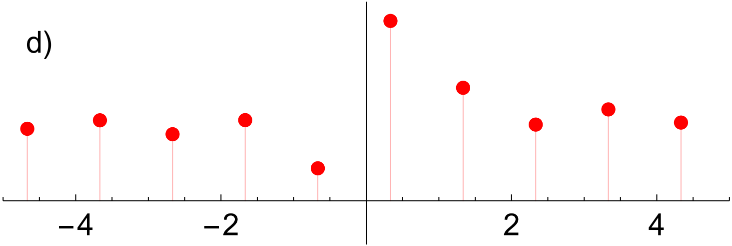
Exact Example

Adding together all of these sinc functions gives back $f_{\mathrm{B}}(x)$ with no approximation



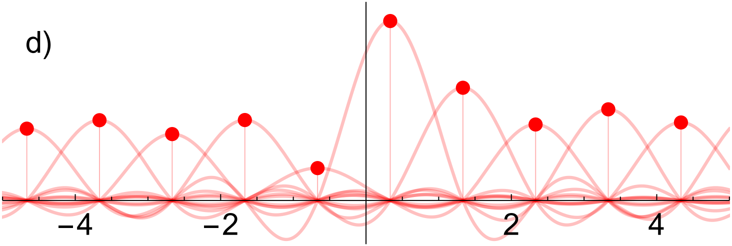
Exact Example

We oversampled in the previous example. We can recover $f_{\mathrm{B}}(x)$ exactly knowing only its values at $x_n = n\,a + 1/3$ with $a = a^* = 1$



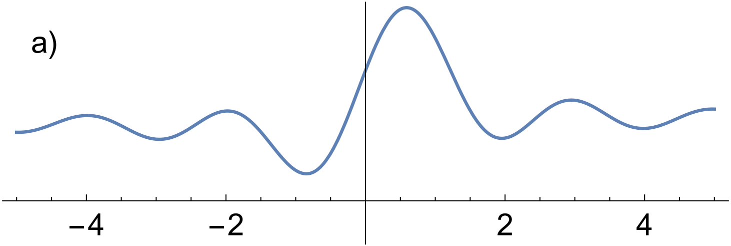
Exact Example

Just as before we recover $f_{\mathrm{B}}(x)$ by associating each x_n with a shifted and rescaled sinc function as



Exact Example

Adding together all of these sinc functions gives back $f_{\mathrm{B}}(x)$ with no approximation



Non-uniform Sampling

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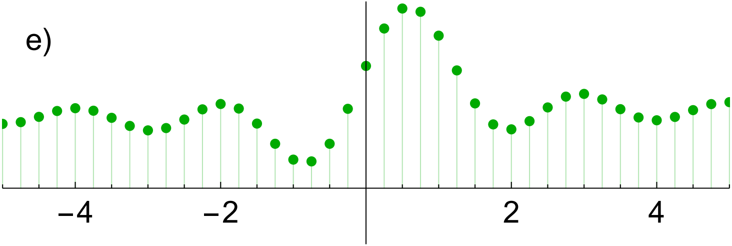
Let's see how this works.

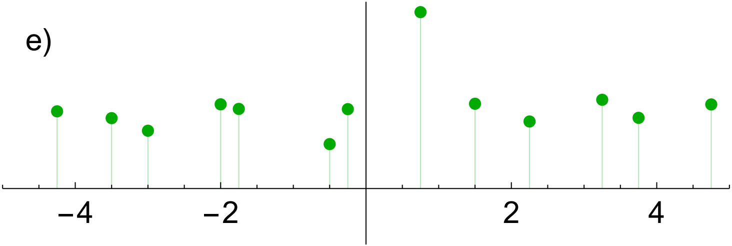
Exact Example

Consider the following oversampling of $f_{\mathrm{B}}(x)$ with $a = 1/4 < a^* = 1$. We do not need all of these sample points to reconstruct (we need approximately one quarter of them).



Exact Example

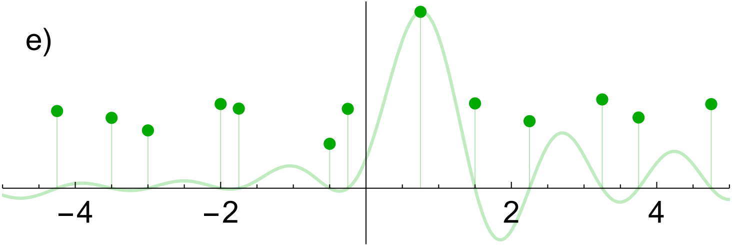
The samples which we drop do not need to be selected uniformly. The following non-uniform sampling works,



Exact Example

The reconstruction function for each sample point is now more complicated. But ultimately,

$$f_{\mathrm{B}}(z) = \sum_{m=-\infty}^{\infty} G_m(z; \{x_n\})\, f_{\mathrm{B}}(x_m) \tag{16}$$



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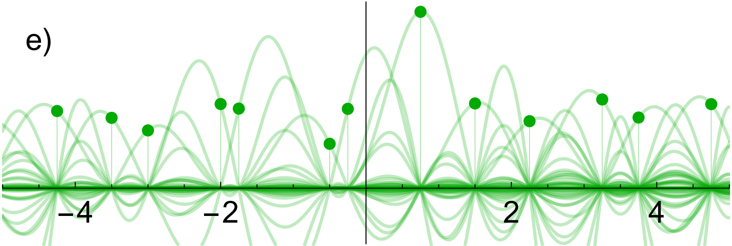

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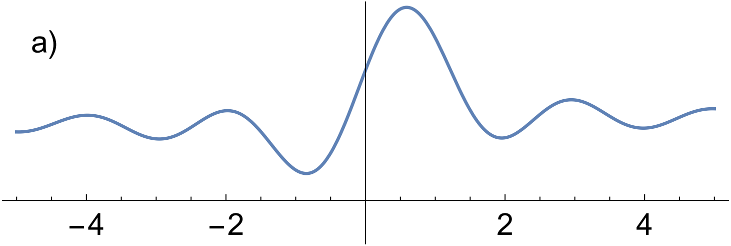

Higher Dimensions

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Consider $f_{\mathrm{B}}(x, y) = J_1(\pi\, r)/(\pi\, r)$ where J_1 is the first Bessel function and $r = \sqrt{x^2 + y^2}$. This function is bandlimited with $\sqrt{k_x^2 + k_y^2} \leq K = \pi$.

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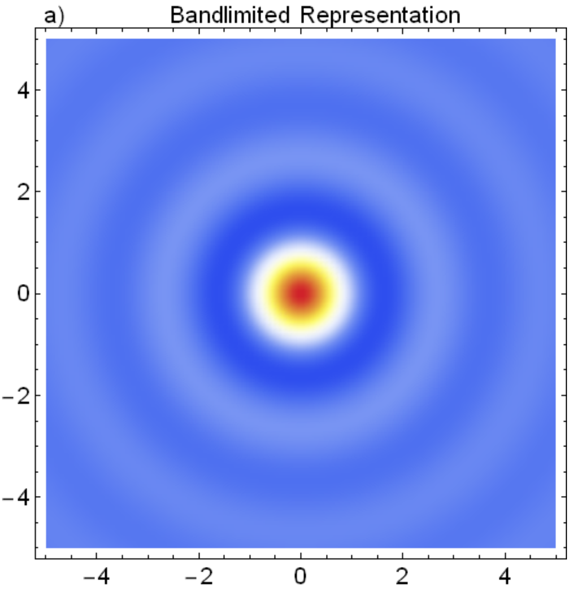
The following figures are all equivalent representations of $f_{\rm B}(x, y)$

Higher Dimensions



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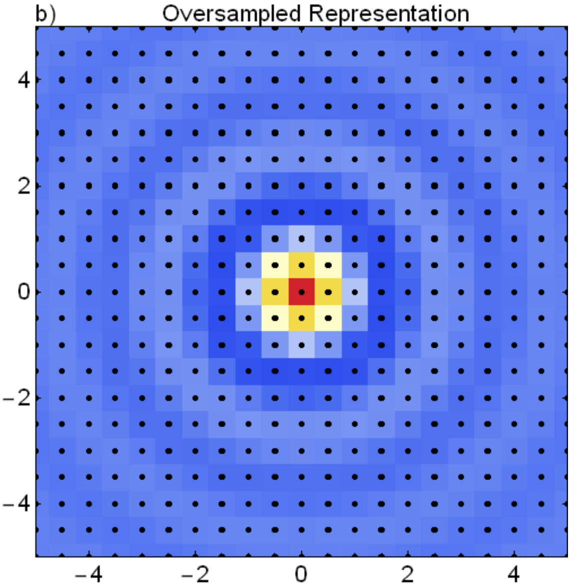

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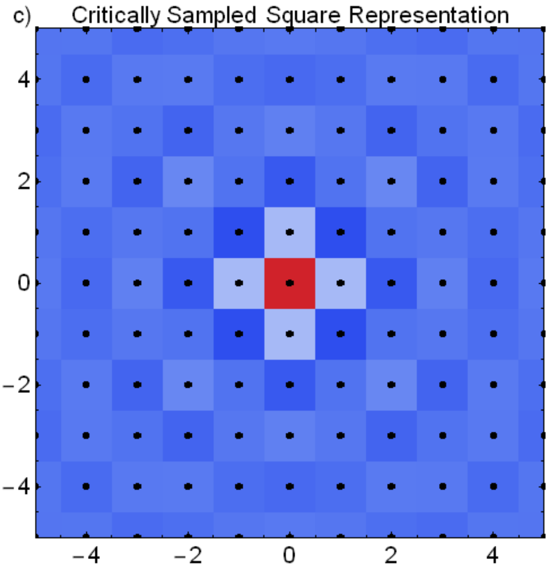

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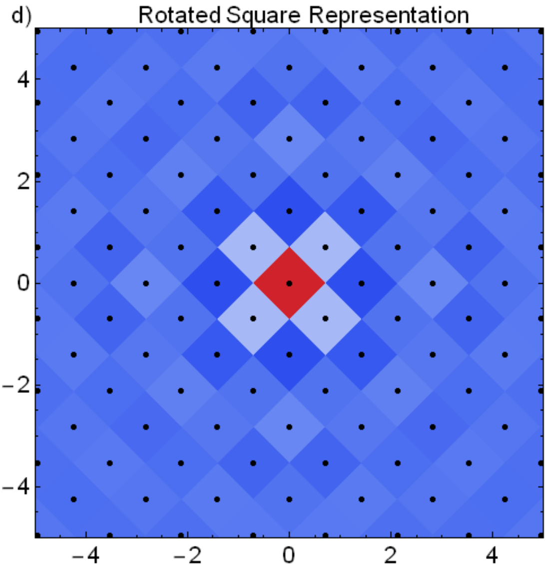

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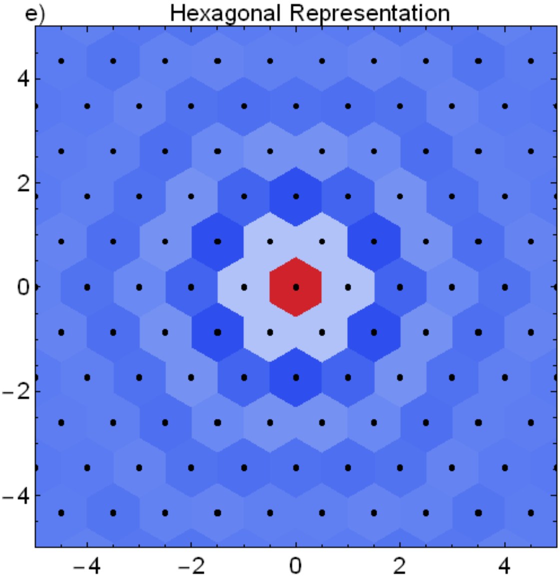

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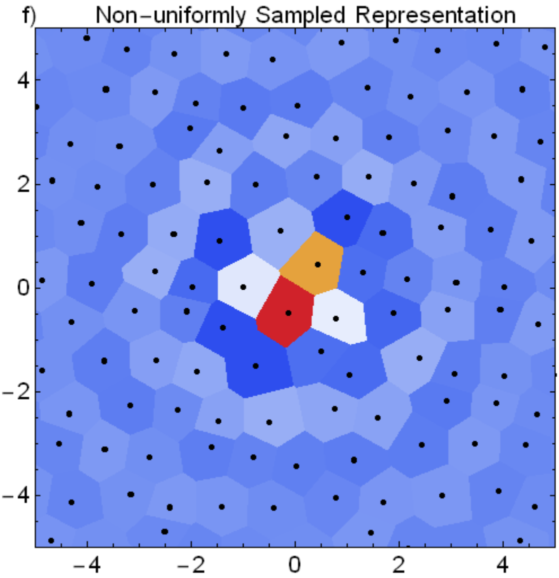

Let's Review

What is remarkable about bandlimited functions is that they have a finite density of degrees of freedom, but these degrees of freedom have no fixed definite location[6].

[6] Achim Kempf, "Spacetime could be simultaneously continuous and discrete, in the same way that information can be" New J. of Physics (2010). arXiv:1010.4354

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Questions before we move on to Part 3?

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Outline

1. Review: What is special about GR? (20 min)
   - General Covariance?
   - Diffeomorphism Invariance?
   - Background Independence?
2. Review: Nyquist Shannon Sampling Theory (15 min)
   - Bandlimited Functions
   - Uniform Sampling
   - Non-uniform Sampling
3. Discrete General Covariance (25 min)
   - Put the above two together

Part 3: Discrete General Covariance

Recall the proposed analogy:

Coordinate Systems	$\leftrightarrow$	Lattice Structure
Changing Coordinates	$\leftrightarrow$	Nyquist-Shannon Resampling
Gen. Covariant Formulation	$\leftrightarrow$	Bandlimited Formulation

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So far we have started with a bandlimited function and induced discrete lattice representations from it. We can also start from a discrete representation and from it find a bandlimited formulation. We can then resample into other discrete representations.

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So far we have started with a bandlimited function and induced discrete lattice representations from it. We can also start from a discrete representation and from it find a bandlimited formulation. We can then resample into other discrete representations.

Moreover, we will do some physics by adding dynamics. We can then make concrete our questions about symmetry and background independence in this discrete context.

Example: 1D Nearest-Neighbor Heat Equation

Consider the 1D nearest-neighbor heat equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_n(t) = \alpha\,\frac{\psi_{n+1}(t) - 2\psi_n(t) + \psi_{n-1}(t)}{a^2} \tag{19}$$

[6] Note, there is no manifold in this formulation of the theory.

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or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}(t) = \frac{\alpha}{a^2}\,\Delta^2_{(1)}\boldsymbol{\psi}(t) \tag{20}$$

where $\Delta^2_{(1)}$ is the nearest neighbor approximation to the second derivative and $\boldsymbol{\psi}(t) = (\dots, \psi_{-1}(t), \psi_0(t), \psi_1(t), \dots)$.

[6] Note, there is <u>no manifold</u> in this formulation of the theory.

Example: 1D Nearest-Neighbor Heat Equation

At each time we can take these discrete values $\psi_n(t)$ and imagine them as samples which are drawn from a bandlimited function $\psi_{\rm B}$ as,[7].

$$\psi_n(t) = \psi_{\rm B}(t, x_n), \qquad x_n = n\,a. \tag{21}$$

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We can then use these samples to reconstruct $\psi_{\text{B}}(t, x)$ as

$$\psi_{\text{B}}(t, x) = \sum_{n=-\infty}^{\infty} S_n(x/a)\;\psi_n(t). \tag{22}$$

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Adding Dynamics

In addition to moving the state-of-the-world at each time into the bandlimited setting we can also move the dynamics,

$$\frac{\partial}{\partial t}\psi_{\mathrm{B}}(t,x) = \sum_n S_n(x)\ \frac{\mathrm{d}}{\mathrm{d}t}\psi_n(t) \\ = \dots \\ = \frac{\alpha}{a^2}\ \frac{\cosh(a\,\partial_x) - 1}{1/2}\ \psi_{\mathrm{B}}(t,x) \tag{23}$$

The complicated cosh term is the continuum analog of $\Delta^2_{(1)}$. Note $\exp(a\,\partial_x)f(x) = f(x+a)$.

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$$\frac{\cosh(a\,\partial_x) - 1}{a^2/2} = \partial_x^2 + \frac{a^2}{12}\partial_x^4 + O(a^4) \tag{24}$$

Three Different Dynamics

H1: $$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\alpha}{a^2}\,\Delta^2_{(1)}\,\psi(t) \tag{25}$$

$$\partial_t \psi_{\mathrm{B}}(t,x) = \frac{\alpha}{a^2}\,\frac{\cosh(a\,\partial_x) - 1}{1/2}\;\psi_{\mathrm{B}}(t,x)$$

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H2:
$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\alpha}{a^2}\,\Delta^2_{(2)}\,\psi(t) \tag{26}$$
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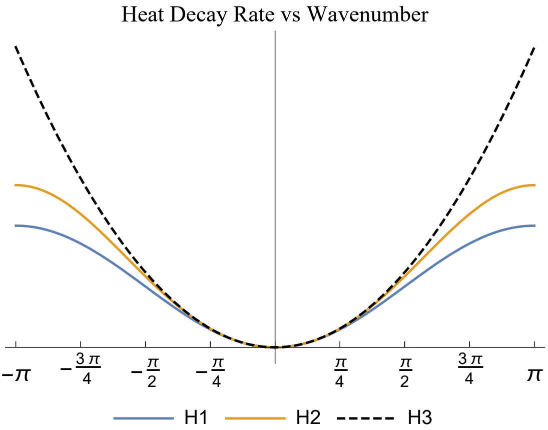
H3: $$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\alpha}{a^2}\,D^2_\mathrm{B}\,\psi(t) \tag{27}$$

$$\partial_t\psi_\mathrm{B}(t,x) = \alpha\,\partial_x^2\psi_\mathrm{B}(t,x)$$

where $D^2_\mathrm{B} = \lim_{n\to\infty}\Delta^2_{(n)}$ is the infinite-range derivative approximation.

Eigen Analysis

In each of these cases the eigensolutions are planewaves, with $|k| \leq K$, which decay exponentially at some rate.



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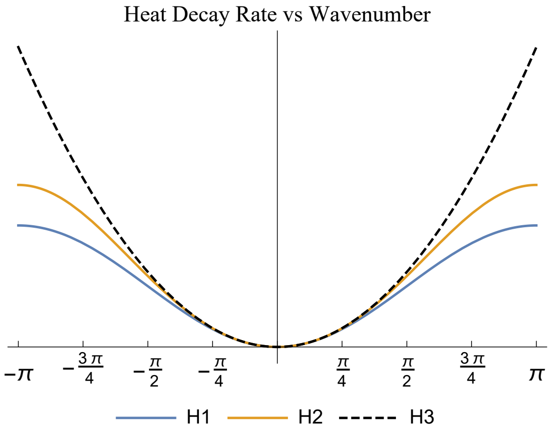
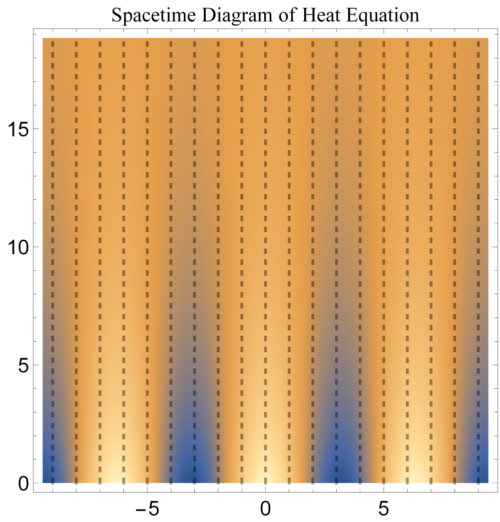


Lesson 1: There is no reason that a lattice theory needs to have different dynamics than the continuum theory (at least not below the bandwidth, K).

Dynamic Resampling



We can plot the discrete values $\psi_n(t)$ and the bandlimited function $\psi_B(t, x)$ in a spacetime diagram.

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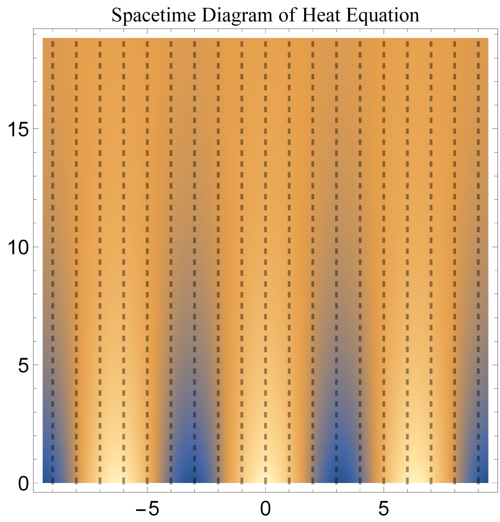
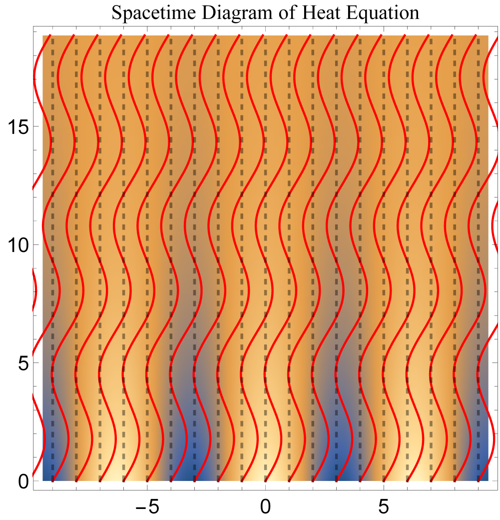


We can plot the discrete values $\psi_n(t)$ and the bandlimited function $\psi_B(t,x)$ in a spacetime diagram.

We can then pick new sample point at each time.

Resampling and Symmetry



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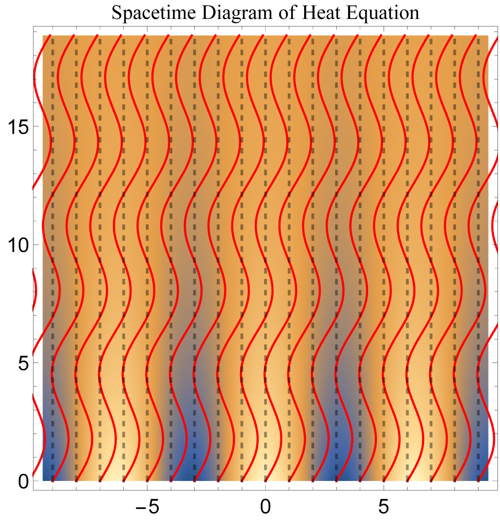
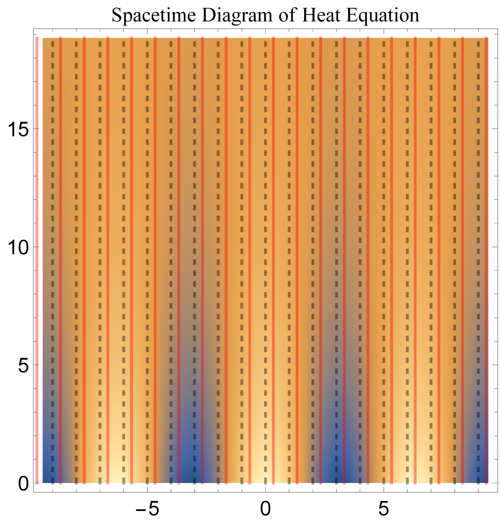


We can pick new sample point at each time.

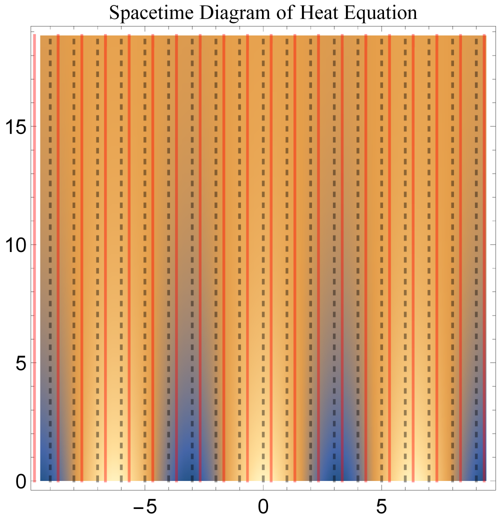
These new sample values will not obey the same equation that the old ones did. But they are completely sufficient to represent the dynamics.

Resampling and Symmetry



What about this resampling? Do the shifted red sample values obey the same equations as the original dashed sample points?

Resampling and Symmetry



What about this resampling? Do the shifted red sample values obey the same equations as the original dashed sample points?

Indeed they do. Lesson 2: There is no reason that a lattice theory can't have a continuous symmetry.

Aside on Representation Theory

You may think I am saying that

$$\text{H1:} \qquad \partial_t \psi_{\mathrm{B}}(t,x) = \frac{\alpha}{a^2}\,\frac{\cosh(a\,\partial_x) - 1}{1/2}\;\psi_{\mathrm{B}}(t,x) \tag{28}$$

has a continuous translation symmetry, $\psi_{\mathrm{B}}(t,x) \to \psi_{\mathrm{B}}(t,x+\epsilon)$.

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It does, but I am also saying that

$$\text{H1:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}(t) = \frac{\alpha}{a^2}\,\Delta^2_{(1)}\,\boldsymbol{\psi}(t) \tag{29}$$

has this symmetry, even when there is no manifold underlying it. The symmetries of a theory do not depend on the how we represent that theory.

Aside on Representation Theory

But how does

$$\text{H1:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\alpha}{a^2}\,\Delta^2_{(1)}\,\psi(t) \tag{30}$$

have a continuous translation symmetry?

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Concretely, there is a representation of the translation group acting on $\mathbb{R}^{\mathbb{Z}}$.

That is, There is some family of linear maps $T(\epsilon) : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ which maps solutions to solutions $\psi(t) \to T(\epsilon)\psi(t)$. Note $T(1)$ is translation by one lattice site.

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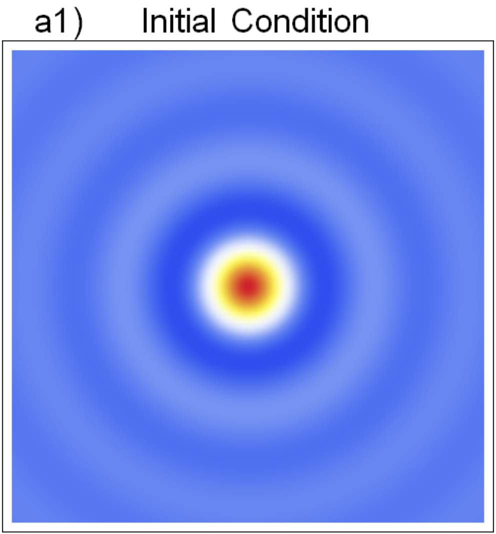
Moreover, the group structure of these $T(\epsilon)$ is the translation group. E.g., $T(1/2)\,T(1/2) = T(1/4)\,T(3/4) = T(1)$.

Another Example: 2D Heat Equation



Consider this initial condition for the 2D Heat Equation.

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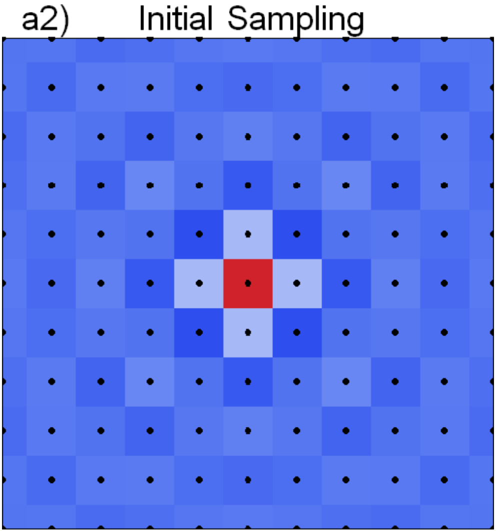

We can sample this initial condition and then evolve it via one of our discrete dynamical equations.

Nearest Neighbor Der.:
H4 : via $\Delta^2_{(1),x} + \Delta^2_{(1),y}$

Bandlimited Derivative:
H5 : via $D^2_{\mathrm{B,x}} + D^2_{\mathrm{B,y}}$.

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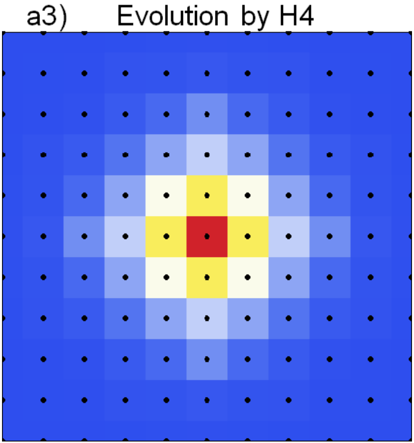

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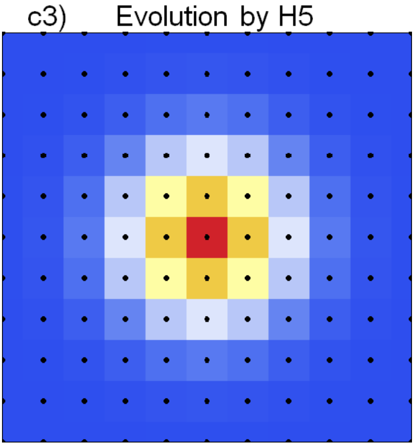

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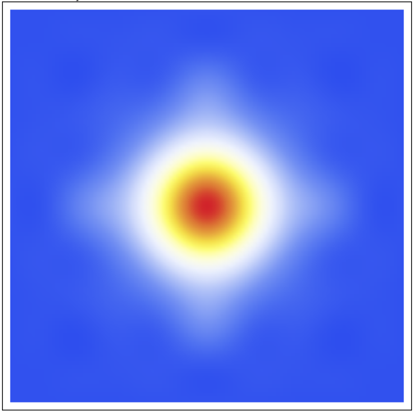
a4) Reconstruction



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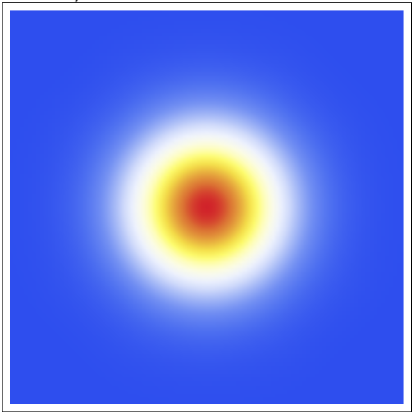

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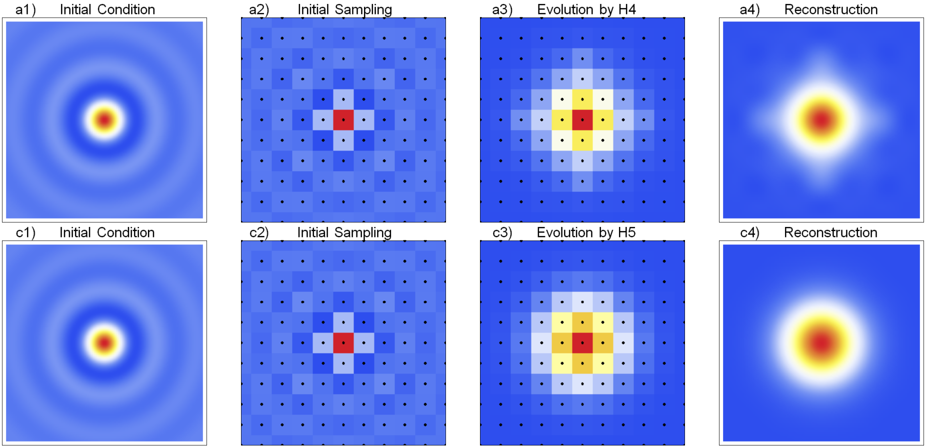

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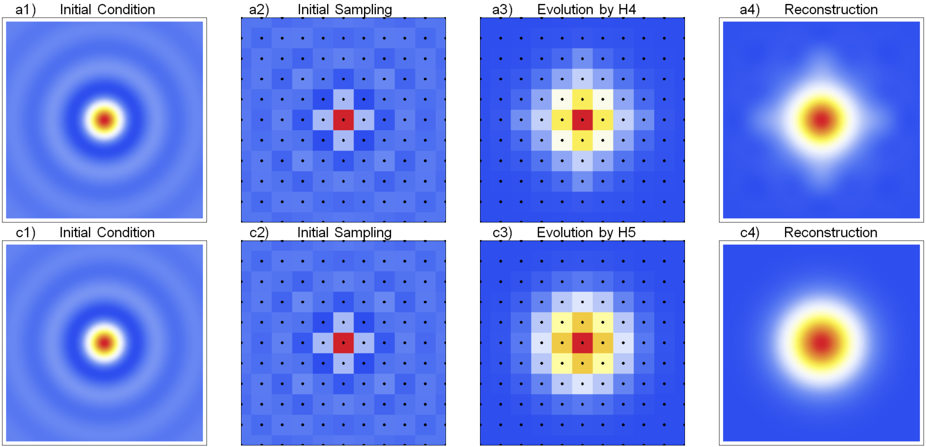

Lesson 2: We can have rotation invariant dynamics described in terms of a square lattice.

Aside on Representation Theory

Its worth stressing that there is a representation of the rotation group acting on $\mathbb{R}^{2\mathbb{Z}} = \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}$.

That is, There is some family of linear maps $R(\theta) : \mathbb{R}^{2\mathbb{Z}} \to \mathbb{R}^{2\mathbb{Z}}$ which maps 2D-arrays to 2D-arrays. Moreover, the group structure of these $R(\theta)$ is the rotation group.

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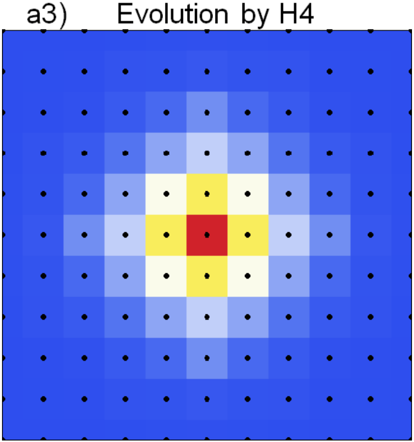
E.g., $R(2\pi) =$ id. and $R(\pi/2) =$ quarter turn, and $R(\pi/4)R(\pi/4) = R(\pi/2)$.

Another Example: 2D Heat Equation

Using this $\mathbb{R}^{2\mathbb{Z}}$ representation of the rotation group, we can judge the rotation invariance of the state without reconstructing ψ_{B}.

Is not rotation invariant



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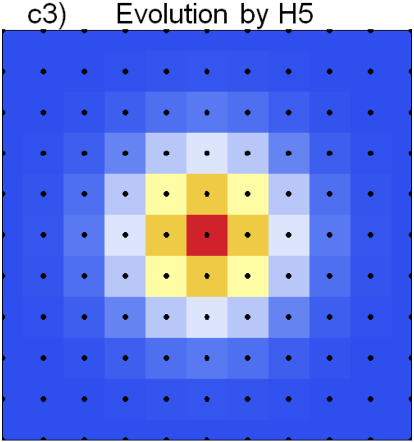

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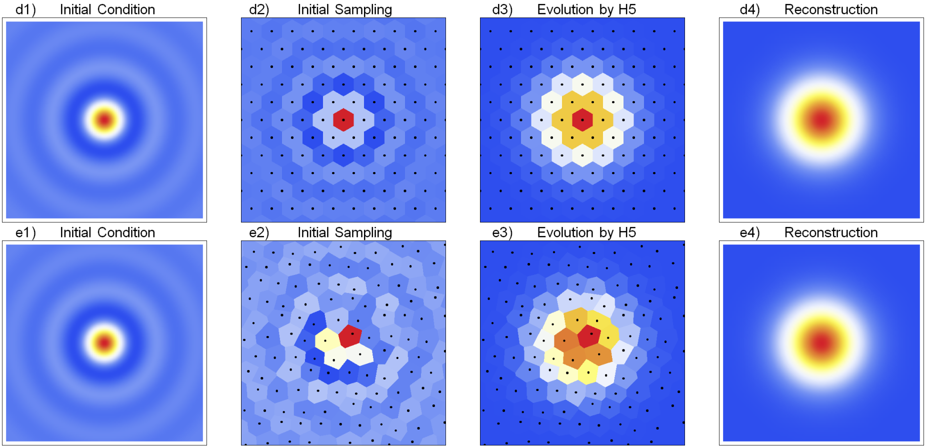
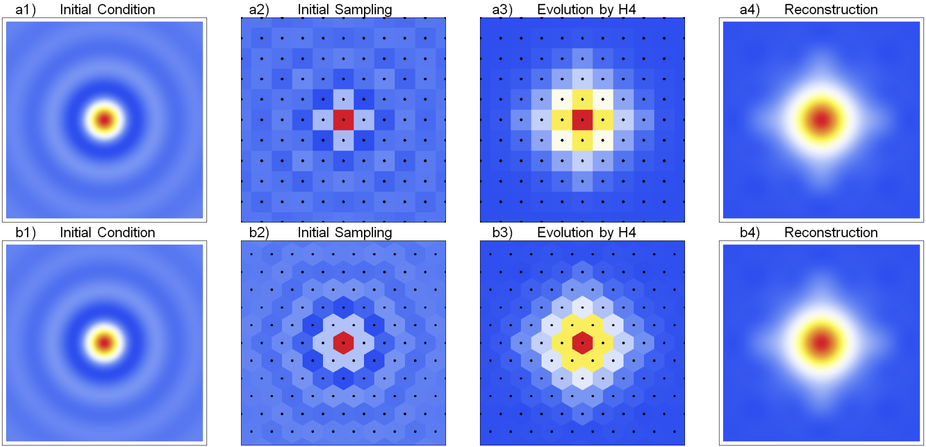


Lesson 2: We can have rotation invariant dynamics described in terms of any lattice.

Another Example: 2D Heat Equation



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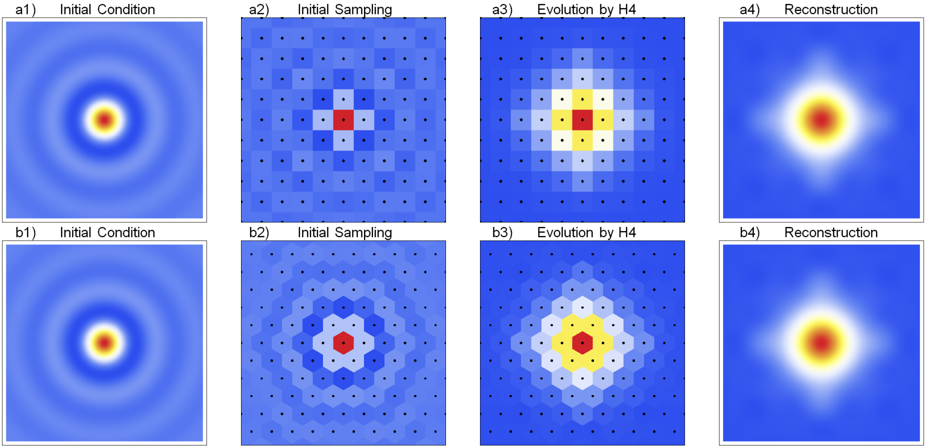

Lesson 3: The 4-fold symmetry of the H4 dynamics has nothing to do with the dynamics being represented in terms of a square lattice.

Bandlimited and Generally Covariant Heat Equation

The discrete 2D heat equation on a square lattice,

$$\text{H5:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \frac{\alpha}{a^2}\left(D_{\mathrm{B,x}}^2 + D_{\mathrm{B,y}}^2\right)\psi(t), \tag{31}$$

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We can further reformulate this in a generally covariant way as,

$$\begin{aligned} \text{H5} \qquad \text{KPMs:} \quad & \langle \mathcal{M}, t_{\mathrm{ab}}, h^{\mathrm{ab}}, \nabla_{\mathrm{a}}, T^{\mathrm{a}}, \psi_{\mathrm{B}} \rangle \\ & \text{with } t_{\mathrm{ab}}, h^{\mathrm{ab}}, \nabla_{\mathrm{a}}, T^{\mathrm{a}} \text{ fixed} \\ \text{DPMs:} \quad & T^{\mathrm{a}}\,\nabla_{\mathrm{a}}\psi_{\mathrm{B}} = \alpha\, h^{\mathrm{bc}}\nabla_{\mathrm{b}}\nabla_{\mathrm{c}}\psi_{\mathrm{B}} \end{aligned} \tag{33}$$

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$$\text{Discrete KG:} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi(t) = (\frac{1}{a^2} D^2_{\mathrm{B,x}} + \frac{1}{a^2} D^2_{\mathrm{B,y}} + \frac{1}{a^2} D^2_{\mathrm{B,z}} - M^2)\,\phi(t), \quad (36)$$

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has the full* Poincare symmetry group.

*with one slight exception. The value of the bandwidth K depends on which flat space-like hypersurface you compute it on.

Conclusion

I have argued for the following analogy:

Coordinate Systems	$\leftrightarrow$	Lattice Structure
Changing Coordinates	$\leftrightarrow$	Nyquist-Shannon Resampling
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Note: Once a “lattice” theory has been given a bandlimited reformulation it can then be given a generally covariant reformulation as well.

Conclusion

We have seen that the lattice structure underlying a "lattice" theory has the same level of physical import as coordinates do, i.e., none at all.

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(Ask me about how this changes for non-linear theories.)

In particular, the lattice does not need to cause modified heat decay rates or modified dispersion relations.

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C2) The lattice does not restrict in any way which symmetries our theory can have.

The symmetry that our dynamics is completely independent of the symmetries of any given lattice structure. We can have:

4-fold rotation symmetric dynamics on a hexagonal lattice.
Continuous rotation symmetric dynamics on a irregular lattice.
Poincare-invariant dynamics on a square lattice.

Questions this Raises for Me

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Q2) What is local in the lattice formulation (nearest neighbor, $\Delta^2_{(1)}$) is non-local in terms of the bandlimited formulation ($\cosh(a\,\partial_x)$).

Likewise, What is local in terms of the bandlimited formulation (∂_x) is non-local in terms of the lattice formulation (infinite range, D_B).

If we care about locality, which of these notions should we prefer?

Questions this Raises for Me

Q2) Partial Answer: If we care about maximizing symmetry in our future theories (necessary to minimize background structure) then the bandlimited locality seems to be preferred.

[8] Achim Kempf, "Spacetime could be simultaneously continuous and discrete, in the same way that information can be" New J. of Physics (2010). arXiv:1010.4354

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Q2) Partial Answer: If we care about maximizing symmetry in our future theories (necessary to minimize background structure) then the bandlimited locality seems to be preferred.

Q3) What possibilities are there for a bandlimited theory of gravity[8][9]? E.g., Bandlimited Newton Cartan. What about a bandlimited background independent theory? E.g., Bandlimited GR.

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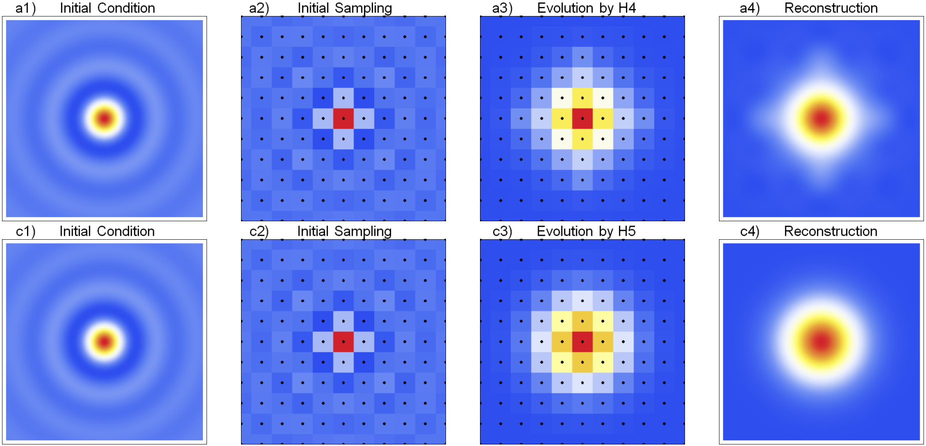
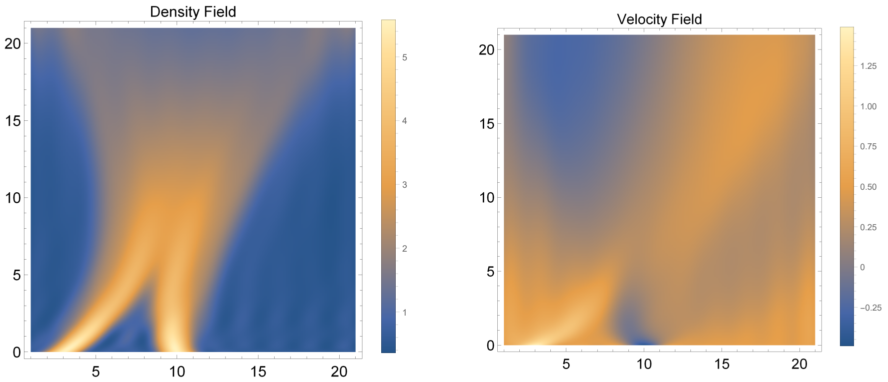
Thanks for your attention

Slides available at: http://users.ox.ac.uk/∼pemb6003/talks.html



Bandlimited Self-grav Navier Stokes

Both the heat equation and the Klein Gordon equation were linear. This stuff works for non-linear dynamics too (with a bit of work). Here is some bandlimited self-gravitating Navier Stokes dynamics.



Self-gravitating fluid

Consider this model of a self-gravitating fluid,

$$\text{KPMs:} \quad \langle \mathcal{M}, t_{\mathrm{ab}}, h^{\mathrm{ab}}, \nabla_{\mathrm{a}}, \varphi, \rho, u^{\mathrm{a}} \rangle \tag{42}$$

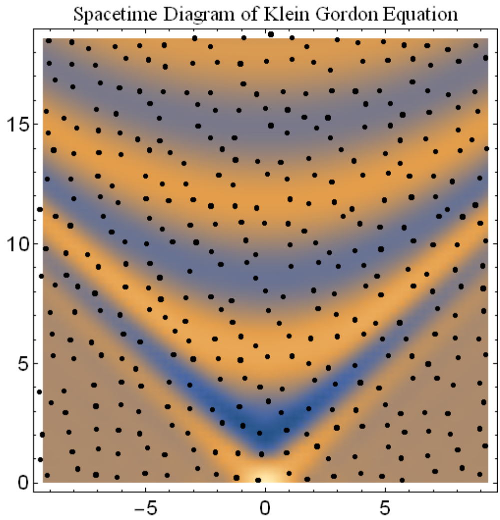
where φ is the grav. potential, ρ is the density, u^{a} is the time-like velocity

$$\begin{aligned} \text{DPMs:} \quad h^{\mathrm{ab}} \nabla_{\mathrm{a}} \nabla_{\mathrm{b}} \varphi &= 4\pi G\, \rho \\ \mathcal{B}_K[u^{\mathrm{a}} \nabla_{\mathrm{a}} u^{\mathrm{b}}] &= \nu\, h^{\mathrm{cd}} \nabla_{\mathrm{c}} \nabla_{\mathrm{d}} u^{\mathrm{b}} - \beta\, h^{\mathrm{bd}} \nabla_{\mathrm{d}} \rho - h^{\mathrm{bd}} \nabla_{\mathrm{d}} \varphi \\ \mathcal{B}_K[\nabla_{\mathrm{a}}(\rho\, u^{\mathrm{a}})] &= 0 \end{aligned} \tag{43}$$

ν is the viscosity and pressure is $p = \beta \rho^2/2$.

$\mathcal{B}_K$ applies a bandlimit with bandwidth K. Something like this is needed because products of bandlimited function can have up to the sum of their bandwidths.

Bandlimited in Time too



If the initial condition $\phi(0, x)$ of the Klein Gordon equation is bandlimited in space, then the full solution $\phi(t, x)$ is bandlimited in time.

As such we can describe it in both space and time via some sufficiently dense sample points.

Does this have anything to do with causal sets? I don't know.