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Localizing the Singular Structure of Spacetime

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ABSTRACT

I argue that the singular structure of spacetime is best conceptualized as an extrinsic, non-local, but localizable, property of spacetime regions (of any size), and that there is no unique, best definition of a singular region of spacetime. There is, however, a schema that any adequate definition instantiates. Moreover, the question of what kind of incomplete curves witness singular structure is intimately connected to the question of what kind of curves represent possible trajectories of observers in 'physically reasonable' motion.

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1 Introduction

Singular structure in general relativity is a rather elusive phenomenon.[1] On one hand, it is expected to be exhibited not only in particular relativistic spacetimes, but also in particular regions of a given spacetime, where (say) some curvature scalar invariant tends to infinity along a curve in finite proper time, or where some kind of curve abruptly comes to an end. On the other hand, it seems that we have no fully satisfying way of 'localizing' singularities within such spacetimes; for: (a) they are not points (nor regions) of Lorentzian manifolds that generally represent spacetimes, (b) the curve incompleteness properties that characterize singular structure come out as non-local on all reasonable definitions of 'local' properties, and (c) many attempts to extend Lorentzian manifolds to appropriate topological spaces including points representing singularities have resulted in spaces with unsavoury topological pathologies.

But perhaps this collection of desiderata for what it takes for singular structure to be localizable is still too demanding. First, it seems that one could usefully distinguish between the notions of 'localizability' and 'locality', as applied to properties of spacetime. Second, once localizability and locality are distinguished, one might challenge the view that singularities must be localizable in the sense that they should be represented by points of some extended space that is structurally apt to represent spacetime structure. For one might distinguish a weaker understanding of localizability, applicable to properties of spacetime, namely that a property F of spacetime is localizable just in case: (a) there is a determinate answer to the question 'Is F exhibited in such-and-such spacetime region?', for any (appropriately structured) spacetime region, in any given spacetime, and (b) that the answer is somehow non-trivial and informative.[2]

Following this line of thought, I will make two claims. First, that there are many kinds of singular structure, each associated with a type of curve incompleteness that provides witnesses for it, and this pluralism is a result of vagueness associated with the notion of 'physically reasonable motion' in general relativity. Second, that one should think of any type of singular structure as an extrinsic and non-local property that can be nevertheless 'localized'—in one

1 And it has a fascinating history (Earman [1995], ch. 1; Earman and Eisenstaedt [1999]; Norton [2024]).

2 This idea is not entirely novel. In an extended version of his illuminating paper on singular spacetimes, Curiel ([unpublished], 4, n. 6) mentioned this sense of localizability in a footnote containing a brief discussion of a pseudo-Riemannian surrogate for the concept of a 'bounded subset'. Here, my aim is not to propose a pseudo-Riemannian surrogate of the concept of a 'bounded subset', but—in the spirit of Curiel's remark—to propose a better sense of 'localizability' of the singular structure of spacetime: that is, as an answer to the question 'Where is the singular structure of spacetime exhibited?', at least insofar as general relativity is concerned.

interesting sense of this word—in an arbitrary (appropriately structured) spacetime region in virtue of certain facts about its topological closure.

Here's the plan for the paper. In Section 2, I provide an overview of various attempts to characterize singular structure in general relativity, both in terms of incomplete curves and boundary points of extended spaces. I conclude that the former are preferable, and that the criteria for singular structure that lie on the spectrum between bounded acceleration incompleteness and *b*-incompleteness are the best formal explications of the notion of singular structure that we have. Then, in Section 3, I offer some remarks about the metaphysics of spacetime's singular structure—I distinguish between the nominal, the adjectival, and the extended adjectival conceptions of singular structure (with some critical remarks about the first two), thereby extending the taxonomy offered by Earman ([1995], pp. 28–9). In Section 4, I distinguish between the notions of 'locality' and 'localizability' as applied to spacetime properties, as well as between different senses of 'locality', and choose the most apt sense in which singular structure can be 'localizable' in spacetime. Then, I offer a definition schema that specifies exactly under what conditions a spacetime region is singular (according to any particular explication of singular structure), and present some of its consequences. Finally, in Section 5, I compare my proposal with an approach inspired by Geroch ([1970]) and Manchak ([2021]).

2 Defining Singular Structure

Let me begin with a quick clarification. A 'relativistic spacetime' is a pair $(M, \mathbf{g})$, where M is a connected, Hausdorff, smooth 4-manifold, and $\mathbf{g}$ is a smooth Lorentz metric of signature $(3,1)$.[3] In what follows, I treat facts about relativistic spacetimes as representing, in a way perspicuous enough for our purposes, various physical facts associated with physical spacetimes that are possible according to general relativity (unless I explicitly say otherwise).

In worryingly vague terms, a relativistic spacetime is said to exhibit singular structure whenever the geometry of that spacetime, in some sense, 'breaks down'. Even though the notion of singular structure has become an indispensable piece of conceptual machinery in general relativity, it is exceedingly difficult to pin down the exact meaning of this vague characterisation.

In Section 2.1, I argue that singular structure is witnessed by incomplete curves, often

[3] See O'Neill ([1983]) and Wald ([1984]) for an account, and further discussion, of these notions. I use regular brackets for n-tuples of models and boldface font for tensors following Hawking and Ellis ([1973]).

independently of facts about curvature; but which incomplete curves witness singular structure depends on what kind of motion one classifies as 'physically reasonable' (in fact, what kind of acceleration one classifies as 'physically reasonable'). Some proposals about this issue fare better than others, but there seems to be plenty of room for disagreement which, perhaps, can only be resolved in a context-dependent manner. Consequently, there is no single best candidate for an explication of singular structure in terms of curve incompleteness.

In Section 2.2, I briefly discuss various types of mathematical constructions that aim to represent singular structure as points on the boundaries of appropriately extended spacetimes. I will conclude that all of them suffer from serious defects.

2.1 Curve incompleteness and curvature growth

Initially, one might suggest that the singular structure of spacetime is witnessed by the existence of a region where some curvature scalar invariant, such as the Ricci scalar or the Kretschmann scalar, goes to infinity, and which can be reached by some observer attaining only a finite affine parameter. This kind of behaviour occurs in cases that physicists paradigmatically classify as 'singular', such as the self-gravitational collapse scenarios, or in some regions of the Kruskal–Schwarzschild and the FLRW solutions to the Einstein's field equations.

Of course, there is no well-defined point on the spacetime manifold where the value of any curvature scalar is infinite; so, strictly speaking, it makes no sense to say that an observer will 'reach' the region of infinite curvature.[4] Rather, in these cases, there are causal geodesics which are incomplete—that is, are inextendible and of finite affine parameter—and along which the relevant curvature scalar goes to infinity.[5] However, it might seem that the existence of such incomplete causal geodesics is, independently of facts about curvature, a mark of some 'breakdown'—or, perhaps less dogmatically, an 'unpleasant feature'—of spacetime's geometry. For, at least in the timelike case, they represent possible trajectories of observers in free fall that, as a matter of spacetime geometry, only record finite proper time on their clocks

[4] One can contrast this with singular behaviour of a matter field defined on spacetime, where the value of the field grows without bound as one approaches some point of spacetime, but the spacetime structure remains well-defined. Similar behaviour can occur in general relativity in the form of 'sudden singularities', which are characterized by the divergence of some physically salient quantity, such as pressure or energy density, in finite time—see, for example, Cattoën and Visser ([2005]), and Curiel ([2023], sec. 1.4) for a philosophical introduction. These phenomena are very interesting on their own, but—since the metric tensor remains well-defined at such singular points, and so the geometry does not 'break down'—they present interpretative difficulties of a different kind, and I set them aside here.

[5] A continuous curve $\gamma : I \to M$, with $I \subseteq \mathbb{R}$, is inextendible (or, equivalently, has no endpoint) iff there is no point $p \in M$ such that for every neighbourhood U of p there exists $t \in I$ where $\gamma(t') \in U$ for all $t' > t$.

during their existence.[6]

Indeed, I take it to be one of the constitutive features of the concept of 'singular structure' that it designates something physically troublesome. And, in this vein, I think it is apt to think that a necessary condition for singular structure is that it somehow 'disrupts' possible trajectories of observers in some physically reasonable motion (such as free fall), where the 'disruption' is exhibited by the fact that they travel along worldlines of finite affine parameter.[7] Now, I say 'physically reasonable motion', because some constraints on the type of motion disrupted by singular structure must be put in place, at least if one wishes to stay reasonably close to the physicists' use of the concept and retain a non-trivial distinction between singular and non-singular spacetimes. For otherwise, there would be no reason to discount, say, timelike curves of finite proper length and arbitrary unbounded acceleration as witnesses of singular structure. Doing this, however, would classify Minkowski spacetime, which contains such curves, as singular—an unfortunate outcome for most, since Minkowski spacetime doesn't seem physically troublesome at all. I shall come back to this question of 'physical reasonability' of different kinds of motion, but, before that, let me note two things.

First, geodesic incompleteness and unbounded growth of curvature do not always go hand-in-hand. It's not difficult to find spacetimes with incomplete causal geodesics, where no curvature scalar goes off to infinity, and all physical components of the Riemann tensor (in all frames) are well-behaved—one example is Minkowski spacetime with an excised point.[8] One might be worried that such examples involve physically and philosophically unmotivated restrictions on the size of spacetime manifold, and thus it seems appropriate to restrict our attention to inextendible spacetimes (that is, spacetimes that cannot be isometrically embedded into another spacetime that is also their proper superset). What's more, every extendible spacetime contains an incomplete timelike geodesic (Clarke [1993], p. 8), so one might also worry that extendible spacetimes give us singular structure 'on the cheap'. Indeed, many treatments of singular structure in general relativity either stipulate at the outset that they restrict

6 Admittedly, it's more difficult to attribute similar significance to null geodesic incompleteness. One might argue that the affine parameter for null curves must have genuine physical significance, since it's an essential component of a non-trivial definition of averaged null energy condition (Visser [2023]); yet, the physical significance of such averaged conditions is not well-established either (Curiel [2017], p. 60). I set this issue aside.

7 In fact, this is very similar to the second of two desiderata of singular structure given by Geroch: 'In a nonsingular spacetime, observers who follow "reasonable" (in some sense) world lines should have an infinite total proper time' ([1968b], p. 532). I would like my point to be read as a reminder of this wisdom.

8 Physical components of the Riemann tensor **Riem** along a curve γ, relative to a choice of frame field $\{\mathbf{e}_i(t)\}$ with $i \in \{1,2,3,4\}$ on $\gamma(t)$, are given by $Riem_{ijkl}(t) := \mathbf{Riem}(\mathbf{e}_i(t), \mathbf{e}_j(t), \mathbf{e}_k(t), \mathbf{e}_l(t))$.

their attention to inextendible spacetimes (Hawking and Ellis [1973], p. 58; Ellis and Schmidt [1977], p. 920), or make a distinction between merely 'incomplete' spacetimes and genuinely 'singular' ones (Clarke [1993], p. 10; Landsman [2022], p. 132).

I am not fully convinced by this kind of reasoning, but, for the sake of argument, I will assume that extendible spacetimes are irrelevant for the question of singular structure.[9] For, in any case, there are geodesically incomplete spacetimes with no curvature scalar blow-ups—and with well-behaved physical components of the Riemann tensor in all frames—that, unlike the previous example, are inextendible. One such spacetime is the universal cover of Minkowski spacetime with an excised timelike two-plane (Ellis and Schmidt [1977], p. 923), another is the spacetime with a 'conical singularity' obtained from Minkowski spacetime with an excised point by cutting out a wedge containing points of a given azimuthal coordinate range and gluing the edges back together (Wald [1984], p. 214). That being said, the relationship between singular structure and inextendibility is subtle—both technically and philosophically—and I will return to it in Section 5 below.

Second, it is very plausible that not all 'physically reasonable' motion is free fall, and thus even geodesic incompleteness (in any of its forms) might undercount the instances of singular spacetimes. Indeed, Geroch ([1968b], pp. 534–40) provides an example of a geodesically complete spacetime which contains an inextendible timelike curve of bounded total acceleration with finite proper length. Such a curve represents a worldline of an observer in a rocket ship with finite thrust and finite amount of fuel, whose existence ends in finite time—again, as a matter of spacetime's geometry. All hands agree that bounded acceleration is physically reasonable, so such curves should certainly count as witnesses of spacetime's singular structure.

So, to accommodate the cases of bounded acceleration as witnesses of singular structure, one needs a criterion weaker than causal geodesic incompleteness. One immediate idea, mentioned by Earman ([1995], p. 35), is 'bounded acceleration [in]completeness', which classifies a spacetime as singular just in case it contains inextendible causal curves of finite affine parameter and bounded acceleration (this, of course, includes geodesics).[10]

9 Clarke ([1975], pp. 65–6, [1993], p. 8) and Ellis and Schmidt ([1977], p. 920) argue for inextendibility on that grounds that no known physical process could, so to speak, 'stop spacetime short of becoming', whereas Earman ([1995], pp. 32–3) motivates the same conclusion with the appeal to the 'dynamical version of the principle of sufficient reason'. For some arguments against the demand of inextendibility, see Manchak ([2011], [2016]).

10 To be precise, Earman ([1995], p. 35) only discusses timelike curves of bounded acceleration. But encompassing all *causal* curves is necessary if one wants bounded acceleration completeness to *imply* causal geodesic completeness—Kundt ([1963], p. 489) provided an example of spacetime which has no incomplete timelike curves of bounded acceleration, but does have incomplete null (and spacelike) geodesics. There's also

Another proposal for a necessary criterion for singular structure uses the notion of 'generalized affine parameter' (Schmidt [1971]). Generalized affine parameter is definable for any continuously differentiable curve by, roughly speaking, integrating over the sum of components of the tangent vector along that curve, where such components are defined relative to a choice of parallely propagated frame field. However, even though this definition is frame-dependent, whether a curve has finite generalized affine length is frame-independent (Hawking and Ellis [1973], p. 259). A curve $\gamma : I \to M$ that has finite generalized affine length and no endpoint is said to be '*b*-incomplete', and a spacetime $(M, \mathbf{g})$ is '*b*-incomplete' iff it contains a *b*-incomplete curve. The *b*-incompleteness criterion is strictly weaker than bounded acceleration incompleteness, even if one remains interested only in causal curves.

To establish whether timelike *b*-complete curves always represent physically reasonable trajectories, one should first look at the (somewhat subtle) relationship between generalized affine length and proper time. Every curve of infinite proper time will have infinite generalized affine length, but the converse is not true—Minkowski spacetime has no *b*-incomplete curves (Beem et al. [1996], p. 215), and yet has curves of finite proper length and unbounded acceleration.

A sufficient condition for a timelike curve of finite proper time to have finite generalized affine length has been established by C.J.S. Clarke (Sussman [1988], p. 967): a timelike curve of finite proper length will have finite generalized affine length if the acceleration does not grow 'too rapidly'. More precisely, any timelike curve $\gamma : [0, \tau_{\text{fin}}) \to M$ of finite proper length also has finite generalized affine length whenever after some proper time $\tau > 0$, there exists $L \in (0, 1)$ such that the magnitude of acceleration $a(\tau')$ is 'Lipschitz-bounded' by $a(\tau')(\tau_{\text{fin}} - \tau') < L$ for all $\tau_{\text{fin}} > \tau' \geq \tau$.[11] Thus, the increase in acceleration must be appropriately offset by the decrease in the remaining 'lifetime' of the observer travelling along that curve.

It's clear that this criterion classifies curves of bounded acceleration and finite proper length as *b*-incomplete, and thus as witnesses of singular structure (contrary to the claim of Curiel ([2023])). But it also classifies some curves of unbounded acceleration and finite proper length as *b*-incomplete (not all such curves, however—recall the case of Minkowski spacetime). Thus, insofar as one wishes to remain committed to the idea that relevant curves witness singular behaviour exactly when they represent possible trajectories of observers in physically reasonable

a family of conditions intermediate between geodesic completeness and bounded acceleration completeness, which is discussed in Avez ([1963], pp. 140–1) and Geroch ([1970], pp. 272–5).

11 The acceleration vector $\mathbf{A}|_{\gamma(t)}$ of a curve γ at a point $\gamma(t)$ is defined as $(\nabla_{\mathbf{V}}\mathbf{V})|_{\gamma(t)}$, where $\mathbf{V}$ is the tangent vector field on $\gamma[I]$ and ∇ is the Levi-Civita connection. The magnitude of acceleration $a(t)$ at $\gamma(t)$ is $\sqrt{\mathbf{g}(\mathbf{A},\mathbf{A})}|_{\gamma(t)}$.

motion, the choice between b-completeness and bounded acceleration completeness boils down to the question: Is 'Lipschitz-bounded' acceleration physically reasonable?[12]

An immediate worry is that for an observer to travel along a curve of unbounded acceleration seems to require an infinite amount of energy. The trouble, of course, is that it's notoriously difficult to make such claims precise, because 'energy' is a tricky concept in general relativity, with no frame-independent definition either for a point particle or for the stress-energy field (Curiel [unpublished], p. 25). Perhaps, as a matter of epistemic humility, this should make us open to the idea that some curves with unbounded acceleration might represent physically reasonable motion; but, I submit, to conclude from this that there cannot be a useful distinction regarding physical reasonability of different kinds of motion in general relativity, so that any inextendible timelike curve of finite proper time witnesses singular structure, would be an overreaction.[13]

One might refine this initial worry about infinite energy with the claim that the integrated acceleration (that is, the possibly improper integral of the magnitude of acceleration with respect to proper time) along some such curves doesn't converge.[14] Indeed, Chakrabarti et al. ([1983]) argue that any physically reasonable observer must traverse some future-directed timelike curve $\gamma : I \to M$ of finite integrated acceleration. First, they show that this condition is equivalent to the existence of a 'mass' function $m : I \to \mathbb{R}^+$ such that: (a) there exists $K \in \mathbb{R}^+$ such that $m(\tau) > K$ for all $\tau \in I$, and (b) the rate of change of $-m\mathbf{V}$ (where $\mathbf{V}$ is the tangent vector

12 Some authors object to the b-completeness criterion on more pragmatic grounds. Curiel, for example, says that 'perhaps the most damning fact about b-completeness is that, so far as I know, it is never used in the statement or demonstration of any result of physical interest' ([unpublished], p. 8). A similar attitude is exhibited by Manchak, who calls it 'mathematically cumbersome and physically dubious' ([2014], p. 1071). With regard to the former complaint, one might remark that generalized affine parameter is very useful for formulating averaged the null energy condition for null non-geodesics (Curiel [2017], p. 57), and it has also been recently applied in the context of investigating (non-)uniqueness of spacetime extensions (Sbierski [2024]). In any case, my main reply to these complaints is this: the primary aims of an adequate criterion for singular structure are: (a) to correctly classify paradigmatic cases of singular structure, and (b) to sharpen our understanding of what is at stake in more debatable cases of singular structure. Arguably, b-completeness meets these aims, and is no worse than its best competitors. This, on its own, seems to me like 'a result of physical interest'. The fact that the concepts employed in the development of the notion of b-completeness are not as widely applicable, or as mathematically elegant, as one might wish, seems like a secondary concern that can only motivate a search for more useful, more elegant, tools, but which in no way discredits the b-completeness approach *tout court*.

13 Curiel ([unpublished], p. 26) takes this line of reasoning, and suggests that '[s]ingular structure would simply be one more type of global structure that all spacetimes necessarily had' and that '[o]nce so much was settled, then one could further classify spacetimes, according to the needs of the project at hand, by satisfaction of more restrictive types of curve-completeness'. My disagreement with him on this point, however, is more in letter than in spirit, as will be more explicit in the following paragraphs.

14 For that integral to converge, one would require a stricter offset of acceleration growth, for example something like $a(\tau')\sqrt{(\tau_{\text{fin}}\tau' - \tau'^2)} < K$, for some $\tau' \geq \tau_{\text{fin}}/2$ and some constant $K \in \mathbb{R}^+$. Unfortunately, I am unaware of any type of parameter whose value is finite iff such a condition is met.

field to γ) along γ is future-directed and causal (this quantity supposedly represents the rate of discharge of energy-momentum by the observer, whereas m is the observer's mass). They also show that: if a spacetime admits a Killing field $\mathbf{K}$ that is non-spacelike on γ, then γ has finite integrated acceleration iff the 'effective energy' $\mathbf{g}(\mathbf{K}, \mathbf{V})$ is also finite along γ.

Chakrabarti et al. ([1983]) bolster this proposal by showing that observers traversing timelike curves of finite integrated acceleration avoid some physically dubious scenarios that are possible for observers traversing generic timelike curves.[15] These are certainly interesting results, but I'm sceptical whether the justification of their proposal is sufficiently general to provide a once-and-for-all criterion that demarcates between reasonable and unreasonable motion.

First, the equivalence between finite integrated acceleration and the existence of an appropriate 'mass' function relies on the assumption that spacetime is time orientable.[16] Although this assumption is common among relativists, it's not always clear on what experimental or conceptual grounds is it justified, for: (a) we have no conclusive empirical evidence that our spacetime is time orientable (Lemos et al. [2023]; Bielińska and Read [2023]), and (b) it's not obvious—contrary to the received wisdom—whether global time non-orientability is inconsistent with time asymmetric laws.[17] Perhaps the strongest argument for time orientability comes from semi-classical gravity: for time orientability is a consequence of a reasonable assumption about the algebra of observables on a possibly non-globally hyperbolic spacetimes, namely that it 'locally' behaves like the algebra of observables obtained though the standard quantisation procedure known from quantum field theory on globally hyperbolic spacetimes (Kay [1992]).[18] While this undoubtedly provides some motivation for the claims that physically reasonable spacetimes are time orientable, I'd prefer to set considerations from semi-classical gravity aside insofar as we wish to study singular structure as a purely classical relativistic

15 This includes reaching null infinity in Minkowski spacetime and reaching an arbitrarily low r-value in Reissner–Nordström and negative mass Schwarzschild spacetimes. Sometimes, however, curves with finite integrated acceleration do exhibit physically unsavoury behaviour—for example, they can be closed in Gödel spacetime.

16 Without this assumption, one can at most derive that $a(\tau) \leq |\nabla_{\mathbf{V}} \ln(m(\tau))|$, whereas what is needed for the equivalence is $a(\tau) \leq -\nabla_{\mathbf{V}} \ln(m(\tau))$ (Chakrabarti et al. [1983], p. 597).

17 The most convincing argument for their inconsistency, due to Earman ([2002], p. 257), relies on the assumption that a time non-orientable spacetime has closed causal curves. Yet this isn't true of all such spacetimes: the elliptic version of de Sitter spacetime is time non-orientable, but has no such curves (Belot [2023], p. 64).

18 More precisely, Kay ([1992]) shows that time orientability is a consequence of a condition on the global algebra of observables that he calls 'F-locality', which demands that every neighbourhood of every point of spacetime has a globally hyperbolic subneighbourhood such that the restriction of the global algebra to that sub-neighbourhood coincides with the algebra obtained from the standard quantisation procedure for globally hyperbolic spacetimes applied to that subneighbourhood *qua* spacetime on its own. For an overview of recent developments in quantum field theory on non-globally hyperbolic spacetimes, see (Kay [2025], sec. 10). I thank an anonymous referee for bringing this issue to my attention.

phenomenon. Second, and more importantly, the result relating integrated acceleration with 'effective energy' is quite limited, for the definition of 'effective energy' relies on the existence of a (causal) Killing field, which one cannot expect to find in a generic relativistic spacetime, not even locally (Beig et al. [2005]).

All of this leads to a somewhat pluralistic conclusion: there is arguably no single notion of curve incompleteness that constitutes both a sufficient and a necessary condition for the existence of singular structure, although there is a spectrum of plausible candidates. This, however, is to be expected, for in order not to trivialize the notion of singular structure, we should impose a constraint that incomplete curves representing physically unreasonable motion should not count as witnesses of singular structure. Yet the notion of 'physically reasonable motion' is itself vague and dependent on concepts that cause a lot of trouble in general relativity, such as 'energy'. Since bounded acceleration is certainly a physically reasonable kind of acceleration, and it seems unlikely that unbounded acceleration that leads to b-incomplete curves is physically reasonable, we can think of two most-discussed candidates—bounded acceleration completeness and b-completeness—as points at the edges of the spectrum of reasonably adequate explications of the notion of singular structure that should be implemented, depending on a particular context, based on their practical and conceptual merits.

2.2 Boundary constructions

There are many proposals that attempt to represent singular structure as elements of boundaries of suitably extended spacetimes.[19] I will not explicitly present any of these proposals, but will merely report that all of them face serious technical obstacles and offer some words of motivation why such technical obstacles have interpretational significance.

There are specific problems associated with each of these boundaries constructions. Sometimes they lose generality by presupposing some particular constraints on the causal structure on spacetime;[20] sometimes they misclassify cases of intuitively singular structure as non-

19 The most-well known ones include: the g-boundary (Geroch [1968a]); the b-boundary (Schmidt [1971]); the c-boundary (Geroch, Kronheimer et al. [1972]); and the a-boundary (Scott and Szekeres [1994]). See García-Parrado and Senovilla ([2005], sec. 6) for a review.

20 In particular, the c-type boundaries have so far only been defined for strongly causal spacetimes (Szabados [1988]). More generally, it's worth noting that the causal conditions presupposed by the strongest singularity theorems—which show that singular structure is, in some sense, 'generic' and not a special feature of highly symmetric solutions—are, in fact, fairly weak. Whereas Penrose's ([1965]) original theorem presupposed, among other things, global hyperbolicity, recent results weaken this condition significantly (Lesourd [2018]; Minguzzi [2020]). Since these results provide evidence that singular structure might be a 'generic' feature of spacetimes obeying very mild causal conditions, they cast further shadow on any boundary constructions that

singular;[21] and sometimes they yield serious topological pathologies.[22] These problems are well-known and going through them would not be particularly illuminating for my main claims announced in Section 1. So let me just focus on two most general issues, emphasized by Curiel ([unpublished], pp. 15–6), that apply to most possible proposals, and seriously weaken the prospects for an adequate boundary construction for spacetime that accounts for its singular structure.

Firstly, it's a fact of mathematics that no compact manifold can be embedded as an open submanifold of any Hausdorff manifold (with or without boundary), just because any compact subset of a Hausdorff space is closed. Since there are examples of geodesically incomplete compact Lorentzian manifolds, such as the Clifton–Pohl torus (O'Neill [1983], p. 193), this shows that any boundary construction must face the following dilemma: either it 'gives up' on being Hausdorff, or it no longer has the original spacetime as an open submanifold of the extended spacetime that includes singular points.[23] The former choice leads to a rather pathological topological structure. The latter choice seriously undermines what one might call 'the representational adequacy' of the boundary construction, for we would have no reasons to believe that it adequately represents the singularity-free part of physical spacetime (insofar as the standard of 'adequate representation' is set by the original spacetime manifold itself). Admittedly, this is an issue of a philosophical, rather than physical, nature, for it does not directly affect the issue of 'doing physics at a singularity'; it does, however, weaken any support that such a construction might provide for the views that 'singularities are objects', or 'singularities correspond to missing points', which will be discussed in Section 3.

Secondly, and more importantly, Geroch, Can-bin et al. ([1982]) showed that any extended spacetime will be non-Hausdorff (in fact, non-Fréchet), provided it satisfies the following two conditions: (a) every incomplete geodesic terminates at a point of the boundary, and (b) the extension of the exponential map from the original spacetime manifold to the extended spacetime is continuous.[24] And all hands agree that failure of sufficient separation for a boundary

rely on substantial assumptions about causal structure.

21 In particular, the a-boundary proposal misclassifies the singular structure of Taub spacetime associated with two families of incomplete null geodesics as non-singular (Curiel [unpublished], p. 20).

22 In particular, the b-boundary of the FLRW spacetime contains one point that is not Hausdorff-separated from any point in the interior (Bosshard [1976]; Johnson [1977]).

23 One can also discount singular structure witnessed by curves contained within compact sets as 'genuine' (Scott and Szekeres [1994], p. 34). I agree with Curiel's ([unpublished], p. 20) argument for the conclusion that this move is not well-motivated.

24 Informally, the exponential map $\exp_p$ at point $p \in (M, \mathbf{g})$ takes a vector $\mathbf{X}|_p \in T_pM$ and returns a point $q \in M$ that lies unit affine length away from p on the geodesic tangent to $\mathbf{X}|_p$ at p whose affine length exceeds

construction would render it inadequate, both physically and philosophically; so the Geroch–Can-bin–Wald result looks like a 'no-go' theorem for boundary constructions based on the notion of curve incompleteness, such as the *g*-boundary or the *b*-boundary, at least insofar as the conditions featured in it are necessary for adequacy of any such construction (and insofar one wishes to have a criterion applicable to all relativistic spacetimes).[25]

Indeed, this appears to be the case. Condition (a) is arguably constitutive of what a boundary condition is supposed to do about spacetime's singular structure, so violating it would seriously undermine the motivation for the boundary construction in the first place. Condition (b) is likewise reasonable, although I have not found an explicit defence of it anywhere. But one can offer an intuitive motivation for it along the following lines: if condition (b) were violated, then we would not be guaranteed that any particular way of narrowing down the point and the direction along which a geodesic can start from that point in spacetime will be mapped onto a corresponding way of narrowing down the point (possibly on the boundary) that the relevant geodesic passes at unit affine length.[26] Since, for any point in spacetime, any tangent vector at that point corresponds to a unique geodesic satisfying the conditions of the exponential map (as a consequence of the existence and uniqueness of solutions to ordinary differential equations), this loss of ability to 'localize' the point unit affine length away—with means that are available 'at the origin'—strikes me as quite puzzling. So, while the failures of Hausdorff separability are often interpreted as showing that the boundary constructions attach singular points that are 'too close' to points in the spacetime interior, condition (b) can perhaps be interpreted as demanding that the singular points are not attached 'too far' from points in the interior.

One can also mount a 'transcendental' argument for condition (b); namely, that violating it prohibits the construction of many useful formal tools and thus hinders one of the primary

one. The extension of this map also allows us to take as arguments those tangent vectors at p that generate geodesics whose affine length is exactly one, and stipulates that their values will be the singular points on the boundary that are determined by those geodesics (Geroch, Can-bin et al. [1982], p. 432).

25 It's worth mentioning that Flores et al. ([2016]) offer a general topological way of enforcing Hausdorff separation between the points in a spacetime's interior and those on its boundary (for various boundary constructions) through a canonical minimum refinement of the topology on the boundary. However, when applied to the boundary constructions based on curve incompleteness, such as the *g*- or *b*-boundaries, their refinements still lead to violation of condition (b). Although they do not provide an argument against condition (b), they do note ([2016], p. 13) that the counter-example of Geroch, Can-bin et al. ([1982]) relies on having a non-compact subset of the form $J^+(p) \cap J^-(r)$, and one could in principle circumvent it by restricting one's attention to globally hyperbolic spacetimes (there, all sets of that form are compact). This, however, is a very significant constraint on the causal structure of spacetime that, I think, diminishes the significance of any mathematical construction that assumes it, at least as regards our conceptual understanding of singular structure in general relativity.

26 This intuitive picture has a formal explication in terms of the definition of continuous function between topological spaces in terms of either nets or filters (Willard [1970], pp. 75, 79).

aims of boundary constructions, which is to allow one to 'do physics at a singularity'. In particular, the continuity of the extended exponential map is a weak condition that already does not prevent the map from exhibiting some undesirable lower regularity behaviour (recall that the original exponential map is smooth). For example, a non-differentiable extended exponential map does not have a well-defined differential. This has several consequences. First, the extended exponential map is not a local diffeomorphism, and there is no clear way in which one can use it to define normal neighbourhoods.[27] Second, lack of a well-defined differential jettisons the standard link between the exponential map and Jacobi fields on the manifold (O'Neill [1983], p. 217), which measure the 'spread' between nearby geodesics through the geodesic deviation equation—which itself is of fundamental importance in relativity. These strike me as significant drawbacks for any boundary construction, at least insofar as it aims to allow us to 'do physics at a singularity'. And they are also good reasons to require the extended exponential map to have at least a continuous derivative, let alone be continuous itself.

Hence the condition (b) imposed on extended spacetimes by Geroch, Can-bin et al. ([1982]) is well-motivated and their result is a genuine 'no-go' theorem. Consequently, the prospects for an adequate boundary construction based on the notion of curve incompleteness are dim.

3 Conceptions of Singular Structure

So far, I've discussed various attempts at defining the singular structure of spacetime within the mathematical formalism used in general relativity, and how these attempts have to face the tribunal of adequacy and usefulness. However, one must also note that one's attitude to the question of definition not only influences, but also is influenced by, a more philosophical kind of consideration, namely: what is one's conception of spacetime's singular structure, that is, what do we mean when we talk about singularities or singular structure. The rival conceptions that I will consider are both realist conceptions—they take singular structure to be a genuine physical feature of the world; so, insofar as they disagree, they disagree about what kind of feature singular structure is.[28]

27 For the standard treatment, see O'Neill ([1983], pp. 71–2).

28 It's a common sentiment among some physicists to treat singular structure as an artefact of the formalism that marks the breakdown of general relativity, rather than as a genuine physical feature of the world discovered by the theory. The implicit hope is that a successful quantum theory of gravity would account for quantum phenomena in high energy regimes in a way that prevents bad behaviour of the metric, and thus eliminates singular structure (for further discussion, see Crowther and De Haro ([2022])). However, the philosophical arguments behind that sentiment are not particularly convincing (Earman [1996]; Curiel [unpublished], sec. 2.2). And, moreover, it seems that singular structure could not be so easily removable in the quantum setting

Let me begin with a distinction, due to Earman ([1995], pp. 28–9), between two conceptions of singular structure: the adjectival one, and the nominal one. According to the former, as formulated by Earman, singular structure is a property of spacetime. One can, strictly speaking, only say that such and such spacetime is, or is not, singular, but no sense can be made of the claim that there is an object—a singularity—which is 'missing from spacetime', and in virtue of which the spacetime is singular. This conception can be easily extended to a more permissive one, where singular structure is a property of arbitrary spacetime regions (represented by subsets of the relativistic spacetime that have desired topological properties), which include the whole spacetime itself. Admittedly, I see no metaphysical reason against such an extension; a physico-mathematical reason might be that we have no explicit criterion for determining whether a particular spacetime region exhibits singular structure or not, on any admissible explication of this notion. I agree, but will try to improve the situation in Section 4 (and let me note that the improvement is rather elementary). In any case, I shall refer to this permissive extension of Earman's proposal as 'the extended adjectival conception' of singular structure.

On the nominal view, singular structure is exhibited in the form of objects—singularities. Such objects are not abstract in the sense that (say) numbers are, but they also do not have a spatiotemporal location: that is, they do not occupy a particular region of spacetime (quite memorably, Earman talks about their '∃xistence' ([1995], p. 28)). Borrowing a metaphor from Earman ([1995], p. 27), they can be conceived of as points, or regions, 'missing' from spacetime, which have their own topological features, both intrinsic and in relation to the rest of spacetime.

Even though I do not wish to divert the discussion too much into metaphysics proper, some words of defence must be said about the cogency of this distinction. For a reasonable sceptic might say: 'Well, all of this seems to hinge on the distinction between objects and properties; and I have no clue what that distinction exactly is, especially when applied to the singular structure of spacetime, since all common ways of distinguishing between objects and properties—concrete vs. abstract, sense-perceptible vs. sense-imperceptible, singly located vs. multiply located, located in spacetime vs. not so located, picked out by saturated vs. unsaturated expressions of some privileged language, and so on—either settle the dispute from the outset in some

in light of the Generalized Second Law of thermodynamics (Wall [2013]). In any case, those who share the sentiment can read this section with a larger pinch of salt than usual. Other sections, apart from occasional excursions into metaphysics in Sections 4 and 5, do not hinge on this realist reading of singular structure; they can be treated as addressing the question 'Where does general relativity break down?'.

irrelevant way, or fail to be adequate, since they presuppose some fixed background spacetime structure and thus ignore the possibility that something (meta)physically subtle might be going on with spacetime itself.'

I reply: the sceptic has their heart in the right place, but also demands too much from the distinction to be applicable. For, arguably, humans have a good enough ability to uniformly represent, in both language and thought, many perceived phenomena as cases of 'a thing being thus and so', then extract the 'thing' (an object) and the 'being thus and so' (a property) elements of this experience, and then check if, given another phenomenon, there is another thing involved and whether it is also thus and so, or perhaps whether it is the same thing as before but no longer thus and so. (Indeed, the distinction is commonly presupposed both in philosophical discussions of which properties can be represented in perception (Siegel [2010]), as well as in empirical study of human judgements about object persistence through property change in relation to changes in spatiotemporal location (Scholl [2007], sec. 3).) It is precisely this ability that underpins our attempts to explicate the distinction between the 'thing' and the 'being thus and so' bits of the phenomena we perceive. We might fail to offer an adequate explication that covers all the cases, or is independent of any particular language, but this does not affect the fact that the distinction is useful, and perhaps that by considering particularly troublesome cases, such as spacetime's singular structure, we might improve our grasp of it.[29]

Coming back to the idea of 'missing points', one might expect this view to be vindicated, or at least supported, by some mathematical construction that will unify spacetime points and singularities within one space; yet, as discussed in Section 2.2, almost all hands agree that the topological pathologies (and/or misclassifications) associated with such boundary constructions render them inconclusive as justifications of the nominal view of singular structure. What is the appeal of this conception then? Why does Earman say that it 'is arguably a touchstone of singularities in relativistic spacetimes: spacetime singularities correspond to missing points' ([1995], p. 40)?

It seems to me that there are two motivations behind the nominal conception that are often weaved together.[30] One is that this conception meshes well with the intuitive thought that

29 This story is similar to Armstrong's motivation for what he calls 'a substance-attribute view' ([1989], p. 60) of ontology. I decided to conveniently avoid traditional philosophical terminology, since I don't wish to argue that the naturalness and uniformity of this way of representing phenomena, or conceptualising our perceptions, reveals a fundamental ontological distinction that ought to be named. I merely claim that it is so common that we have good reasons to think it is fruitful.

30 In particular, Curiel ([unpublished], pp. 21–3) seems to mould the two motivations I shall discuss here into one, when he implicitly associates 'localizability' with 'locality'. I discuss these two concepts in more detail

spacetime's singular structure is, in some sense, 'localizable'; but then the question is: in what sense? I shall discuss this question in greater detail in Section 4, but for now, let me just foreshadow that the most adequate sense in which singular structure might be 'localizable' is that, in the spirit of some remarks from Section 1, there is a determinate answer to the question, for any region of spacetime, 'Is the singular structure of spacetime exhibited in such and such spacetime region?'. This meaning of 'localizability', however, is compatible with the extended adjectival conception, so it cannot constitute a reason to adopt the nominal conception.

The second motivation for the nominal conception is that, in contrast to the adjectival conception, it coheres with the possibility of doing (at least theoretical) physics 'at a singularity', with the hope that some formal construction will make this idea work (in an analogous way to Penrose's ([1963]) conformal compactification method that allows one to analyse asymptotically flat spacetimes 'at infinity'). Despite the fact that there is arguably no such construction available so far, the advocate of the nominal conception might insist that mere inconsistency between the adjectival conception and the possibility of such a formal construction leads to an unacceptable *a priori* restriction of physics by purely metaphysical considerations.

I am sceptical of this line of reasoning, however, since it's not clear to me that the extended adjectival conception of singular structure would be at odds with most natural interpretation of such a putative construction—it is simply very difficult to tell in the absence of it. One might speculate that such a construction would invoke a mathematical point that would represent the localized witness of singular structure. But it is not out of the question that such a mathematical point could be most naturally interpreted as a mere idealisation representing, say, a limit of a sequence of intersections of unions of spacetime regions that exhibit singular structure, rather than a point-like singularity.[31] Perhaps explicit details of such a construction could favour one option over the other; but one cannot really tell unless one sees the details.

Dissatisfaction with the nominal conception might lead one to proclaim the victory of something akin to the adjectival conception. For example, after a critical discussion of various attempts to define singularities in terms of points or holes, Lam claims that 'the singular behaviour of spacetime cannot be reduced to (is not supervenient on) some intrinsic properties instantiated at some particular spacetime (boundary) points', and that such behaviour 'seems to constitute an irreducible *nonlocal* feature of spacetime ... [that is, one which is] independent

in Section 4.

31 Such a non-literal interpretation has a precedent in general relativity: a different idealisation (although not different in relevant respects) justifies the 'vacuum approach' to the problem of motion (Lehmkuhl [2017]).

of the existence of any particular spacetime points (and of any intrinsic properties instantiated at a particular spacetime point)' ([2007], p. 718). Then, he remarks that 'the very concept of a spacetime point seems to lie at the heart of the challenge', and that such a 'misleading dependence on the atomistic and local conception of spacetime ... is actually induced by [the] standard differential geometric representation of spacetime' ([2007], p. 722). Consequently, he advises us to pay more attention to the algebraic approach to general relativity initiated by Geroch's [1972] article on Einstein algebras (Lam [2007], p. 719).

I agree with Lam that the adjectival conception of singular structure fares better than the nominal one, but I'd like to qualify two of his claims. First, he takes '*local* entities and properties ... to be those that can be associated with (and determined at) a spacetime points and its (arbitrarily small) neighbourhood' ([2007], p. 717). But when he talks about 'nonlocal' or 'global' structure, he only mentions scalar fields defined on the whole spacetime *qua* what is represented by the elements of a smooth algebra of functions in the algebraic formulation of general relativity (Lam [2007], p. 722). This usage, therefore, overlooks features that might be associated with all (arbitrarily small) neighbourhoods of points, but not with points themselves—although they would count as 'nonlocal' is Lam's sense. This is, of course, one way of specifying what 'local' means, and I have no principled reason to criticize it here; but let me flag that there are others ways, and I will take a closer look at them in Section 4.

Second, I disagree that the singular nature of spacetime reveals some flaws in the standard geometric formulation of general relativity (even those of misleading metaphysical propaedeutics), and thereby prompts us to move to the algebraic formulation. Although various developments regarding the relationship between the algebraic approach to general relativity and singularities mentioned by Lam are deep and interesting (Heller and Sasin [1999]; Heller, Odrzygóźdź et al. [2007]), the standard differential geometric approach does not lead to any confusions. Moreover, as I shall argue in the remaining part of this paper, it gives us all the technical machinery one needs for 'localizing' singular structure in the sense I defend here.

4 Locality and Localizability

Let me now turn to an exegesis of two crucial notions: 'locality' and 'localizability' of properties of spacetime (or spacetime regions). It is often claimed that the failure to characterize singular structure of spacetime in terms of boundary points of some extended space (or, equivalently, a choice to do so in terms of some kind of curve incompleteness) entails that singular

structure is 'not localizable' (Geroch, Can-bin et al. [1982], p. 435), or that it doesn't make sense to think of singular structure as exhibited by some 'localized object', that is, a singularity (Earman [1996], p. 625; Curiel [unpublished], p. 23). The reasoning behind these claims is surely sound, although it is only the latter conclusion that I find unobjectionable; for we've already seen in Section 2.2 that the prospects for reifying singular structure in the form of singularities (that is, localized objects) are weak. So, in this section, I argue that there is a sense in which the singular structure of spacetime, understood in line with the extended adjectival conception introduced in the previous section, can be non-local, and yet—in some interesting sense—localizable.

Before I discuss localizability, however, let me say a few words about another, related, concept that is often mentioned in the discussion of singular structure, namely: locality.[32] Once we're set on the idea that singular structure is a property of spacetime, it remains to be determined whether it is a local property. There is, however, no consensus as to what makes a property of spacetime (or its connected regions) local. The three most rigorous definitions that one can find in the literature are due to, respectively, Krasnikov ([2014], p. 2, [2018], p. 39), Curiel ([unpublished], pp. 21–2), and Manchak ([2009], p. 55).[33] They are as follows:

Definition 4.1 (Krasnikov-locality)**.** A property F of spacetimes is Krasnikov-local iff, for any spacetime $(M, \mathbf{g})$ and any open cover $\{U_\alpha\}$ of M, $(M, \mathbf{g})$ is F iff every connected component of any member of $\{U_\alpha\}$, considered as a spacetime on its own, is F.[34]

Definition 4.2 (Curiel-locality)**.** A property F of spacetimes is Curiel-local iff for every spacetime $(M, \mathbf{g})$, $(M, \mathbf{g})$ is F iff, for every point $p \in M$, every neighbourhood of p has a sub-neighbourhood of p that, considered as a spacetime on its own, is F.[35]

32 The notions of locality and localizability appear frequently in various discussions in the foundations of physics, although usually with a manifestly different meaning. In relativity, the issue of localization of gravitational stress-energy has been thoroughly investigated (Pitts [2010]; Gomes and Rovelli [2024]). In that context, 'localization' means that there is an appropriately 'absolute' object suitable to represent gravitational stress-energy at a point. I set this issue aside, since it's not clear how this sense relates to the senses relevant to singular structure; but I agree that clarifying the status of energy in general relativity would have consequences for the question of 'physically reasonable motion', as discussed in Section 2.1. I thank a referee for bringing up this point.

33 There are also less rigorous definitions, such as Lam's, who defines local features as 'those that can be associated with (and determined at) a spacetime point and its (arbitrarily small) neighborhood' ([2007], p. 717). A very similar definition was offered by Butterfield ([2006], p. 187).

34 Krasnikov's original definition does not require the relevant subsets to be connected, but since it assumes that they can be treated as relativistic spacetimes on their own, and his own definition of relativistic spacetime assumes connectedness, it's evident that some way of imposing connectedness must be implemented. Although there's more than one way to do it, the way I propose here strikes me as yielding the most plausible definition.

35 Curiel's original definition applies more generally to topological spaces, but this is its harmless restriction to

Definition 4.3 (Manchak-locality)**.** A property F of spacetimes is Manchak-local iff, for every two locally isometric spacetimes $(M, \mathbf{g})$ and $(M', \mathbf{g}')$, one of these is F iff the other one is too.[36]

(Let me note that Curiel-locality and Krasnikov-locality invoke the idea that local properties must be 'uniform' across $(M, \mathbf{g})$, whereas Manchak-locality does not. Hence, these definitions do not imply Manchak-locality: for example, the property 'has an open region with negative Ricci curvature scalar' is Manchak-local—for the intersection of such an open region with the region that locally witnesses the isometry is itself an open region with such curvature properties that must be preserved under isometry—but manifestly not Curiel-local nor Krasnikov-local.)[37]

It's quite clear that any criterion for singular structure based on some form of curve incompleteness will fail to be local on any of these definitions. But there are some definitions of locality, according to which the verdict is not so clear. Weatherall, for example, takes F to be a local ('Weatherall-local') property of spacetime when it 'can be associated with some region of space and time [represented by] ... some open set, with compact closure' ([2023], pp. 1342–3).

Whether singular structure characterized by some kind of curve incompleteness can be plausibly classified as Weatherall-local depends on what exactly one means by 'can be associated with some region' (I set aside the issue of relative compactness here). If one means 'determinable from the information available at that region' (or, equivalently, 'determinable from that region considered as a spacetime on its own'), then it's unlikely that singular structure would count as Weatherall-local. But, if one has a weaker reading in mind—where 'associated with some region' means 'determinable for that region from information (possibly) beyond that region, but in a way that allows distinctions between various regions'—then the outcome is not so clear.[38] As I see it, however, it is the former explication that captures the intuitive meaning of 'locality' better, whereas the latter explication fits much better as a re-description of the idea that there is a determinate fact about whether such a property is exhibited in any (sufficiently structured) region of spacetime, and that such ascription can be in some sense informative.

our domain of inquiry. One can also see similarities between this definition and Kay's ([1992], p. 172) notion of F-locality for algebras of observables in semi-classical gravity, mentioned in Section 2.1.

36 $(M, \mathbf{g})$ and $(M', \mathbf{g}')$ are 'locally isometric' iff for every p in M, there is a neighbourhood U of p, an open set U' of M', and an isometry $\phi : (U, \mathbf{g}|_U) \to (U', \mathbf{g}'|_{U'})$, and an analogous condition holds for every p' in M'.

37 Whether Curiel-locality implies Manchak-locality depends on whether F is preserved by isometries (which we might assume is the case, as we're interested in geometric properties of spacetimes). It's also the case that Krasnikov-locality is equivalent to Curiel-locality, so the former also implies Manchak-locality, as claimed by Krasnikov ([2018], p. 39) and Manchak ([2021], 416, n. 12), insofar as F is preserved by isometries. Thanks to JB Manchak for spotting a howler in an earlier version of my discussion of Krasnikov-locality.

38 The last clause—'... but in a way that allows distinctions between various regions'—dismisses a trivial and uninteresting way of defining singular regions of spacetime as, merely, regions of a singular spacetime.

So, we have good reasons to believe that singular structure, on all plausible definitions of this notion, is non-local.

The latter idea—of singular structure being a property of spacetime regions that is determinable for any region, regardless of how small it is, in a way that is informative and allows variation between regions—is interesting on its own, and it signifies the feature of singular structure that, I think, would be most aptly described as 'localizability'.[39] This point elaborates on my remarks from Section 1 that it is often appropriate to say that some regions of a particular singular spacetime exhibit singular structure, whereas others do not: for example, that region II of the Kruskal–Schwarzschild spacetime exhibits singular structure, whereas region I does not. Moreover, it coheres well with the idea present in various treatments of singularity theorems: namely, that specifying a region containing the incomplete curves that witness the singular structure is a desirable aspect of a given singularity theorem.[40]

So far, I argued that it's coherent to think that singular structure is non-local, and yet localizable. But it is one thing to argue that this distinction is coherent, and another thing to argue that it is, in fact, convincing. Whereas the non-locality of singular structure is rather uncontroversial (on all plausible definitions of 'locality' considered above), to show that singular structure is localizable, one needs an adequate criterion demarcating singular regions from non-singular ones, for any given spacetime. Let me turn to this issue now.

Since singular spacetimes were characterized by some form of curve incompleteness, one might wonder if the same cannot be done for singular regions of spacetime. Curve incompleteness properties are not (Curiel-)local, so it's clear that one cannot treat regions as spacetimes on their own, and then apply a particular criterion. But, as we no longer believe that singular structure is local anyway, we can use information from regions other than the region we're examining.

Now, the situation is quite straightforward. For we can check whether every half-curve of the relevant kind, contained in the region we are examining, and with a finite value of the relevant parameter, has an endpoint. Such an endpoint would be an adherent point to our region; so it will, by definition, lie in its topological closure. Thus we can say that, for proper subregions

39 This meshes well with what Earman ([1987], p. 453) calls 'semantic localizability' of physical states. I take up the question of what it means for a singular regions to be 'small' in Cudek ([unpublished]).

40 For example, Lesourd ([2018]) shows that future incomplete null geodesics that witness the singular structure of certain possibly achronological spacetimes obeying null energy condition can be 'localized' within a region that represents the interior of a black hole. In contrast, the classic theorem of Hawking and Penrose ([1970])—which, admittedly, applies to different spacetimes—doesn't do that.

of the whole spacetime, 'being singular' is a an extrinsic (as opposed to an intrinsic) property—whether it holds of a particular spacetime subregion depends on facts about spacetime points that do not belong to it. (Admittedly, there is a lot of debate about the exact formulation of the intrinsic/extrinsic distinction as applied to properties (Marshall and Weatherson [2023]). But I submit, in line with the mainstream metaphysics literature, that: (a) the imprecise distinction is sufficiently intelligible to seek for paradigm cases to which any adequate precise version of it would apply, and (b) it's easier to come up with cases of properties that one would paradigmatically classify as extrinsic. And I claim that 'being singular' is one such case.[41])

If every such curve has an endpoint, then we have reasons to believe that any such 'suspicious' curve can be extended beyond our region, and does not witness singular structure insofar as our region is concerned (even though the extension of that curve might witness singular structure in another region, possibly containing the original one). If, however, the curve does not have an endpoint, then we know that it will also count, as it is, as a witness of singular structure for the whole spacetime; and since it is contained in our region, it seems natural to treat it as a witness of singular structure in that region as well. (So, I admit I am happy to count incomplete null geodesics trapped in a compact region of Taub–NUT spacetime (Hawking and Ellis [1973], sec. 5.8) as witnessing certain types of singular structure.) Indeed, the fact that the existence of singular regions within spacetime implies that the whole spacetime is singular explicates the sense in which 'being singular' is determinable for any open region *in an informative way*.[42]

Throughout this discussion, I have not been specific as to what kind of curve incompleteness signifies singular structure. This is to emphasize that this idea is compatible with the view that I defended in Section 2.1: namely, that there might be multiple adequate explications of singular structure in terms of curve incompleteness. Let me group these explications under a generic term 'Φ-incompleteness', where 'Φ' ranges over different types of incompleteness: 'geodesic', 'bounded acceleration', '$b-$', and so on. I also mentioned, but have not elaborated

41 Notice that for proper spacetime regions, 'being singular' is a 'positive extrinsic' property in the sense of Lewis ([1983], p. 199), that is, a property whose instantiation by any object implies the existence of a distinct contingent object. Here, let S be proper subregion of some spacetime M. If S is singular, then this implies the existence of spacetime points distinct from those that comprise S: namely, those that form the closure of S in M. (The existence of such distinct points follows from the fact that relativistic spacetimes are connected.) Being 'positive extrinsic' in this sense strikes me as a very plausible sufficient condition for a property to be considered extrinsic.

42 Compare this with properties of spacetime regions that are 'uninformative' in the sense that they don't tell us anything about the properties of the whole spacetime. For example: any spacetime will have both simply connected and non simply connected open subsets.

on, the idea that one might require of spatiotemporal regions that are to be classified as (non-)singular, in any above sense, to be represented by subsets of relativistic spacetimes that have certain topological properties which go beyond being open.[43] Let me group subsets that have these properties under the generic term 'Ψ-subset', where 'Ψ' ranges over: 'connected', 'path-connected', 'simply connected', etc. (Willard [1970], ch. 8).

Now, let me propose a formal schema which specifies what it means for a certain region of spacetime to count as singular with respect to a specific kind of curve incompleteness. It goes as follows:

Definition 4.4 (Φ-incomplete singular Ψ-subsets)**.** For any relativistic spacetime $(M, \mathbf{g})$, an open Ψ-subset U of M is a 'Φ-incomplete singular Ψ-subset' of $(M, \mathbf{g})$ iff there is a Φ-incomplete curve $\gamma : [0, a) \to M$ whose image is contained in U.

This schema allows one to distinguish between, for example, various kinds of geodesically incomplete singular subsets, bounded acceleration incomplete singular subsets, and b-incomplete singular subsets. Another pleasant feature of this schema is that the role of the metric (if any) in determining the singular structure of spacetime is contained in the Φ-condition for incompleteness; so, the schema is in principle applicable to spacetime theories that do not posit a Lorentzian spatiotemporal metric, such as the Newton–Cartan theory.

Moreover, the schema meshes well with singularity-related concepts used in general relativity. First, it can be easily extended to accommodate region-based versions of the properties of being 'detectably nakedly singular' and 'evolvedly nakedly singular' (Manchak [2020], p. 24) that are of interest in the discussions of determinism and cosmic censorship (Earman [1995], sec 3.3).[44] More importantly, however, the schema can be put to some formal work: in another paper, I use the schema to discuss what it means for a singular region to be 'small' and offer technical results that relate some open, connected, b-incomplete singular regions of spacetime (in the sense of Definition 4.4) to bounded, Cauchy incomplete regions of the orthonormal frame bundle over that spacetime (Cudek [unpublished]).

[43] There might be various motivations for restricting one's attention to regions with particular topological features. For example, one might be interested only in spacetime regions that can be treated as spacetimes on their own. (Then, to comply with our definition of a *relativistic* spacetime, such regions must be connected. In other spacetime theories, such topological constraints might be different.)

[44] In particular, an open Ψ-subset U of M is 'detectably nakedly Φ-singular' iff there is a point $p \in U$ such that there is a Φ-incomplete curve $\gamma : [0, a) \to M$ whose image is contained in $I^-(p) \cap U$ (where $I^-(p)$ is the set of all points in M that lie in the timelike past of p). And an open Ψ-subset U of M is 'evolvedly nakedly Φ-singular' iff there is an edgeless closed achronal set $S \subset U$ and a point $p \in H^+(S) \cap U$ such that $I^-(p) \cap S$ is relatively compact (where $H^+(S)$ is the future Cauchy horizon of S). For definitions of edge and Cauchy horizon, see Hawking and Ellis ([1973], p. 202).

5 Singular Structure, Extendibility, and Extrinsicality

It might be instructive to compare the approach toward singular structure that I have advocated here with an approach inspired by the proposals of Geroch ([1970]) and Manchak ([2021]) regarding an alternative definition of 'being singular' for extendible spacetimes.[45] First, I will describe Geroch's proposal and Manchak's elaboration of it. Then, I will discuss how this proposal can be modified to the provide a criterion of what makes a particular spacetime *region* singular. Finally, I will argue against this criterion, on the grounds that it misclassifies some intuitively singular regions as non-singular, and implausibly treats singular structure as an intrinsic, rather than an extrinsic, property of spacetime regions.

Recall that every extendible spacetime contains incomplete timelike geodesics, and so will be classified as singular according to any plausible definition based on curve incompleteness. In other words, being extendible is a sufficient condition for being singular. One might find something unpleasant about this fact, and thus many physicists include the requirement of inextendibility into the very definition of 'being singular' (as mentioned in Section 2.1).

Alternatively, one might suggest that extendible spacetimes merit a definition of 'being singular' separate from the one applicable only to inextendible spacetimes. In this spirit, Geroch proposed a three-step procedure for arriving at such a definition:

> ... (1) define 'extendible', (2) define 'singular' for inextendible spacetimes, and (3) define 'singular' for all spacetimes ... [by saying that a] spacetime M will be called *singular* if every extension M' of M which is itself inextendible is singular (as an inextendible spacetime). ([1970], p. 262)

Manchak ([2021], p. 418) notes that Geroch's step (1) might be more difficult to execute than it at first seems if one restricts one's attention to the class of 'physically reasonable spacetimes' (that is, if one requires that an extension counts as permissible only if it also belongs to that class). For different ways of executing step (1) have different effects on the outcomes associated with step (3); in particular, it might happen that some physically reasonable spacetime that has a physically reasonable extension does not have a physically reasonable extension that itself has no physically reasonable extension (Manchak [2016]). For such spacetimes, Geroch's definition of 'being singular' from step (3) would be vacuously satisfied.

45 I'm grateful to an anonymous referee for suggesting this comparison.

To be clear, neither Geroch nor Manchak discuss what makes a spacetime *region* singular; their only concern is the definition of singular extendible spacetimes. But one can adapt their discussion to the former issue without much hassle.[46] Consider the following definition of a singular spacetime region (that I call the 'maximal extensions sensitive' (MES) definition):

Definition 5.1 (the MES definition)**.** An open and connected region U of $(M, \mathbf{g})$ is *singular* iff $(U, \mathbf{g}|_U)$ is a singular spacetime in the sense of Geroch's step (3), that is, iff every maximal extension of $(U, \mathbf{g}|_U)$ is singular.

First, note that the MES definition does not conform to the schema offered above (in Definition 4.4). Moreover, if one wished to impose on this definition a further restriction to the class of 'physically reasonable spacetimes', one would be faced with all the subtleties regarding the existence of maximal extensions with relevant properties (Manchak ([2021])).

I think, however, that the MES definition is not appealing—it misclassifies some intuitively plausible cases and it's also not well-motivated on philosophical grounds. First, notice that the MES definition implies that for any two spacetimes $(M, \mathbf{g})$ and $(M', \mathbf{g}')$ and any regions $U \subset M$ and $U' \subset M'$ such that there is an isometry from U to U', the region U of $(M, \mathbf{g})$ is singular iff the region U' of $(M', \mathbf{g}')$ is too. One can summarize this fact by saying that, on this definition, 'being singular' is preserved by isometries between regions.

One consequence of this fact is that the MES definition classifies certain intuitively singular regions as non-singular. In particular, I submit that an open connected region of an intextendible geodesically incomplete spacetime that contains an incomplete causal geodesic should be considered singular. However, some intextendible spacetimes have incomplete causal geodesics that are locally extendible, that is, there are open connected regions containing those incomplete causal geodesics that are isometric to open connected regions of a geodesically complete (and so inextendible) spacetime (Ellis and Schmidt [1977], p. 928). In those cases, contrary to the initial intuition, the MES definition classifies the former regions as *not* singular, despite the fact that they contain incomplete causal geodesics and are regions of an inextendible spacetime.

From a more philosophical perspective, it seems that the requirement of preservation by isometries between regions amounts to treating 'being singular' as an *intrinsic* property of spacetime regions.[47] However, since the concept of singular structure is so elusive, it is tempting to treat any definition of 'being singular' as giving us not only the necessary and sufficient

[46] That being said, JB Manchak told me that he would not endorse such an adaptation, for the reasons described below (personal communication, 16 May 2025).

[47] There is a case to be made for this claim in the framework of Lewis ([1986]). For Lewis ([1986], pp. 61–2),

conditions for that property to be instantiated by spacetime regions, but also as making a claim about its relative fundamentality (or, similarly, as making a claim in virtue of what a spacetime region is singular).[48] If that were the case, the MES definition would make the claim that an intrinsic property of some spacetime region of a given spacetime is instantiated in virtue of facts about multiple wholly distinct spacetimes; but this seems to clash with intuitions that motivate the explication of intrinsicality in the first place. Among the various ways of resolving this puzzle, rejecting the MES definition strikes me as most convincing.

I admit that neither argument against the MES definition strikes me as conclusive on its own—either can be resisted in some way or another. (I do take the misclassification objection to be more serious.) Nevertheless, I submit that, taken together they make a convincing case against the MES definition, especially in comparison to the definition schema that I proposed in Section 4 (Definition 4.4); for that definition schema is perfectly compatible with the classification of singular structure as extrinsic, and thus suffers from no such problems.

6 Conclusion

Let me close with some remarks about the aims and scope of my discussion here. I argued that there is no one-size-fits-all characterisation of spacetime's singular structure—we are faced with many possible ways of deciding what kind of curve incompleteness witnesses singular structure. In light of this, we have to resort to pragmatic considerations, or restrict ourselves to particular contexts of inquiry. Still, we can, and should, think of different varieties of singular structure as extrinsic, non-local, and yet localizable properties of spacetime regions. This conception can be made precise—in a way that meshes well with the pluralism defended before—by the schema I offered at the end of Section 4; and it is arguably better than an alternative definition, discussed in Section 5, which quantifies over maximal extensions of a given spacetime region, when that region is treated as a spacetime on its own.

> a property F is intrinsic iff any two objects whose parts can be put into a one-one correspondence that agrees on all 'perfectly natural' properties and relations are either both F or both not-F. Setting aside Lewis' own treatment of 'perfectly natural' properties and relations, it strikes me as reasonable to hold that in the context of (vacuum) models of general relativity (and perhaps local spacetime theories more generally) 'perfectly natural' properties and relations are *exactly* those that are (part-wise) preserved by isometries between spacetime regions. (This characterisation does *not* entail that geometric properties of spacetime *points* must be intrinsic, and hence is not committed to any form of *pointillisme*. For further discussion, see Bricker ([1993]) and Butterfield ([2006]).) So, in that framework, the MES definition classifies 'being singular' as an intrinsic property of spacetime regions.

48 For a discussion of fundamentality and grounding, and their applications to physics, see McKenzie ([2022]).

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