On the Nonabsoluteness of Satisfaction

Michał Tomasz Godziszewski

University of Warsaw

The so-called *model–theoretic method* of characterizing the notion of truth consists in defining a general notion of a model of a given formal language L, providing a definition of a binary relation between models of L and the sentences of L, and finally singling out a concrete model as the standard or the intended one and declaring that truth simpliciter (of sentences of L) should be understood as truth in this model. Can we really treat this model–theoretic definition of truth as *the* definition of (mathematical) truth (say, at least with respect to the language of arithmetic)? One of the problems is that it indeed relies on the concept of an intended or standard structure (or the class of intended structures in case of some other theories, e.g. in the case of set theory). Assume then that our metatheory can provide us with a determinate concept of the standard model. Does it follow that then the concept of truth is definite, complete, determinate or absolute?

Some researchers give a positive answer to the last question:

S. Feferman [2]:

> In my view, the conception of the bare structure of the natural numbers is completely clear; and thence all arithmetical statements are definite.

D. Martin [5]:

> What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers.

However, already in 1974, Stanisław Krajewski proved in [4] a theorem on the so-called incompatible satisfaction classes. Let M be a model of arithmetic. A subclass $S \subseteq M$ is a full satisfaction class if every element of S is a sentence in the language of arithmetic and such that S obeys the recursive Tarskian definition of truth. Then Krajewski's result is:

Theorem 1. *Every model of PA that admits an (inductive) satisfaction class has an elementary extension that admits several (actually: arbitrarily many) incompatible (inductive) satisfaction classes.*

The result gives rise to certain questions related to strength of the theory needed to allow for such incompatibility:

- Under what condition can these satisfaction classes become the true theory of arithmetic inside a model of the metatheory?

- What happens then to the nonarithmetic superstructure of a model of the metatheory then?
- Can we then maintain that completeness of arithmetic truth follows from determinateness of arithmetic structure?

J. D. Hamkins and R. Yang proved in [3] the following theorem (generalizing the result of Krajewski):

Theorem 2. *For any theory T extending ZFC there are isomorphic models M_0 and M_1 of this theory such that $\mathbb{N}^{M_0} = \mathbb{N}^{M_1}$ (the natural numbers of the models agree), but $Th(\mathbb{N})^{M_0} \neq Th(\mathbb{N})^{M_1}$ (i.e. the arithmetical satisfaction classes of the models are incompatible).*

claiming that the meaning of the result is that:

> a philosophical commitment to the determinateness of the theory of truth for a structure cannot be seen as a consequence solely of the determinateness of the structure in which that truth resides. The determinate nature of arithmetic truth, for example, is not a consequence of the determinate nature of the arithmetic structure $\mathbb{N} = \{0, 1, 2, \ldots\}$ itself, but rather, we argue, is an additional higher-order commitment requiring its own analysis and justification.

One might first argue: well, if you have the entire strength of ZFC at hand, than it is not surprising that you can cook up such structures. But: the result can be made much stronger, actually. One actually only needs the theory T to be able to interpret arithmetic.

We can thus generalize Hamkins-Yang theorem to the following:

Theorem 3. *For any theory T interpreting PA and for any countable ω-nonstandard model M of this theory such that $\mathbb{N}^M$ (i.e. the numbers of the model) is recursively saturated there is an isomorphic model M' such that $\mathbb{N}^M = \mathbb{N}^{M'}$ (the natural numbers of the models agree), but $Th(\mathbb{N})^M \neq Th(\mathbb{N})^{M'}$ (i.e. the arithmetical satisfaction classes of the models are incompatible).*

The theorem above, combined together with some results of Enayat from [1] gives two important corollaries:

Corollary 1. *For any theory T interpreting PA and defining the arithmetical truth*[1] *and for any countable recursively saturated model $(M, Tr(\mathbb{N})^M) \models T$ there is an isomorphic model $(M', Tr(\mathbb{N})^{M'})$ such that $\mathbb{N}^M = \mathbb{N}^{M'}$ (the natural numbers of the models agree), but $Tr(\mathbb{N})^M \neq Tr(\mathbb{N})^{M'}$ i.e. the arithmetical (inductive) satisfaction classes of the models are incompatible).*

Corollary 2. *For any theory T interpreting PA and for any countable model $M_0 \models T$ there exists an elementary extension M of M_0 and a model $M' \cong M$ such that $Tr(\mathcal{N})^M$ (i.e. arithmetical truth according to the model) satisfies CT^- (i.e. admits a satisfaction class) $\mathcal{N}^M = \mathcal{N}^{M'}$ (the natural numbers of the models agree), but $Tr(\mathcal{N})^M \neq Tr(\mathcal{N})^{M'}$.*

We conclude with a philosophical discussion of the results, with an emphasis on the context of the so-called multiverse view in the foundations of mathematics.

[1] e.g. Z_2 will absolutely do, but curiously, ACA_0 will not

References

[1] Ali Enayat. *On ZF-standard models of PA*. preliminary report, Mittag Leffler Institute, 7 September 2009.

[2] Sol Feferman. *Comments for EFI workshop meeting*. pp. 1-9, 2013. Exploring the Frontiers of Incompleteness, Harvard, August 31-September 1, 2013.

[3] Joel David Hamkins, Ruizhi Yang, *Satisfaction is not absolute*, arXiv: https://arxiv.org/abs/1312.0670, pp. 34.

[4] Stanisław Krajewski, *Mutually inconsistent satisfaction classes*. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 22:983-987, 1974.

[5] Donald A. Martin. *Completeness or incompleteness of basic mathematical concepts*, draft 3/15/12. pages 1-26, 2012. Exploring the Frontiers of Incompleteness, Harvard, April 4, 2012.