Chapter 4

The 1+1-dimensional Ising model

The 1+1-dimensional Ising model is one of the most important models in statistical mechanics. It is an interacting system, and behaves accordingly. Yet for a variety of reasons, it is analytically very tractable because of a variety of special properties. Among them are the facts that it has an exact duality between ordered and disordered phases, and can be mapped exactly (albeit non-locally) onto a system of free fermions. While these properties are special, they are not unique to the Ising model, and so they are important to understand. This chapter therefore contains a detailed analysis of the 1+1-dimensional Ising model in both its classical and quantum versions. Two key references are given by the review article by Schultz, Mattis and Lieb, and the original research article by Kadanoff and Ceva.

Although thus far most of the discussion has been in terms of the square lattice, the Ising model can be defined on any lattice, or for that matter, any graph. A *graph* is a collection of *vertices*, which are connected by *edges*. In the physics literature, when dealing with a lattice, it is common to call the vertices *sites*, and the edges *links*, since the words vertices and edges are often used to mean other things. Ising model can be defined on an arbitrary graph by putting the spins $\sigma_i = \pm 1$ on the sites labeled by i. The most general nearest-neighbor interaction gives an energy

$$E(\{\sigma_i\}) = -\sum_{<ij>} J_{<ij>}\sigma_i\sigma_j \tag{4.1}$$

allowed to depend on arbitrary nearest-neighbor couplings $J_{<ij>}$, where $< ij >$ labels the link between sites i and j.

4.1 The low-temperature expansion and the phase transition

In any interacting system in two dimensions and higher, the classical partition function is a complicated object. To gain some more intuition into it and the resulting physics, it is often useful to develop expansions around the $T \to 0$ and $T \to \infty$ limits. Roughly speaking, in the two limits the dominant effects arise from energy and entropy respectively.

At low temperatures in the Ising model, the configurations with aligned spins have much larger Boltzmann weights than those without. The low-temperature expansion is therefore done by rewriting the states of the system in terms of *domain walls*, as done in chapter 1 to use for the Peierls argument. For a system in d spatial dimensions, the domain walls are $d-1$-dimensional objects. Thus in the two-dimensional classical Ising model the domain walls are lines on the links of the *dual* lattice, and separate regions of spin up and spin down.

The sites of a dual lattice in two dimensions are at the center of the faces (aka "plaquettes") of the original lattice. Two dual sites are connected by a link on the dual lattice if the corresponding faces on the original lattice share a link in common. It is easy to see (draw a picture) that the links on the dual lattice are in one-to-one correspondence with the links of the original lattice. If two faces only touch at a point, there is no corresponding link on the dual lattice. The dual lattice to the square lattice is also a square lattice, but e.g. the dual of the triangular lattice is the honeycomb lattice (where the sites form hexagons).

The Boltzmann weight for any dual link with a domain wall present is $e^{-2\beta J_{<ij>}}$ relative to the link without the domain wall. In the simplest case where the coupling is independent of link, $J_{<ij>} = J$, the two-dimensional Ising partition function

$$Z = \sum_{\sigma_i=\pm 1} e^{\beta J \sum_{<ij>} \sigma_i \sigma_j}$$

can be rewritten in terms of domain walls as

$$Z = e^{\beta J N} \sum_{\sigma_i=\pm 1} e^{-2\beta J \widehat{L}} \ , \qquad (4.2)$$

where $\widehat{L}$ is the number of domain walls, i.e. their length, and N the total number of sites. The hat on $\widehat{L}$ is a reminder that these walls live on links on the dual lattice.

The sum in (4.2) remains over spin configurations, and how it is rewritten in terms of domain walls depends on the boundary conditions. If the boundary conditions are fixed to some value (say up) all around the boundary, then domain walls form closed loops on the dual lattice. In this case, these loops are in one-to-one correspondence with the spin configurations, so Z(fixed) is equivalent to a sum over closed loops with weight $e^{-2\beta J}$ per unit length of loop. For free boundary conditions, the domain walls can end at the boundary. Z(free) is thus twice the sum over all closed loops and loops ending on the boundary; the factor of two is because there are two spin configurations for every domain wall configuration. For periodic boundary conditions in one direction (where space is topologically a cylinder) or in both (a torus), the situation is more intricate, and is discussed below in the context of duality.

The partition function written in the form (4.2) is a *low-temperature expansion*. At low temperature βJ is large, so $e^{-2\beta J}$ is very small, and the sum is indeed dominated by terms with small $\widehat{L}$, i.e. short loops. This result is in harmony with the proof in chapter 1 that the Ising model in two and greater dimensions is ordered. With all the boundary spins fixed to be up, small $\widehat{L}$ means that most spins are in the same cluster as the boundary, and so indeed are spin up.

Writing the partition function in terms of domain walls provides a convenient way of understanding the competition between energy and entropy that causes the phase transition. The partition function can be rewritten in a similar fashion as that done in chapter 1, by breaking it in up into terms of fixed energy. Since all terms with a given total length $\widehat{L}$ of the domain walls have the same energy,

$$Z = \sum_{\widehat{L}} n(\widehat{L}) e^{-2\beta \widehat{L}} \ , \tag{4.3}$$

where $n(\widehat{L})$ is the number of configurations of closed loops on the dual lattice at this particular value of $\widehat{L}$. The energy amounts to an exponentially small suppression of large $\widehat{L}$, but the number of loops grows exponentially with L over a wide range of L. For large number of sites, this number is typically of the form

$$n(\widehat{L}) \propto K^{\widehat{L}} \tag{4.4}$$

The (as yet unknown) constant K is a purely geometrical quantity: it depending on the lattice but *not* on any couplings or temperature in the model.

Written this way, it is clear that which types of configurations dominate the partition function depend crucially on the temperature. The exponent in the argument of the summand is simply $(\ln K - 2\beta J)\widehat{L}$. If $2\beta J > \ln K$, then the coefficient of $\widehat{L}$ in the exponent is negative. The sum therefore is dominated by configurations with short $\widehat{L}$. This is the low-temperature ordered phase, where most spins take on the same value. Conversely, if $2\beta J < \ln K$, then the sum is dominated by long loops. Even in the limit of a large number of sites, a non-vanishing fraction of the dual lattice is covered by loops. Thus even if all the spins are fixed up at the boundary, the partition function is dominated by configurations with the roughly the same numbers of up and and down spins, so that the expectation value of the magnetization per site

$$\frac{\langle N_\uparrow - N_\downarrow \rangle}{N}$$

goes to zero as $N \to \infty$. This is the disordered phase.

The preceding is a strong argument that a phase transition between order and disorder takes place at a single value of the temperature

$$T_c = \frac{2J}{\ln K}$$

(henceforth, Boltzmann's constant is set to 1). A number of loops growing as in (4.4) implies that the transition is at least somewhat abrupt: at T_c the behavior of the system changes qualitatively. This argument does not really indicate what happens precisely at the transition, since among other things what happens here will depend on the $\widehat{L}$-dependent coefficient neglected in (4.4). Two possibilities are that the magnetization per site gradually decreases to zero as T is increased T_c, or that it abruptly drops to zero there from some finite value. As will be shown later in this chapter, it turns out that here the former happens.

Of course, this argument is not a proof that there is no intermediate phase somehow in between order and disorder, because there is no guarantee that the number of loops will depend on the length in a simple a fashion as in (4.4). However, the form (4.4) does apply generically to such geometrical quantities, so this argument frequently works for understanding when such phase transitions between order and disorder happen in classical statistical mechanics.

The low-temperature expansion can be generalized in an obvious way to any spin model without geometric frustration in any dimension. If there are more than two types of spin at each site, then there can be different types of domain walls, but the basic idea is the same.

4.2 The high-temperature expansion and Kramers-Wannier duality

The high-temperature expansion in the Ising model is not as obvious as the low-temperature expansion. The first step in its derivation comes from rewriting the partition function as a product over each link. For a given link, the Boltzmann weight can be written a fashion similar to that done when deriving the quantum Hamiltonian from the transfer matrix in chapter 2:

$$e^{\beta J\sigma_i\sigma_j} = \cosh(\beta J) + \sigma_i\sigma_j \sinh(\beta J) \ .$$

The sum over all spins in the energy can thus be recast as a product as

$$Z = \sum_{\{\sigma_i=\pm 1\}} \prod_{<ij>} e^{\beta J\sigma_i\sigma_j} = \sum_{\{\sigma_i=\pm 1\}} \prod_{<ij>} \left(\cosh(\beta J) + \sigma_i\sigma_j \sinh(\beta J)\right) \ .$$

This product can be expanded out into a sum of 2^{N_l} terms, where N_l is the number of links on the dual lattice. This is conveniently written in terms of a variable $d_{<ij>}$ for each link, where $d_{<ij>} = 0$ if the $\cosh(\beta J)$ terms is on this link, while $d_{<ij>} = 1$ if the $\sigma_i\sigma_j \sinh(\beta J)$ term. Then

$$Z = \cosh(\beta J)^{N_l} \sum_{\{\sigma_i=\pm 1\}} \sum_{\{d_{<ij>}=0,1\}} \prod_{<ij>} \left(\sigma_i\sigma_j \ \tanh(\beta J)\right)^{d_{<ij>}} \ .$$

So far, this looks much more complicated, but the trick in the high-temperature expansion is to interchange the order of the sums over the σ_i and $d_{<ij>}$:

$$Z = \cosh(\beta J)^{N_l} \sum_{\{d_{<ij>}=0,1\}} \tanh(\beta J)^{\sum_{<ij>} d_{<ij>}} \sum_{\{\sigma_i=\pm 1\}} \prod_{<ij>} (\sigma_i \sigma_j)^{d_{<ij>}} \ .$$

The sums over the variables $d_{<ij>} = 0, 1$ is on the same footing as the sum over spin configurations: each is an independent variable. Thus one can fix a "configuration" of the $d_{<ij>}$ and then do the sums over all the $\{\sigma_j\}$. These sums are

$$\sum_{\{\sigma_i=\pm 1\}} \prod_{<ij>} (\sigma_i \sigma_j)^{d_{<ij>}} \quad = \quad \sum_{\{\sigma_i=\pm 1\}} \prod_i (\sigma_i)^{b_i} \ ,$$

where

$$b_i = \sum_{j \text{ next to } i} d_{<ij>} \ .$$

Because

$$\sum_{\sigma_j=\pm 1} \sigma_j = 0$$

and $(\sigma_j)^2 = 1$, these sums are

$$\sum_{\{\sigma_i=\pm 1\}} \prod_i (\sigma_i)^{b_i} = \begin{cases} 0 & \text{any } b_i \text{ odd} \\ 2^{N_l} & \text{all } b_i \text{ even} \end{cases}$$

Thus if *any* b_i is odd for this particular set of $d_{<ij>}$, this particular set has vanishing contribution to the partition function. In other words, the sum over spin configurations forces *all* the b_i to be even to contribute to the partition function. The sum over the the different configurations of $d_{<ij>}$ therefore can be taken to be only over those where all b_i are even:

$$Z = (2\cosh(\beta J))^{N_l} \sum_{\{d_{<ij>}=0,1; b_i=0,2,4,\dots\}} (\tanh(\beta J))^{\sum_{<ij>} d_{<ij>}} \qquad (4.5)$$

One can thus think of the $d_{<ij>}$ as now being the degrees of freedom in the model akin to the spins in the original definition; one important difference however is that these degrees of freedom live on the links of the lattice.

The high-temperature expansion (4.5) has a very nice (and familiar) graphical presentation. Each term in the sum over $d_{<ij>}$ (i.e. a fixed configuration of $d_{<ij>}$ can be represented graphically by drawing a line on the lattice along the link from i to j if $d_{<ij>}$=1, and leaving it empty if $d_{<ij>} = 0$). The rule that b_i must be even now has an obvious graphical meaning – the lines must form closed loops! After summing over spins, the high-temperature expansion is exactly the same form as the low temperature expansion: the remaining sum is over all loop configurations on the lattice. The weight is simply the total length $L = \sum_{<ij>} d_{<ij>}$ of the loops, so

$$Z = (2\cosh(\beta J))^{N_l} \sum_{\text{closed loops}} (\tanh(\beta J))^L \ . \tag{4.6}$$

The sum here is over all closed loops on the original lattice. Comparing the high-temperature expansion (4.6) with the low-temperature expansion (4.2) makes it obvious that the two are the same kind of expansion. Not only are the sums over closed loops, but the Boltzmann weights: up to overall unimportant constants both are of the form

$$\sum_{\text{closed loops}} (w(J))^L \ .$$

In the low-temperature case, the loops are on the dual lattice.

This can be exploited to give an *exact* relation between the partition functions of different Ising models. Consider the Ising model with coupling J and all sites on the boundary fixed to be spin up. A fixed boundary condition means that no domain walls in the low-temperature expansion can end, and so form closed loops. Getting rid of the constant in front by shifting the energy gives its partition function to be

$$Z(J;\ \text{fixed}) = \sum_{\widehat{\mathcal{L}}} e^{-2\beta J \widehat{L}} \ ,$$

where the notation in sum indicates it is summed over all closed loops on the *dual* lattice. Now consider an Ising model where the spins live on the *dual* lattice of the original and the coupling is $\widehat{J}$, and the boundary conditions are free along the boundary. If the original is a square lattice, then its dual is also a square lattice with sites at the centers of the squares of the original. Doing the high-temperature expansion for this model on the dual lattice and getting rid of the overall constant gives

$$\widehat{Z}(\widehat{J};\ \text{free}) \sum_{\widehat{\mathcal{L}}} (\tanh(\beta \widehat{J}))^{\widehat{L}} \ .$$

The hat on Z emphasizes the fact that this is an Ising model with spins on the dual lattice. The reason the sum on the right-hand side is over the dual lattice is that the high-temperature expansion has loops connecting the sites of the lattice the spins with the spins. The two are obviously identical if the couplings obey the relation

$$e^{-2\beta J} = \tanh(\beta \widehat{J}) \ . \tag{4.7}$$

This is known as the *Kramers-Wannier duality* in the Ising model.

A *duality* is a transformation, typically non-local, that maps a given model onto another. Here it shows that

$$\widehat{Z}(\widehat{J};\ \text{free}) = Z(J;\ \text{fixed}) \ .$$

This relation is truly remarkable, because a model with βJ large (low temperature) is equivalent to a model on the dual lattice with $\beta\widehat{J}$ small, high temperature. As an important check, a little algebra shows that the relation (4.7) can be rewritten as

$$e^{-2\beta\widehat{J}} = \tanh(\beta J) \ .$$

This means that taking the dual of the dual gives the original model back again.

The duality remains valid even if the couplings J vary from link to link, and for arbitrary two-dimensional lattices (or for that matter, graphs). This is because both the low-temperature and high-temperature expansions can be built up one nearest-neighbor pair $< ij >$ at a time, so the above arguments can be rerun for arbitrary $J_{<ij>}$. The resulting expressions are virtually identical, and so another Ising model can be defined on the *dual* lattice with the same partition function (up to the usual unimportant overall constant) when

$$e^{-2\beta\widehat{J}_{<\hat{i}\hat{j}>}} = \tanh(\beta J_{<ij>}) \tag{4.8}$$

The square lattice is special because it is self-dual, so even with varying couplings the duality takes an Ising model on the square lattice to another on the square lattice. An important example is the anisotropic case considered when deriving the quantum Hamiltonian in chapter 2, where the couplings only depend on the direction of the link, so $J_{<ij>} = J_x$ or $= J_y$ when the links are along the x and y directions respectively. Since on the dual lattice, a given link forms a right angle with the corresponding link on the original lattice, the duality (4.8) implies that

$$e^{-2\beta\widehat{J}_x} = \tanh(\beta J_y) \ , \qquad e^{-2\beta\widehat{J}_y} = \tanh(\beta J_x). \tag{4.9}$$

In chapter 1 it was proven that the Ising model orders at sufficiently small βJ. It is also easy to prove that i cannot order at sufficiently large βJ. Thus an obvious question is: how can an ordered model be equivalent to a disordered model? Recall the definition of order in terms of the asymptotic value of the two-point $\langle\sigma_a\sigma_b\rangle$. The duality mapping is highly non-local in terms of the spins: one must sum over the spins in order to demonstrate it. A non-vanishing spin-spin correlator corresponds to an non-vanishing correlator of the *dual spins* $\langle\mu_{\hat{a}}\mu_{\hat{b}}\rangle$, where any product of dual spins can be found by taking products of the nearest-neighbor relation $\mu_{\hat{i}}\mu_{\hat{j}} = 1 - 2d_{<ij>}$ and exploiting the fact that $(\mu\hat{i})^2 = 1$.

4.3 Duality in the quantum model

The Hamiltonian of the quantum Ising chain is found by taking the strongly anisotropic limit of the transfer matrix of the two-dimensional classical Ising model. It is thus natural to expect that the duality has consequences for this Hamiltonian. In this subsection, I show that the duality indeed arises very elegantly for the quantum chain. It not only gives an exact relation between the spectrum of the low-temperature phase and that of the high, but also introduces the important concept of disorder operators in quantum spin chains.

The geometrical degrees of freedom in the high-temperature expansion of the two-dimensional classical Ising model contribute to the partition function only when they form closed loops. The duality arises because the low-temperature expansion in terms of domain walls is automatically written in terms of closed loops as well.

This correspondence suggests that it would be useful to rewrite the degrees of freedom of the quantum chain in terms of the domain walls. In the quantum chain written in a basis where all the σ^z_j are diagonal, having a domain wall corresponds to adjacent sites having different eigenvalues, i.e. $\sigma^z_j\sigma^z_{j+1} = -1$ when a domain wall is present between sites j and $j+1$, while it $= 1$ if none is present. The operator

$$\mu^z_{j+1/2} = \sigma^z_j \sigma^z_{j+1} \tag{4.10}$$

therefore is naturally defined on the dual site halfway between sites j and $j+1$. This operator thus is the analog of the disorder field defined in the classical model. It obeys $(\mu^z_j)^2 = 1$, and so its eigenvalues are ± 1. Moreover, just as the energy term in the classical Ising model measures the length of the domain walls, the $\sum_j \sigma^z_j\sigma^z_{j+1} = \sum_j \mu^z_{j+1/2}$ term in the Ising quantum Hamiltonian effectively counts (minus twice) the number of domain walls in a given configuration.

Half the quantum Ising Hamiltonian is therefore simply rewritten in terms of domain walls. The other half is the sum of the spin-flip operators σ^x_j. Because σ^x and σ^z anticommute,

$$\{\sigma^x_j, \mu^z_{j+1/2}\} = \{\sigma^x_j, \mu^z_{j-1/2}\} = 0 \ . \tag{4.11}$$

This means that acting with σ^x_j on an eigenstate of $\mu^z_{j-1/2}$ and $\mu^z_{j+1/2}$ results in a state where *both* eigenvalues are flipped. Thus a single spin flip on site j has the consequence of “flipping” the *two* domain walls on sites $j+1/2$ and $j-1/2$; by flipping a domain wall I mean that if a domain wall is present it is removed, if not present it is created. This suggests defining an operator $\mu^x_{j+1/2}$ so that

$$\mu^x_{j-1/2}\mu^x_{j+1/2} = \sigma^x_j \ . \tag{4.12}$$

How this is achieved depends slightly on the boundary conditions. For simplicity, consider free boundary conditions on the original Ising chain; others will be discussed later. Then define $\mu^x_{1/2} = 1$. Because $(\sigma^x_j)^2 = 1$ for all j, the remainder of the μ^x_j are then given as the product of the spin flips

$$\mu^x_{j+1/2} = \prod_{k=1}^{j} \sigma^x_k \ . \tag{4.13}$$

With these definitions, the quantum Ising Hamiltonian with free boundary conditions can be rewritten in terms of domain-wall operators as

$$H = -\sum_{j=1}^{N} \sigma_j^x - \lambda \sum_{j=1}^{N-1} \sigma_j^z \sigma_{j+1}^z \tag{4.14}$$

$$= -\sum_{j=1}^{N} \mu_{j-1/2}^x \mu_{j+1/2}^x - \lambda \sum_{j=1}^{N-1} \mu_{j+1/2}^z \ . \tag{4.15}$$

The latter form often is called the "dual Hamiltonian", but it is important to remember that it is the same Hamiltonian – it is just rewritten in terms of different operators.

The dual Hamiltonian looks quite similar to the original. In fact, up to boundary conditions and an overall unimportant rescaling, the former is given in terms of the latter by sending

$$\sigma^x \to \mu^z, \qquad \sigma^z \to \mu^x, \qquad \lambda \to 1/\lambda.$$

The coupling λ was defined originally from the anisotropic limit $J_x \to \infty$, $J_y \to 0$ of the transfer matrix while keeping $\lambda \equiv e^{2\beta J_x} \tanh(\beta J_y)$ fixed. Thus if a dual coupling $\widehat{\lambda}$ is defined as

$$\widehat{\lambda} \equiv e^{2\beta \widehat{J}_x} \tanh(\beta \widehat{J}_y)$$

the duality relation (4.9) for the anisotropic model means that

$$\widehat{\lambda} = \frac{1}{\lambda} \ .$$

Thus to establish duality in the quantum Ising chain, one therefore must show that the operators μ^x and μ^z are equivalent to σ^z and σ^x. More precisely, one must show that the operators $\mu_{\hat{j}}^x$ and $\mu_{\hat{j}}^z$ obey the *same algebra* as σ_j^x and σ_j^z. The basic properties of Pauli matrices mean that

$$(\sigma_j^a)^2 = 1; \quad \{\sigma_j^z, \sigma_j^x\} = 0; \quad [\sigma_j^a, \sigma_k^b] = 0$$

for $j, k = 1 \ldots N$, $k \neq j$, and $a, b = x$ or z. It is then simple to check that with the definitions (4.10) and (4.13),

$$(\mu_{\hat{j}}^a)^2 = 1; \quad \{\mu_{\hat{j}}^z, \mu_{\hat{j}}^x\} = 0; \quad [\mu_{\hat{j}}^a, \mu_{\hat{k}}^b] = 0$$

for $\hat{j}, \hat{k} = 3/2, 5/2, \ldots N - 1/2$ and $\hat{k} \neq \hat{j}$.

SInce the μ operators satisfy the same algebra as do the σ operators, the last thing to understand is the Hilbert space on which they are acting. The entire 2^N-dimensional Hilbert space is spanned by the eigenstates of the N operators σ_j^z, and any of these basis states can obtained by acting on any of the other ones by a suitable product of the σ_j^x. In less mathematical language, any spin configuration can be obtained from any other one by suitable spin flips. The situation for the μ operators is slightly different. Since the role of σ^z in the original is played by μ^x in the dual, it is natural to then work in a basis where all the $\mu^x_{\hat{j}}$ are diagonal. This the state where the eigenvalue of μ^x_{j} is 1 or -1 can be referred to as having a dual spin up or down respectively. When the original spins have free boundary conditions, by definition $\mu^x_{1/2}=1$, so this corresponds to a *fixed* boundary condition for this dual spin. Moreover, the dual spin at the other end

$$\mu^x_{N+1/2}=\prod_{j=1}^{N}\sigma^x_j$$

flips all of the original spins and commutes with the Hamiltonian; this symmetry is a consequence of the original $\mathbb{Z}_2$ symmetry of the classical model. Acting with the Hamiltonian therefore does not change the value of $\mu^x_{N+1/2}$, and so the boundary condition at this end is therefore fixed in terms of the dual spins as well. Free boundary conditions in the original model therefore correspond to fixed in the dual. The $N-1$ operators $\mu^z_{\hat{j}}$ acting on a sector with a given fixed boundary condition $\mu^x_{N+1/2}=\pm 1$ then give all 2^{N-1} basis states in this sector. In less mathematical language, any dual spin configuration with a particular fixed boundary condition can be obtained from any other in that sector by dual spin flips.

The dual Hamiltonian therefore describes a quantum Ising chain with fixed boundary conditions and coupling $\widehat{\lambda}=1/\lambda$. The Hilbert space of the Ising chain with fixed boundary conditions on $N+1$ sites is 2^{N-1}-dimensional; the original 2^N-dimensional Hilbert space is recovered by considering both $++$ and $+-$ boundary conditions for $(\mu^x_{1/2},\mu^x_{N+1/2})$. Note that while $\prod_{j=1}^{N}\sigma^x_j$ commutes with the Hamiltonian for free boundary conditions, it cannot be utilized for fixed, because acting with it changes the boundary conditions. Thus the Hamiltonian with free boundary conditions can be made block diagonalized into two $2^{N-1}\times 2^{N-1}$ dimensional matrices using this symmetry operator, each corresponding to a given fixed boundary condition in the dual model. Letting $\{E_\pm(\lambda)\}$ be the sets of energies of the quantum Ising Hamiltonian with free boundaries and $\prod_{j=1}^{N}\sigma^x_j=\pm 1$ and $\{E(\lambda\ +\pm)\}$ be the sets of energies with fixed boundaries,

$$\{E_\pm(\lambda)\}=\{\lambda E(1/\lambda;+\pm)\}$$

Up to an unimportant overall rescaling, the spectra are the same!)

4.4 Fermions from the Jordan-Wigner transformation

Objects with nice algebraic properties.)

Pauli matrices are naturally associated with fermions: they obey anticommutation relations. Moreover, *spin-statistics theorem* for a Lorentz-invariant theory in spatial dimensions two and higher requires that any particle of half-integer spin is a fermion, while those of integer spin are bosons. In one dimension, there is no notion of statistics, since there is no way of taking a particle around another without coming close. Nonetheless, one can still discuss fermions in one dimension, in terms of operators that have the same properties as fermionic operators in higher dimensions. For example, the Pauli exclusion principle makes sense in any dimension: no identical particles can occupy the same state. This is apparent in the fermionic operator commutation relations, since $(c^\dagger)^2 = 0$.

Another hint that fermions are present in the 1+1d Ising model is in the fact that the spin and disorder operators obey some interesting anticommutation relations (**??**). In fact, because of the "string" in μ^x, these can hold true even for operators far from each other, for example

$$\{\sigma_i^z, \mu_{j+1/2}^x\} = 0 \qquad \text{for } i \le j \ .$$

Full-fledged fermionic anticommutation relations can be found by defining the combinations

$$\chi_j =$$

This leads to

$$\chi_j = \left(\prod_{k<j} \sigma_k^x\right) \sigma_j^z, \qquad \psi_j = \left(\prod_{k<j} \sigma_k^x\right) \sigma_j^y \tag{4.16}$$

These are what are usually known as "Majorana" or "real" fermions, because they are hermitian operators: $\chi_j^\dagger = \chi_j$ and $\psi_j^\dagger = \psi_j$. Their anticommutators obey the algebra

$$\{\chi_j, \chi_k\} = 2\delta_{jk}; \quad \{\psi_j, \psi_k\} = 2\delta_{jk}; \quad \{\chi_j, \psi_k\} = 0 \tag{4.17}$$

for any j and k. To obtain the standard "complex" fermion algebra described in chapter 2, one simply defines the combinations

$$c_j = \chi_j + i\psi_j; \qquad c_j^\dagger = \chi_j - i\psi_j \ . \tag{4.18}$$

Note that there two states at every site here just as in the earlier fermion case, i.e. the Hilbert space is 2^N-dimensional for N sites.

translation symmetry, momentum space, still satisfies commutation relations. Just a change of basis. Bogoliubov transformation. same "reference" state eigenstates

4.5 "Solving" the quantum Ising chain

Could have directly diagonalized the Hamiltonian via the Bogoliubov transformation, and ended up with the same results,

vanishing of gap as a function of $\lambda = f/J$.

The free energy of the two-dimensional classical Ising model can also be found using free fermions. Since the transfer matrix acts on the same vector space as the quantum Hamiltonian, the same change of basis into fermions can be used. The transfer matrix is more complicated, however, so finding the free energy takes more work. The analysis does simplify on the honeycomb lattice...

The universal results, of course are the same. So the vanishing of the gap...

4.6 The two-point function

dynamical critical exponent.