

Chapter 5

The XXZ chain and the six-vertex model

The purpose of this chapter is to focus on two models fundamental to the study of both 1d quantum systems and 2d classical systems. They are the XXZ chain (and its 2d classical analog, the six-vertex model), and the 2d classical rotor or “XY” model. The XXZ model is a deformation of the Heisenberg model breaking the $SU(2)$ symmetry down to a $U(1)$ subgroup. The degrees of freedom of the XY model are classical fixed-length “spins” pointing anywhere in a *plane*, and so can be represented by an angle $0 \leq \theta < 2\pi$. The latter also has a $U(1)$ symmetry given by shifting θ by a constant mod 2π .

The physics of the two models is closely related; in fact in a sense to be described precisely, it is identical. Both are worth studying in their own right. From the physics point of view, the rotor model describes for example the transition in superfluids: the value θ is the phase of the expectation value of the wave function. Even though the existence of superfluids of course is a quantum-mechanical effect, quantum effects are essentially negligible when studying the transition between the superfluid and the normal phase. Thus the classical rotor model provides a way of quantitatively understanding this transition. Physically, the XXZ chain is one of the simplest models of a magnet. The free parameter describing the anisotropy (the breaking of rotational symmetry in spin space) also provides a very useful tool in describing the behavior; one can continuously tune in between ferromagnets and antiferromagnets.

From the formal point of view, both models are also of fundamental importance. The rotor model provides a direct and obvious connection with the physics of a free bosonic field. This provides a simple intuitive way of understanding the Mermin-Wagner theorem, which shows that in two-dimensional classical and one-dimensional quantum systems, a continuous symmetry cannot be spontaneously broken. However, a phase transition of Kosterlitz-Thouless transition still occurs, and the rotor model provides a very clear qualitative understanding of the transition. Moreover, when the Boltzmann weights are chosen to take a particular (Villain) form, the analysis of the different operators and

their scaling dimensions becomes very transparent.

Intuitive arguments can also be applied usefully to the one-dimensional XXZ chain and the six-vertex model. However, a real merit of studying these models is that *exact* computations can be done. The behavior is much richer (and the analysis more elaborate) than that of the Ising case. Models of this type are sometimes referred to as “exactly solvable”, but a better word than solvable is “integrable”. This implies that there are an infinite number of conserved charges, i.e. explicit quantities that commute with the Hamiltonian. Not only does this allow for example the free energy to be computed, but it allows one to show that the 1d XXZ Hamiltonian and the transfer matrix of the six-vertex model have the *same* eigenvectors, thus cementing the connection between the two.

An unusual feature of these models is they exhibit a *line* of critical points with continuously varying critical exponents. This behavior is almost impossible outside 1+1 dimensions, and not even that common there.

5.1 The XXZ Hamiltonian

Both are built from spin operators S_i^x , S_i^y and S_i^z at each site i satisfying $su(2)$ commutation rules

$$[S_j^a, S_k^b] = i\delta_{jk}\epsilon_{abc}S_j^c, \quad (5.1)$$

with spin operators on different sites commuting. The XXZ model however allows an arbitrary coupling between neighboring S_i^z , namely

$$H = - \sum_{\langle jk \rangle} [J_{\perp} (S_j^x S_k^x + S_j^y S_k^y) + J_z S_j^z S_k^z]. \quad (5.2)$$

Although this is no longer $SU(2)$ symmetric, the z component of the total spin

$$S^z = \sum_j S_j^z$$

still commutes with H_{XXZ} , generating a $U(1)$ symmetry. This $U(1)$ symmetry. Even though the full $SU(2)$ symmetry is broken, the degrees of freedom are still typically referred to as “spins”, and the dimension- $2s + 1$ representation of the spin operators is referred to as spin- s .

The sign of J_{\perp} is unimportant on any bipartite lattice; redefining the states by $S^x \rightarrow -S^x$ and $S^y \rightarrow -S^y$ on half the sites leaves the algebra (??) unchanged, but flips J_{\perp} . The sign of J_z therefore determines whether the model is ferromagnetic ($J_z > 0$) or antiferromagnetic ($J_z < 0$). The physically meaningful coupling is therefore

$$\Delta = \frac{J_z}{|J_{\perp}|}, \quad (5.3)$$

so that $\Delta = 1$ and $\Delta = -1$ are the Heisenberg ferromagnet and antiferromagnet respectively. In the $\Delta \rightarrow \pm\infty$ limit, only the J_z term remains, so the model is effectively classical. For the ferromagnetic case $J_z > 0$, all the spins simply line up with the maximum value of S^z . In the antiferromagnetic case $\Delta \rightarrow -\infty$ on a bipartite lattice, the spins take their maximum opposite values on every other site.

For the remainder of this chapter, I focus only on the spin-1/2 case in one dimension. The spin-1/2 XXZ Hamiltonian on a chain in terms of raising/lowering operators then can be rescaled to be:

$$H_{\text{XXZ}} = - \sum_{i=1}^N \left((\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) + \frac{\Delta}{2} \sigma_i^z \sigma_{i+1}^z \right) \quad (5.4)$$

For spin 1/2, this is the most general nearest-neighbor Hamiltonian consistent with $U(1)$ symmetry, but for higher spins, there are many possible nearest-neighbor couplings one can add consistent with this symmetry.

The Heisenberg spin-1/2 ferromagnet $\Delta = 1$ was already explained in chapter 3. Its Hamiltonian can be written as a sum of projection operators with positive coefficients. Moreover, the completely ferromagnetic states (and the other parts of the $N + 1$ -dimensional spin- N multiplet) are annihilated by each projector, so it must be an exact ground state. The excitations obey the dispersion relation $E \propto k^2$, so whereas this theory is gapless, it is not Lorentz invariant in the continuum limit.

This result for the ferromagnet can be extended to $\Delta > 1$, since increasing Δ favors spin alignment even further. The Hamiltonian H_{XXZ} can be rewritten (after rescaling and shifting) as

$$H = \sum_{i=1}^N \left(2P^{(0)}(\vec{S}_i + \vec{S}_{i+1}) + \frac{\Delta - 1}{2}(1 - \sigma_i^z \sigma_{i+1}^z) \right).$$

with $\vec{S} = \vec{\sigma}/2$ as always for spin-1/2 particles. The last term, while breaking the $SU(2)$ symmetry, is also a projector times the coefficient $(\Delta - 1)$, since

$$\left[\frac{1}{2}(1 - \sigma_i^z \sigma_{i+1}^z) \right]^2 = \frac{1}{2}(1 - \sigma_i^z \sigma_{i+1}^z)$$

This projector annihilates the completely ferromagnetic states with all spins aligned (although not the full spin- $N + 1$ multiplet). Thus for $\Delta > 1$ the state with all spins up and that with all spins down remain ground states of the system. The system is gapped for $\Delta > 1$: the other ground states at $\Delta = 1$ now are excitations with energy $\propto (\Delta - 1)$ over the ground state. In general, a spin wave has energy shifted by $2(\Delta - 1)$ in this normalization. Thus for $\Delta > 1$, the XXZ chain is in a gapped, ordered phase.

5.2 The XX chain

The point $\Delta = 0$ is very special. By using the Jordan-Wigner transformation described in chapter 4, the free fermion system in chapter 2 maps directly on to it. A redefinition of the Jordan-Wigner relations (exchanging $\sigma^x \leftrightarrow \sigma^z$ and sending $\sigma^y \rightarrow -\sigma^y$) leaves the $su(2)$ algebra unchanged, and gives

$$c_j = \left(\prod_{k < j} \sigma_k^z \right) \sigma_j^-, \quad c_j^\dagger = \left(\prod_{k < j} \sigma_k^z \right) \sigma_j^+ \quad (5.5)$$

Ignoring the boundary conditions, the hopping Hamiltonian for spineless fermions then becomes exactly the XXZ Hamiltonian at $\Delta = 0$:

$$H_{\text{XX}} = \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) \quad (5.6)$$

$$= \sum_j (\sigma_j^+ \sigma_{j+1}^- + \sigma_{j+1}^+ \sigma_j^-). \quad (5.7)$$

The latter model is sometimes called the XX chain, for obvious reasons. The entire spectrum of the XX chain can therefore be computed exactly, by the same methods as for free fermions.

For any boundary condition, the XX chain is gapless, with dispersion relation $E \propto |k - k_F|$ at each of the two Fermi points $k_F = \pm\pi/2$. Like with the hopping fermion chain

A further elegant result is that the XX chain, up to boundary conditions, decouples into two copies of the critical Ising model. This fits with the fermionization result, since the hopping Hamiltonian has two Fermi points (i.e. two points in momentum space around which there are gapless excitations), where as a single Ising model has only one. The XX Hamiltonian can trivially be rewritten as

$$\begin{aligned} H_{\text{XX}} &= \sum_i (\sigma_{2i-1}^x \sigma_{2i}^x + \sigma_{2i}^y \sigma_{2i+1}^y) + \sum_i (\sigma_{2i}^x \sigma_{2i+1}^x + \sigma_{2i-1}^y \sigma_{2i}^y) \\ &= H_1 + H_2 \end{aligned}$$

where H_1 and H_2 are the two respective sums. It does not require fermionization to see the decoupling, merely the observations that

$$[\sigma_j^x \sigma_{j+1}^x, \sigma_{j+1}^x \sigma_{j+2}^x] = [\sigma_j^y \sigma_{j+1}^y, \sigma_{j+1}^y \sigma_{j+2}^y] = 0; \quad [\sigma_j^x \sigma_{j+1}^x, \sigma_j^y \sigma_{j+1}^y] = 0,$$

the latter following from the fact that different Pauli matrices at the same site anticommute. Thus $[H_1, H_2] = 0$ and so the eigenstates of H_{XX} can be made into eigenstates of both H_1 and H_2 . With an appropriate choice of boundary conditions, H_1 and H_2

each have the same spectrum of a single Ising chain. This follows from fermionization, yielding e.g.

$$H_1 = \sum_i (\psi_{2i-1}\chi_{2i} + \chi_{2i}\psi_{2i+1}) .$$

This is equivalent to the critical Ising Hamiltonian on *half* the sites, since the only Majorana fermions appearing in H_1 are ψ_{2i-1} and χ_{2i} . One can then reverse the Jordan-Wigner transformation using only these fermions, i.e. define $\tau_i^x \equiv \psi_{2i-1}\chi_{2i}$ and $\tau_i^z\tau_{i+1}^z = \chi_{2i}\psi_{2i+1}$. Then τ_i^x and τ_i^z indeed anticommute, and so with appropriate boundary conditions the spectrum of H_1 is indeed that of a single critical Ising chain on $N/2$ sites. Likewise, the only Majorana fermions appearing in H_2 are ψ_{2i} and χ_{2i-1} , so corresponding operators $\tilde{\tau}^x$ and $\tilde{\tau}^z$ can be defined analogously, with the same result that the spectrum of H_2 with appropriate boundary conditions is that of a critical Ising chain on $N/2$ sites.

This can be extended to different couplings in the two terms in XX chain; the resulting Ising models are then no longer identical or critical. It is also straightforward to relate correlation functions of the spin operators in the XX chain to those in the Ising case. The operator σ^z here has dimension $1/4$...

5.3 The six-vertex model

The quantum-classical correspondence applies to the Heisenberg and XXZ chains as well. In this section I will show how the XXZ chain can be obtained by taking an anisotropic limit of the classical *six-vertex* model on the square lattice. The relation, however, goes much deeper than that. With an appropriate identification of parameters, the XXZ chain and the transfer matrix of the six-vertex model have the *same eigenvectors*. Thus this shows that the quantum-classical correspondence can be much more than a heuristic statement about a certain limit, but instead allow an exact relation to be made between physical quantities in the two models, even without taking the anisotropic and/or continuum limit.

Figure 5.1: The Boltzmann weights for the six-vertex model

The degrees of freedom in the six-vertex model are placed on the *links* of the square lattice. There are two states per link, which are most conveniently presented as an arrow pointing in one of the two directions along the link. Allowed configurations must have the same number of arrows pointing into each site as they do outward. This can be thought of as forbidding “sources” or ”sinks”. There are thus six possible configurations for each vertex, illustrated in fig. 5.1, hence the name of the model. One can allow for an arbitrary Boltzmann weight for each vertex, but it is simplest and most common to consider the “zero-field” case, where the weights are invariant under reversing all the

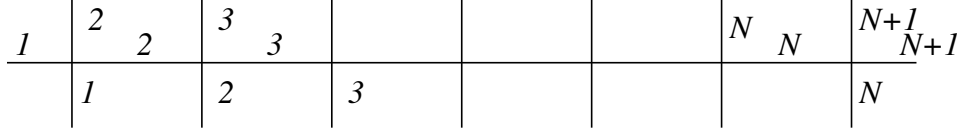


Figure 5.2: The six-vertex model transfer matrix.

arrows. This gives three distinct types of weights, conventionally labelled a , b and c . The model is isotropic (i.e. invariant under 90-degree rotations) when $a = b$. The partition function is then simply

$$Z_{6v} = \sum a^{n_a} b^{n_b} c^{n_c} , \quad (5.8)$$

where n_i is the number of vertices of type i .

The transfer matrix can be built up a single vertex at a time. Consider first just a single vertex, where the transfer matrix takes a system of two links to another system of two links. Labeling the two links i and $i + 1$, the single-vertex transfer matrix is then the 4×4 matrix

$$\begin{aligned} R_{i,i+1} &= \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= a(1 + \sigma_i^z \sigma_{i+1}^z)/2 + c(1 - \sigma_i^z \sigma_{i+1}^z)/2 + b(\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^-) . \end{aligned} \quad (5.9)$$

where for the initial state, an arrow *in* to the vertex corresponds to $\sigma^z = 1$, while an arrow out of the vertex corresponds to $\sigma^z = -1$. For the final state, these conventions are reversed so that an arrow *out* of the vertex is $\sigma^z = 1$. This convention means that $U(1)$ charge

$$\mathcal{S}^z = \sum_j \sigma_j^z = \#(\text{up arrows}) - \#(\text{down arrows})$$

is conserved: $[R_{i,i+1}, \mathcal{S}^z] = 0$ for all i . The transfer matrix for open boundary conditions can be pictured as in fig. 5.2, and is simply

$$T(\text{open})_{ab} = (R_{1,2} R_{2,3} \dots R_{N,N+1})_{ab} . \quad (5.10)$$

The labels in the picture indicate which vector space the R -matrix acts on; when the vertices are drawn like this, R acts up and to the left. The indices $a = \pm 1$ and $b = \pm 1$ represent respectively the end states at the left and right, i.e. the final value of σ_1^z and the initial value of σ_{N+1}^z . For periodic boundary conditions,

$$T = \sum_{a=\pm 1} T(\text{open})_{aa} . \quad (5.11)$$

This is a $2^N \times 2^N$ matrix, and commutes with $\mathcal{S}^z = \sum_{j=1}^N \sigma_j^z$.

Finding the relation of this transfer matrix to the XXZ chain is now easy. From (5.9), it is obvious that the transfer matrix is close to the identity if a and c are close to 1 and b very small. Writing

$$a = 1 + \delta a, \quad b = \delta b, \quad c = 1 + \delta c ,$$

the transfer matrix for one vertex is

$$R_{i,i+1} \approx \mathbb{I} + \delta b(\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^-) + \frac{\delta a - \delta c}{2} \sigma_i^z \sigma_{i+1}^z .$$

Then for δa , δb and δc small, expanding out the product in T gives

$$T = \mathbb{I} + H_{XXZ} + \dots , \tag{5.12}$$

where $J_{\perp} = 2\delta b$ and $J_z = 2(\delta a - \delta c)$ and the boundary conditions on H_{XXZ} are periodic. Thus the correspondence between the XXZ parameter Δ and the six-vertex parameters in this limit is

$$\Delta = \frac{\delta a - \delta c}{\delta b} . \tag{5.13}$$

From this one expects the physics of the two models to be related; in the next section it will be shown that the relation is much deeper.

Before moving on to a detailed analysis of the six-vertex model, it is worth noting that a special case long ago in a context completely different from that of magnets: ice! In the hexagonal or cubic phase of an ice crystal (the stuff that is found in a freezer), each oxygen atom is surrounded by four other oxygen atoms forming a tetrahedron. The Bernal-Fowler ice rules¹ state that each hydrogen atom is associated with two oxygen atoms, one bonded strongly and the other weakly. Only one hydrogen atom is associated with each linkage, so that means that each oxygen atom is associated with four hydrogen atoms, one bound weakly, and the other strongly. (Thus indeed there are two hydrogens for each oxygen.) is thus the oxygen and its two strongly-bonded partners.) Pauling observed that there were thus multiple ways of that the hydrogens can pair with the oxygens, even at zero temperature. This is called the *residual* or *geometric entropy*. In fact, this provided the first known natural example of geometric frustration! Each bond can be labelled by an arrow along each link pointing in the direction of the oxygen atom with the strong bond. The ice rules mean that are two arrows pointing in and two arrows pointing out at each oxygen atom, i.e. the same six vertices as in fig. 5.1. Computing the geometric entropy is tantamount to computing the six-vertex model partition function on this three-dimensional lattice with weights $a = b = c = 1$.

The three-dimensional problem cannot be solved exactly, although Pauling² estimated the residual entropy by noting that for A oxygen atoms there are 2^{2A} configurations

¹J.D. Bernal and R.H. Fowler, J. Chem. Phys. **1**, 515 (1933)

²L.J. Pauling, Am. Chem. Soc. **57** : 2680-2684 (1935)

without the six-vertex restriction, and that only 6 of the 16 vertices are allowed. Thus the total number of configurations is approximately

$$2^{2A} \left(\frac{6}{16} \right)^A = \left(\frac{3}{2} \right)^A .$$

This estimate turns out to be remarkably close, and in fact a lower bound, as shown by Onsager and Dupuis. An improved estimate $(1.50685 \pm .00015)^A$ was derived by Nagle³ using series expansions. This gives for the entropy (in chemist's units) $S = .8145 \pm .0002$ cal/deg/mole. The 1936 experimental value⁴ is $.82 \pm .05$ cal/deg/mole; shockingly this does not seem to have been improved upon!

The six-vertex model on the square lattice with $a = b = c = 1$ can be thought of as "square ice". Obviously, it is not the real thing, but not only does the entropy behave simply, but it can be derived exactly for M large. This will be done below.

5.4 Connecting XXZ and 6v exactly: Commuting transfer matrices

The six-vertex model is *integrable*. A lattice model is integrable if one can find an series of local quantities Q_n commuting with the transfer matrix: $[Q_n, T] = 0$. Baxter recognized that such commuting quantities could be generated if *commuting transfer matrices* could be found. Any transfer matrix depends on some parameters; for the Ising model these are J_x and J_y , while for the six-vertex model these are a , b and c . Commuting transfer matrices require finding a special parametrization for the Boltzmann weights such that transfer matrices for different values of one of these new parameters commute. For example, for the six-vertex model, the Boltzmann weights can be written in terms of new variables ρ , λ and u , given in (5.18) below. What Baxter showed was that here (and in many other cases), the transfer matrices obey

$$[T(\rho, \lambda, u), T(\rho, \lambda, u')] = 0 \tag{5.14}$$

for *any* u and u' . To make this equation general, it is convenient to suppress all parameters other than u so that

$$[T(u), T(u')] = 0 ;$$

all other parameters are the same in both transfer matrices. By expanding say $T(u')$ in powers of u' , each term in the resulting expansion commutes with $T(u)$. If u is defined so that $T(0) = \mathbb{I}$, then a quantum Hamiltonian is defined by taking

$$H = \lim_{u \rightarrow 0} \frac{1}{u} (T(u) - \mathbb{I}) .$$

³J.F. Nagle, J. Math. Phys. **7**, 1484 (1966)

⁴W.F. Giauque and J.W. Stout, J. Am. Chem. Soc. **58**, 1144 (1936)

Commuting transfer matrices follow from requiring that the Boltzmann weights satisfy the *Yang-Baxter equation*

$$R_{i,i+1}R'_{i+1,i+2}R''_{i,i+1} = R''_{i+1,i+2}R'_{i,i+1}R_{i+1,i+2} \quad (5.15)$$

This is a matrix equation; for the six-vertex model, before symmetries are exploited this results in 64 separate equations. The $U(1)$ symmetry in the six-vertex model is very powerful, and this reduces to three equations

$$\begin{aligned} a''ca' &= b'cb'' + c'ac'' \\ a''bc' &= c'ab'' + b'cc'' \\ c''ba' &= b'ac'' + c'cb'' \end{aligned} \quad (5.16)$$

Using these equations to eliminate a'', b'', c'' gives the simple relation

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{a'^2 + b'^2 - c'^2}{2a'b'}$$

Defining

$$\Delta \equiv \frac{a^2 + b^2 - c^2}{2ab} \quad (5.17)$$

The Yang-Baxter equation is therefore satisfied if $\Delta = \Delta'$. The use of the same parameter Δ here and in the XXZ chain is not a notational oversight. By taking

$$a = 1 + \delta a; \quad b = u\delta b; \quad c = 1 + u\delta c$$

as before, in the limit of u small

$$\Delta = \frac{2\delta a + 2\delta c}{2\delta b} = \frac{J_z}{|J_\perp|}$$

indeed. Going back to the original equations shows that they are satisfied $\Delta'' = \Delta = \Delta'$ as well.

The simplicity of this relation hides its profound nature. Most models do not possess Boltzmann weights that solve the Yang-Baxter equation, as one easily finds by trying to solve the equation – there are far more equations than unknowns. For the six-vertex model, any value of a , b and c still results in commuting transfer matrices. The only constraint is that the different transfer matrices have the same value of Δ . There are essentially two unknown Boltzmann weights (since one can always rescale all weights to make any single non-vanishing weight to be 1), so this still gives a one-parameter family of commuting transfer matrices.

A very useful and illuminating parametrization for the Boltzmann weights and transfer matrices is given by

$$\begin{aligned} a &= \rho \sin(\lambda - u) , \\ b &= \rho \sin(u) , \\ c &= \rho \sin(\lambda) . \end{aligned} \quad (5.18)$$

The parameter ρ is an unimportant overall constant and can be set to any convenient value. Such Boltzmann weights are real, if all of λ , u and ρ are real and positive and $0 \leq u \leq \lambda$. They are also real and positive if λ , u and ρ are imaginary, so that \sin can be replaced with \sinh . The limit $\lambda \rightarrow 0$ makes sense if one takes $\rho = 1/\sin(\lambda)$ and rescales u appropriately. The parameter u is sometimes called “anisotropy” or “spectral” parameter. Rotations by 90 degrees interchange the a and b vertices, while leaving c invariant, so sending $u \rightarrow \lambda - u$ is equivalent to rotating the lattice by 90 degrees. Taking $u = \lambda/2$ gives the isotropic case $a = b$, while taking $u \rightarrow 0$ gives the very anisotropic XXZ limit.

This parametrization is very useful because Δ is independent of u . Using a few trigonometric identities gives

$$\Delta = -\cos(\lambda) , \tag{5.19}$$

Transfer matrices at different values of u but the same values of λ commute: (5.14) holds as claimed!

This means that the six-vertex model has an infinite sequence (as $N \rightarrow \infty$) of charges commuting with the transfer matrix. Since they share the same eigenvectors, this infinite sequence also commutes with the XXZ Hamiltonian. For example, a local conserved charge can be found by taking the logarithmic derivative:

$$H^{(2)} \propto \left(\frac{\partial^2}{\partial u^2} \ln T(u) \right)_{u=0}$$

This reduces to commutators of individual terms in H . For $\Delta = \pm 1$, this can be rewritten in the simple form

$$H^{(2)} \propto \sum_j \vec{S}_j \cdot (\vec{S}_{j+1} \times \vec{S}_{j+2}) = \sum_j \epsilon_{abc} S_j^a S_{j+1}^b S_{j+2}^c$$

does indeed commute with H_{XXZ} .

The parameters u and λ play very different roles. The physics strongly depends on λ ; for example universal quantities such as critical exponents depend on it but not on u . Intuitively, this comes from the fact that u is governing a purely lattice property, the relative couplings in the x and y directions. When the continuum limit of the lattice model is a rotationally invariant field theory, then u only governs the anisotropy of the discretized version. Thus the field theory depends only on λ , as will be discussed in this example in the next chapter. Varying u does not change for example the phase structure of the model. For these reasons, u is also often called “spectral” parameter, because while the eigenvectors of $T(u)$ and so much of the physics of the problem is independent of it, the eigenvalues (the spectrum) of course do depend on it.

Determining the phase diagram of the six-vertex model is therefore essentially equivalent to determining that of the XXZ chain. Naming the quantity Δ in the six-vertex model as (5.17) and in the XXZ model as (5.13) was not being lax; the former reduces to the latter as δa , δb and δc are made small. Thus the six-vertex model should have the

same phase diagram as the XXZ chain as a function of Δ . The fact that the Boltzmann weights change form at $\lambda = 0$ from trigonometric to hyperbolic is a hint that a phase transition is taking place here. Thus a putative phase diagram...

The six-vertex model is somewhat unusual in that it has a two-parameter family (both λ and u) of solutions to the Yang-Baxter equation. The resulting line of critical points with continuously varying critical exponents are somewhat unusual in 1+1 dimensions, and very rare in higher dimensions.

An important (seemingly technical but in fact very fundamental) point is in order. Once the six-vertex weights are parametrized in terms of λ and u , solving the Yang-Baxter equation (5.15) requires that $u'' = u - u'$. Thus the YBE can be rewritten as

$$R_{i,i+1}(u - u')R_{i+1,i+2}(u)R_{i,i+1}(u') = R_{i+1,i+2}(u')R_{i,i+1}(u)R_{i+1,i+2}(u - u') \quad (5.20)$$

Most (but not all) integrable models obey the YBE in this “difference” form. However, it is important to note that all that is required to have a commuting transfer matrix is the original relation (5.15). When the YBE is obeyed in the difference form (5.20), additional special properties follow. A simple but profound one is that if the equation is drawn as triangles as in figure ???, u , $u - u'$ and u' can be interpreted as the angles between the lines in the picture, a fact that means that the six-vertex model and its generalization the eight-vertex model are integrable on any planar lattice comprised of lines intersecting so that only two lines ever meet at a point (sometimes called the “Baxter” lattice).

The Yang-Baxter equation in difference form also arises in 1+1 dimensional integrable continuum theories with exact scattering matrices. The YBE is a consistency condition on the requirement that the three-particle scattering matrix factorizes into a product of two-body ones; the two sites of the equation represent the two ways this is possible. The lines in figure ??? can be interpreted as the world lines of the particles, and the parameters u , $u - u'$ and u' as the difference of rapidities of the each of the pairs of particles. The fact that the scattering matrices depend only on the rapidity differences is a consequence of Lorentz invariance.

Writing the Boltzmann weights as a function of the anisotropy/spectral parameter u uncovers a host of profound results that can be derived simply by assuming certain analyticity properties in u . In the next section, the leading contribution to the free energy will be found by this method.

5.5 The exact free energy in the large-volume limit

On an $N \times M$ lattice with periodic boundary conditions in the direction of transfer matrix evolution.

$$Z = \text{tr } T^M .$$

In the limit of a large number of sites, this is therefore dominated by the largest eigenvalue of T . The logarithm of this eigenvalue is proportional to the free energy, which is $\propto N$. In general, the N -dependent contribution to the free energy and hence the largest eigenvalue is independent of boundary conditions, since the boundaries will not affect an N -dependent quantity. Defining this largest eigenvalue to be $\kappa(u)^N$ gives

$$Z \approx \kappa(u)^{NM} . \quad (5.21)$$

as the number of sites NM gets large.

The free energy per site (divided by the temperature) in this limit is therefore independent of N , so

The trick of Baxter's *inversion relation method* is to derive several relations for $\kappa(u)$. These relations are sufficient to find an exact expression for $\kappa(u)$ if certain analyticity assumptions are made about the behavior with u complex. The first relation for $\kappa(u)$ follows from the observation that rotating by 90 degrees is the same as sending $u \rightarrow \lambda - u$. Thus

$$\kappa(\lambda - u) = \kappa(u) \quad (5.22)$$

To find the first inversion relation, note that

$$W_{i,i+1}(u)W_{i,i+1}(-u) = \rho^2 \sin(\lambda + u) \sin(\lambda - u) \mathbb{I} .$$

This can be derived directly, or it follows from the Yang-Baxter equation by setting $u' = 0$ and noting that $W(0) \propto \mathbb{I}$. (The property is thus not a fluke of the six-vertex model, and so this trick can be applied to many integrable models; in knot theory this turns out to be related to the second Reidemeister move.) Since $\kappa(u)$ is independent of boundary conditions... Defining $T_{\text{refl}}(u)$ here with "reflecting" boundary conditions

$$T_{\text{refl}}(u)T_{\text{refl}}(-u) = (\rho^2 \sin(\lambda + u) \sin(\lambda - u))^N \mathbb{I} .$$

Notice that the Boltzmann weights are periodic if $v \rightarrow v + 2\pi i$. and make the following assumption:

$\ln(\kappa(v))$ is analytic in an open region containing the vertical strip $0 \leq \text{Re}(v) \leq \lambda$, and is periodic under shifts $v \rightarrow v + \pi i$.

This assumption can be checked by analyzing the low- and high-temperature expansions of the model (and of course by using the exact Bethe ansatz result). Even number of sites, even number of c vertices with periodic boundary conditions.

$$\kappa(v)\kappa(-v) = \rho^2 \sinh(\mu - v) \sinh(\mu + v) \quad (5.23)$$

$$\kappa(\mu - v) = \kappa(v) \quad (5.24)$$

Expand $\ln(\kappa(v))$ in a Laurent series inside the region of analyticity:

$$\ln(\kappa(v)) = \sum_{n=-\infty}^{\infty} c_n e^{-2nv} .$$

which indeed is periodic under shifts of $v \rightarrow v + \pi i$. The rotation relation (5.24) therefore requires that

$$c_{-n} = e^{-2\mu n} c_n$$

and

$$\ln(\sinh(\mu - v) \sinh(\mu + v)) = \sum_{n=1}^{\infty} \frac{1}{n} (e^{-2(\mu+v)n} + e^{-2(\mu-v)n}) = \sum_{n \neq 0} \frac{1}{|n|} e^{-2vn} e^{-2\mu|n|}$$

Therefore for $n > 0$

$$c_n = \frac{e^{-2\mu n}}{n(1 + e^{-2\mu n})} = \frac{e^{-\mu n}}{2n \cosh(\mu n)} \quad (5.25)$$

Taking $\rho = 1/\sinh(v)$ gives then

$$\begin{aligned} \ln(\kappa(v)) &= \ln(\rho^2/4) + 4\mu + \sum_{n=1}^{\infty} \frac{e^{-\mu n}}{2n \cosh(\mu n)} (e^{-2nv} + e^{2n(v-\mu)}) \\ &= 2 \sum_{n=1}^{\infty} \frac{e^{-2n\mu} \sinh(nv) \sinh(n(\mu - v))}{n \cosh(n\mu)} \end{aligned}$$

which converges in the strip.

For the case $|\Delta| < 1$, there is not periodicity in imaginary u like in the $\Delta > 1$ case. Instead, the weights a and b grow as $\text{Im}(u) \rightarrow \pm\infty$ as $e^{\mp i\mu}$ whereas the c weight remains finite. The partition function can easily be computed then, and so $\kappa(u)$ grows as $e^{\mp i\mu}$ in these limits. Therefore the second derivative can be Fourier integrated, i.e. define a function $k(t)$ so that

$$\frac{d^2 \ln(\kappa(u))}{du^2} = \int_{-\infty}^{\infty} k(t) e^{2ut} dt$$

The function $k(t)$ is then analytic in some strip about the real t axis, and the integral converges for u in the domain.

$$\ln(\kappa(u)) = \int_{-\infty}^{\infty} \frac{\cosh(\pi - 2\lambda)t \sinh ut \sinh(\lambda - u)t}{t \sinh \pi t \cosh(\lambda t)} .$$

Then after Fourier transforming the inversion equations

$$\begin{aligned} c(t) + c(-t) &= -4t \frac{\cosh(\pi - 2\lambda)t}{\sinh(\pi t)} \\ c(t) + e^{-4\lambda t} c(-t) &= -4te^{-2\mu t} \frac{\cosh(\pi - 2\lambda)t}{\sinh(\pi t)} \end{aligned}$$

5.6 KT transition

The energy of a single vortex is

$$E \approx J \ln(R/a)$$

while the number of vortices possible is $(R/a)^2$.