

A conformal field theory primer

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1 Some general comments

A central tenet of physics is that one should exploit symmetry as much as possible. A major innovation in theoretical physics in the last 30 years has come from the recognition that the presence of *conformal symmetry* is a powerful constraint in many theories. Conformal invariance is a generalization of the invariance under scale transformations, the fundamental property of critical points. A conformal transformation allows space to not just be rescaled, but to be twisted such that angles are preserved. What one typically needs to have conformal invariance is *scale invariance* (the hallmark of critical points) plus rotational invariance in a classical theory, or Lorentz invariance in a quantum theory¹ Conformal invariance is particularly powerful in classical theories in two spatial dimensions, or quantum theories in one spatial dimensions. Henceforth, this is what I will focus on: whenever I say CFT, I mean those in two or 1+1 dimension. However, it is worth noting that it is useful in any dimension, and also quite powerful in higher dimensions when combined with supersymmetry.

1.1 Introducing the free boson

The canonical example of a field theory with conformal invariance is that of a free massless boson, a field theory often called the *Gaussian model* in condensed-matter physics. It is very familiar there, because it describes the scaling limit of a many systems of fundamental interest. A free massless scalar boson $\phi(x, y)$ in two dimensions has partition function

$$Z = \int [D\phi] e^{-S}$$

with action S given by

$$S = \frac{g}{4\pi} \int dx dy \nabla\phi \cdot \nabla\phi \tag{1}$$

with the coupling g generally called the “stiffness”. Thus in terms of classical statistical mechanics, e^{-S} is the Boltzmann weight. The Boltzmann weight is thus maximized by “flat” configurations, where the field is constant in space. However, there are many more configurations that are not flat, so entropy is maximized by not having flat configurations, so the physics of this model is far from trivial.

The “path” or “functional” integral amounts to summing over all configurations of the bosonic field. This sum is often difficult or impossible to define precisely and rigorously

¹There is a perturbative proof that scale invariance plus certain other reasonable conditions implies conformal invariance in $d = 4$; see arXiv:1302.0884. There are currently no known counterexamples.

in the continuum in an interacting theory. In condensed matter physics we typically appeal to some underlying lattice model, which of course is well defined, but rarely useful when trying to do a calculation outside free theories – continuum calculations are almost always easier. Usually field theorists define interacting theories via perturbation theory, which is a perfectly reasonable thing to do if i) perturbation theory is valid and ii) you're not interested in non-perturbative effects. Modern condensed matter physics is very concerned in low-dimensional physics, and the most interesting physics there is inherently non-perturbative. One of the glories of conformal field theory is that it gives a completely alternative way of defining the theory and in doing computations.

To give an idea of why conformal invariance is so powerful in 2d, let's understand how it appears in the free-boson theory. The action (with space the infinite plane) is obviously invariant under the scale transformation $x \rightarrow \lambda x$ and $y \rightarrow \lambda y$; the rescaling of the derivatives is canceled by the rescaling of the measure $dx dy$. The easiest way to see the full set of conformal transformations is to rewrite space in *complex coordinates* $z = (x + iy)$, $\bar{z} = (x - iy)$. One thing that pops out easily is that the equation of motion for the field ϕ

$$(\partial_x^2 + \partial_y^2) \phi = 0$$

becomes

$$\partial_z \partial_{\bar{z}} \phi = 0 .$$

The general solution to this is obvious:

$$\phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z}) \tag{2}$$

for *any* functions $\varphi(z)$ and $\bar{\varphi}(\bar{z})$ satisfying

$$\partial_{\bar{z}} \varphi(z) = \partial_z \bar{\varphi}(\bar{z}) = 0 ,$$

i.e. they are analytic and anti-analytic functions respectively. If this were instead a boson in 1+1 dimensional Minkowski space-time instead of two-dimensional real space, the two functions would be the familiar left- and right-moving solutions of the wave equation. In general conformal field theories, the decoupling between analytic and anti-analytic parts is not so simple, but studying analytic and anti-analytic sectors separately is one of the key parts of conformal field theory. In fact, in the edge theory of the simplest fractional quantum Hall effect edge states, only one of the sectors appears.

Rewriting the action in complex coordinates gives

$$S = \frac{g}{2\pi} \int dz d\bar{z} \partial_z \phi \partial_{\bar{z}} \phi . \tag{3}$$

The conformal symmetry is now apparent: this action is under the transformation

$$z \rightarrow f(z) , \quad \bar{z} \rightarrow \bar{f}(\bar{z}) \tag{4}$$

for *any* analytic function $f(z)$ and anti-analytic function $\bar{f}(\bar{z})$; the transformation of the derivatives is cancelled by the change in the measure. This is an infinite-dimensional symmetry! Namely, the infinitesimal transformations are by $z \rightarrow z - z^{n+1}$ and $\bar{z} \rightarrow \bar{z} - \bar{z}^{n+1}$, for any integer n . The case $n = -1$ is ordinary translation, while $n = 0$ is a scale transformation, but the rest are new.

To understand what these new “conformal” transformations are in general, consider the metric of space $g_{\mu\nu}$, defined so that the square of the “distance” between two nearby points is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The conformal group (in any dimension) by definition is comprised of coordinate transformations that just rescale the metric of space: $ds^2 \rightarrow \Omega(x^\mu)ds^2$. Less formally, what this means is that while conformal transformations deform space, they preserve the *angle* between any two vectors starting at a given point. In complex coordinates, the metric $g_{\mu\nu} = \delta_{\mu\nu}$ becomes

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 ; g_{z\bar{z}} = g_{\bar{z}z} = 2 .$$

The transformations simply rescale the metric by the factor $|f(z)|^2$.

1.2 Some important questions to ask, and a moral

I will discuss the 2d free boson in the most depth, because it is the simplest case, and because it is so useful in condensed matter. This may seem to belie my point that conformal symmetry is useful and powerful, since in a free field theory one can seemingly compute everything without having to use fancy theoretical techniques such as CFT. But I said “seemingly”, because even though in principle one can compute all correlation functions with sufficient effort, the more difficult and much more interesting question is: what good are these correlators? More precisely,

- Do any describe the continuum limit of some interesting quantity in a lattice model?
- Which are universal, i.e. which are independent of microscopic details of some underlying model?
- Is there a systematic way of approaching the theory?
- Do any quantities characterize the theory in some fundamental fashion?

I’m going to try to repeatedly address these questions as we proceed.

I will explain how conformal invariance provides great insight into these questions even in the case of a free boson. For example, one result of central importance is that strongly interacting fermions in 1+1 dimensions (the Luttinger liquid) can be described in terms of 2d free bosons. The procedure in showing this is often called bosonization, and while this can be done without recourse to more general ideas, it is more precise and systematic to use conformal field theory. Even more importantly, I believe that in the end it is simpler. Of course there’s an initial hump to get over, but just as an experimentalist

would be foolish to ignore a new piece of technology because the instruction manual looked unfamiliar, we shouldn't be afraid to learn new things. To climb onto the pulpit for a moment, here's an important moral principle for theoretical physicists:

If it's worth doing, it's worth doing right.

To climb back down, it is important though to recognize that conformal field theory is very much its own beast. What I mean by this is that although in some of the models such as the free boson, more traditional methods can be used profitably to analyze the system, such approaches are either CFT in disguise (as with bosonization), or just plain different. This is a good thing, but the fact that is that most of what you learned in solid-state theory is mainly useful as a complementary approach that works occasionally. This I think is the main reason why CFT has a (mostly but not completely unwarranted) reputation as being difficult. As in any field, it gets complicated after a while, but the basics are no more difficult than in any other theoretical field. They are, however, somewhat different.

For example, most of the time in field theory the starting point is the action or Hamiltonian. In CFT, the action is typically not used directly, except in very special cases like free bosons and fermions. (I promise never to say "gauged Wess-Zumino-Witten action" again.) And whereas one may in a sense know what the Hamiltonian of a CFT is, what is most important is how the Hamiltonian fits into the conformal symmetry algebra (i.e. the algebra of conformal transformations). One reason one can do such marvelous things in CFT is that the Hamiltonian is just one part of an infinite-dimensional symmetry algebra.

1.3 An overview, via the math involved

One of the great things about being a theoretical physicist is that we get to dip into whatever mathematical field seems to be useful at the moment. Moreover, when we do something new, typically it will have an impact in very different mathematical fields. So a cute way of outlining some of what we do in CFT is to chart out how it fits into the math world.

CFT starts in *geometry*, since it exploits the fact that the model is invariant under certain kinds of geometric transformations of two-dimensional space, those that preserve angles. This then quickly moves onto *complex analysis*, since the natural way of expressing these transformations is in terms of complex coordinates $z = x + iy$. Then the math starts getting serious, because this leads naturally to finding the (infinite series of) generators of conformal transformations. In a 1+1-dimensional ("quantum") picture these satisfy an *algebra* with some very nice properties. These are so nice that much can be understood about the *representations* of this algebra, and so for example all the states in the Hilbert space and their energies can be found exactly. Moreover, the representation theory of the algebra allows *differential equations* to be derived for the correlators, which allows all of them (at least in principle) to be computed exactly.

What I just described was the state of the art of CFT in 1988, as reviewed in the superb lectures by Ginsparg and Cardy in the Les Houches 1988 summer school volume. These notes will borrow heavily from these much more thorough introductions. An equally excellent set of lectures in the same volume that we won't have time to get to are those by Affleck, where some important applications of CFT to spin systems are described. But starting that same year, a whole new set of interesting results emerged. In understudying the operators and their correlators, an essential part is in understanding their behavior when locations of different operators are rotated around each other (their monodromy). The monodromy is easiest to express as *braiding* in 2+1 dimensional space-time; the new time coordinate just being thought of labeling the position of the operators as we move them around each another. This in turn can be expressed very elegantly in terms of properties of a three-dimensional *topological* field theory, with remarkable consequences in *knot theory*. In understanding how all this works, a whole new algebraic structure of the operators, the fusion algebra, emerges, with major consequences for topology and physics.

This path from geometry to algebra to topology in fact takes us back to condensed-matter physics again, because this physics and mathematics turns out to be essential in understanding how *anyonic* systems behave. Finding the statistics of a particle in a two-dimensional *quantum* system is formally the same problem as computing the monodromy of an operator in conformal field theory. The fact that there are lots of interesting CFTs means that there are much more intricate kinds of statistics possible for particles in two dimensions than bosonic or fermionic. Two-dimensional quantum condensed matter thus can have a variety of exotic behaviors unseen in higher dimensions, the fractional charge and statistics present in the fractional quantum Hall effect being the first prominent experimental example. More recent excitement has come from the prospect of using anyons to do quantum computation where protection against errors may be strong due to the robustness of topology. This too is proving experimentally interesting; in three words: Majorana, Majorana, Majorana. To understand the physics of these systems, it is absolutely essential to understand some things about CFT.

This I hope illustrates the value of my above moral. CFT began by studying 1d quantum or 2d classical systems at critical points. It has now been applied successfully to both classic problems of condensed matter physics (e.g. the Kondo problem), classic problems in statistical mechanics (e.g. percolation), as well as a host of recent experimental systems (see Giamarchi's book for examples). However, smart people trying to understand this as well as they could uncovered a host of interesting results of relevance in arenas (two-dimensional gapped quantum systems) seemingly far from this origin.

Of course what I'll do here is a tiny fraction of what is known. What I will cover is the free boson in a little detail, and then apply this to the Kosterlitz-Thouless phase transition. Then I will describe some of the central characteristics of CFT, including the meanings of the central charge and the fusion algebra. If I have any time left, I will then say a little bit about the Ising model, the traditional name of what these days is called in certain circles the Majorana chain.

2 A free massless boson

2.1 The XY and XXZ models

When condensed matter theorists calculate something, it's usually in the context of some field theory. However, no matter how formally you're inclined, it's a good idea to try to stay grounded in reality by keeping some very concrete examples in mind. Quite frequently, in condensed matter theory, such examples are lattice models. So before we start discussing the 2d free bosonic field theory in more detail, I think it's important to give several examples of important lattice models whose continuum limit is described by this field theory.

The most obvious one is the XY model. This is a classical 2d system whose degrees of freedom are classical fixed-length “spins” or “rotors” at each point on some 2d lattice. The rotors pointing anywhere in a *plane*, and so can be represented by an angle $0 \leq \theta < 2\pi$. The interaction is ferromagnetic in spin language, favoring nearest-neighbor rotors to be pointing in the same direction. The simplest action with the desired properties is

$$S = -\frac{J}{T} \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) .$$

The XY model describes for example the transition in superfluids: the value θ is the phase of the expectation value of the wave function. Even though the existence of superfluids of course is a quantum-mechanical effect, quantum effects are essentially negligible when studying the transition between the superfluid and the normal phase. Thus the classical rotor model provides a way of quantitatively understanding this transition.

For J/T large enough (i.e. large coupling or low temperature) the dominant field configurations in the partition function of the XY model are those where θ varies slowly over space. In this limit it is then reasonable to replace θ with a continuous field ϕ governed by the action (1). This is the simplest action consistent with the symmetry of the original model under a global rotation of the rotors $\theta_i \rightarrow \theta_i + a$ for any constant a . The fact that ϕ came from an underlying periodic θ is extremely important; the resulting vortices cause a phase transition named after its discoverers Kosterlitz and Thouless.

A less obvious example of a model described by the 2d free boson is the antiferromagnetic Heisenberg quantum spin-1/2 chain. The Heisenberg spin chain is built from spin spin operators S_j^x , S_j^y and S_j^z at each site j satisfying $su(2)$ commutation rules

$$[S_j^a, S_k^b] = i\delta_{jk}\epsilon_{abc}S_j^c , \tag{5}$$

with spin operators on different sites commuting. In the spin-1/2 case, the spins are represented by Pauli matrices via $\vec{S} = \vec{\sigma}/2$. The Hamiltonian of the slightly more general XXZ model is

$$H_{\text{XXZ}} = -\sum_{i=1}^N \left((\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) + \frac{\Delta}{2} \sigma_i^z \sigma_{i+1}^z \right) ; \tag{6}$$

the Heisenberg case is $\Delta = \pm 1$, with $\Delta = -1$ corresponding to the antiferromagnet.² For spin 1/2, this is the most general nearest-neighbor Hamiltonian consistent with the $U(1)$ symmetry generated by z component of the total spin

$$\mathcal{S}^z = \sum_j S_j^z .$$

The presence of the $U(1)$ symmetry is a hint that a free boson might be a good description of the XXZ spin-chain. A way to see it much more convincingly is to first study a two-dimensional classical lattice model called the six-vertex model. This is discussed in chapter 5 of my “book”. On one hand, this model can be directly mapped onto a “height”, an integer-valued degree of freedom on the sites of a lattice. In the continuum, one expects this to turn into a free boson if one can ignore the discrete nature of the height. This turns out to be the case when the temperature in the lattice model is large; CFT gives a nice way of understanding why this is so. On the other hand, one can show that the Heisenberg and XXZ Hamiltonians can be obtained by taking a particular limit of the six-vertex model transfer matrix. In fact, with an appropriate identification of parameters, the XXZ chain and the transfer matrix of the six-vertex model have the *same eigenvectors*. (This is a consequence of the integrability of the problems, a fascinating issue that there’s no way I’ll have time to discuss in these lectures!) Thus the physics of the XXZ chain, at least for a certain parameter range, is that of the free boson. This parameter range (again coming from integrability) is $-1 \leq \Delta < 1$.

An unusual feature of these models is they exhibit a *line* of critical points with continuously varying critical exponents. This behavior is almost impossible outside 1+1 or 2 dimensions, and not even that common there. The free boson field theory has precisely this behavior; this line is parametrized by g .

2.2 Operators and correlators

Since the action (3) is quadratic, all correlators can be computed exactly. Doing the Gaussian integral gives the two-point function on the infinite plane to be

$$G(z, \bar{z}) = \langle \phi(z, \bar{z}) \phi(0, 0) \rangle = -\frac{1}{2g} \ln \left(\frac{z\bar{z}}{a^2} \right) \quad (7)$$

where a is a short-distance cutoff (e.g. a lattice spacing) necessary to define the theory. This cutoff is what will make life interesting, because it is not invariant under conformal transformations: e.g. a scale transformation takes $a \rightarrow \lambda a$. This is the Green function for the 2d Laplacian:

$$\nabla^2 G(z, \bar{z}) = \partial_z \partial_{\bar{z}} G(z, \bar{z}) = \frac{2\pi}{g} \delta(z) \delta(\bar{z}) .$$

²The minus sign is to conform with the conventions of Baxter’s book. To recover the usual form of the antiferromagnet, one can rotate every other spin to flip the sign of the first two terms in (6).

(This is easiest to verify by going to polar coordinates.)

Since we can split the field itself into two pieces, we can do the same for the correlator:

$$\langle \varphi(z)\varphi(w) \rangle = -\frac{1}{2g} \ln \left(\frac{z-w}{a} \right) ; \quad \langle \bar{\varphi}(\bar{z})\bar{\varphi}(\bar{w}) \rangle = -\frac{1}{2g} \ln \left(\frac{\bar{z}-\bar{w}}{a} \right) .$$

Note that these correlators *increase* with distance between operators. If ϕ itself were a physically measurable operator, this could lead to trouble. However, it typically is not; this is related to the fact that there are no Goldstone bosons in two space-time dimensions, a consequence of the famed Mermin-Wagner theorem, and explained in field theory language by Coleman. However, since all the operators in the theory can be expressed in terms of φ and $\bar{\varphi}$, this correlator forms the foundation of all correlators. For example, in the XXZ chain the $U(1)$ symmetry under shifts of ϕ is generated by $J(z) + \bar{J}(\bar{z})$, where $J = \partial\varphi$ and $\bar{J} = \bar{\partial}\bar{\varphi}$. The two-point function of the former is simply

$$\langle J(z)J(w) \rangle = \frac{1}{(z-w)^2} ; \tag{8}$$

and likewise for \bar{J} . In a chiral fractional quantum Hall edge (chiral meaning only say left movers appear), J is proportional to the electric current operator.

Other physically interesting operators are given by *exponentials* of φ and $\bar{\varphi}$. (In string theory language, exponentials of free bosons are called “vertex operators”.) By using the fact that S is Gaussian and completing the square, or simply by expanding out the exponents and using Wick’s theorem, it is easy to show that

$$\langle e^{i\alpha\varphi(z)}e^{-i\alpha\varphi(w)} \rangle = e^{\alpha^2\langle\varphi(z)\varphi(w)\rangle} = \frac{a^{\alpha^2/2g}}{(z-w)^{\alpha^2/2g}} .$$

The cutoff a can be absorbed in a rescaling of the operators $e^{i\alpha\varphi}$, and so typically these operators are normalized so that the constant of proportionality is 1. (In canonical quantization this corresponds to normal-ordering the annihilation and creation operators, and is sometimes called “multiplicative renormalization”.) The correlator of this “renormalized” operator is then independent of the cutoff, and so still makes sense when $a \rightarrow 0$. The presence of this cutoff, however has a lasting effect. The field ϕ must be dimensionless to make the action dimensionless, and so its exponentials are dimensionless as well. Nevertheless, absorbing the factor $a^{-\alpha^2/4g}$ into each operator effectively gives it a dimension $\alpha^2/4g$.

In fact, a general consequence of scale invariance is that the two-point function falls off at large distance as a power law:

$$\langle A(\vec{r}_1)B(\vec{r}_2) \rangle \propto \frac{1}{|\vec{r}_1 - \vec{r}_2|^{2x_A}} \tag{9}$$

where $x_A = x_B$ is called the *scaling dimension* of the operator, or usually just “dimension” for short. In a CFT, one can define left and right *conformal* dimensions, usually denoted

(h_A, \bar{h}_A) , so that $x_A = h_A + \bar{h}_A$. Thus in a general CFT

$$\langle A(z, \bar{z})B(w, \bar{w}) \rangle \propto \frac{1}{(z-w)^{h_A}(\bar{z}-\bar{w})^{\bar{h}_A}} . \quad (10)$$

Under the scale transformation $\vec{r} \rightarrow \vec{r}' = \lambda \vec{r}$, any operator with dimension must transform as well. For example, because $\partial_z = \frac{\partial z'}{\partial z} \partial_{z'} = \lambda \partial_{z'}$, the operator $J = \partial_z \phi$ transforms as $J(z) \rightarrow \lambda J(z')$. Thus for arbitrary conformal transformations $z \rightarrow z' = f(z)$ and $\bar{z} \rightarrow \bar{z}' = \bar{f}(\bar{z})$, the simplest way for operator A to transform under conformal transformations is as

$$A(z, \bar{z}) \rightarrow f'(z)^{h_A} \bar{f}'(\bar{z})^{\bar{h}_A} A(z', \bar{z}') . \quad (11)$$

As a side remark, conformal invariance in any dimension also requires that the three-point function of any three operators be of the form

$$\langle A(\vec{r}_1)B(\vec{r}_2)\mathcal{O}(\vec{r}_3) \rangle = \frac{C_{AB\mathcal{O}}}{r_{12}^{x_A+x_B-x_{\mathcal{O}}} r_{23}^{x_B+x_{\mathcal{O}}-x_A} r_{13}^{x_A+x_{\mathcal{O}}-x_B}}$$

where $r_{ij} \equiv |r_i - r_j|$. Thus each three-point function is determined uniquely from the dimensions of the operators, up to a single number, known for reasons I will explain below as the operator product coefficient.

Going back to the bosonic case, the operators J and \bar{J} have conformal dimensions $(1, 0)$ and $(0, 1)$ respectively, because ϕ is dimensionless, while the derivative has dimension 1 (note dimensions are in powers of energy). The operators $e^{\pm ib\phi}$ have conformal dimensions $(b^2/4g, b^2/4g)$, so that $x_b = b^2/2g$. An arbitrary vertex operator can be written as

$$V_{\alpha\beta}(z, \bar{z}) \equiv a^{-(\alpha^2+\beta^2)/4g} \exp(i\alpha\varphi(z) + i\beta\bar{\varphi}(\bar{z})) .$$

Its two-point function is

$$\langle V_{\alpha\beta}(z, \bar{z})V_{-\alpha, -\beta}(w, \bar{w}) \rangle = \frac{1}{(z-w)^{\alpha^2/2g}(\bar{z}-\bar{w})^{\beta^2/2g}} . \quad (12)$$

so that it has dimensions $(\alpha^2/4g, \beta^2/4g)$. For these operators to be local, they must return to the same value when they are exchanged, i.e. the position of one is rotated by π around the other. This means that locality requires that the ‘‘conformal spin’’ $h - \bar{h}$ be an integer. It frequently is useful, however, to consider more general operators. For example, a fermionic operator has conformal spin a half integer, so that one obtains the familiar fermionic minus sign when the two are exchanged. It is only slightly more work to use the Green function (7) to compute arbitrary correlators of the vertex operators. For each pair of operators, there is a singular contribution analogous to (12), giving

$$\langle V_{\alpha_1\beta_1}(z_1, \bar{z}_1)V_{\alpha_2\beta_2}(z_2, \bar{z}_2) \dots V_{\alpha_N\beta_N}(z_N, \bar{z}_N) \rangle = \prod_{i < j} (z_i - z_j)^{\alpha_i\alpha_j/2g} (\bar{z}_i - \bar{z}_j)^{\beta_i\beta_j/2g} \quad (13)$$

when $\sum_i \alpha_i = \sum_i \beta_i = 0$; if these conditions do not hold, the correlator vanishes. Notice that this reduces to (12) when setting $\alpha_2 = -\alpha_1$ and $\beta_2 = -\beta_1$.

2.3 The Coulomb gas and the operator product expansion

Using this free-boson field theory as an effective field theory for statistical-mechanical problems is often called the ‘‘Coulomb-gas approach’’. This is because the two-point function for ϕ is the Green function for the electrostatic potential, and the exponential operators create electric and magnetic charges. An electric charge is of the form $V_{\alpha,\alpha}$, whereas a magnetic one is $V_{\alpha,-\alpha}$; in both cases the conformal spin is zero.

This bosonic field theory contains in it many theories; a particular theory requires specifying the allowed values of the exponents. As pioneered by Kadanoff and collaborators, in the Coulomb-gas approach one uses the underlying physical model of interest to identify which vertex operators are allowed, and to give them a physical interpretation.

This approach is nicely illustrated in the XY model, where it lets us understand the Kosterlitz-Thouless transition in CFT language. The lattice variable θ turns into the field ϕ in the continuum limit. A key piece of physics one gets from the lattice model is that θ is identical to $\theta + 2\pi$, so ϕ is only meaningful mod 2π . This means that the physically meaningful functions of ϕ are exponentials of ϕ with integer coefficients. This means that the ‘‘electric’’ operators are $V_{e,e}$, where e is an integer. At sufficiently large values of J/T , the fact that these decay algebraically instead of exponentially means that while ultimately the system is disordered, the correlations are quite long ranged.

To understand what the magnetic operators are, it is useful to introduce a deep concept essential to the study of conformal field theory (and for that matter, useful for field theory in general). This is called the *operator product expansion* (OPE). Like many brilliant ideas, it is simple to explain, but the consequences are profound. The idea is that one can treat the product of two operators $A(z)B(w)$ brought close together in terms of an expansion of operators at a single point. In an equation,

$$A(z, \bar{z})B(w, \bar{w}) \sim \sum_n C_n(z-w, \bar{z}-\bar{w}) \mathcal{O}_n((z+w)/2, (\bar{z}+\bar{w})/2), \quad (14)$$

where the $C_n(z-w)$ are possibly singular coefficients depending only on the difference $z-w$, and the $\mathcal{O}_n(w)$ are a complete set of local operators. It is arbitrary whether on the right-hand-side one expands around the point z or w or any point nearby; this will serve only to modify the coefficients. The important point is that as z gets close to w , most of the terms in this expansion will vanish; one needs only to keep the most singular terms. Typically, very few of them need to be kept. Moreover, in a CFT, dimensional analysis determines the form of the singularity: one must have

$$C_n \propto (z-w)^{h_n-h_A-h_B} (\bar{z}-\bar{w})^{\bar{h}_n-\bar{h}_A-\bar{h}_B} .$$

It is easy to work out the leading term in the OPE of electric and magnetic operators in the free-boson theory. Looking at the field-theory formulation makes it obvious one simply adds the exponents to get the operator product, and using the correlator (13) gives the coefficient to be

$$V_{\alpha\beta}(z, \bar{z})V_{\alpha'\beta'}(w, \bar{w}) = (z-w)^{\alpha\alpha'/2g} (\bar{z}-\bar{w})^{\beta\beta'/2g} V_{\alpha+\alpha', \beta+\beta'} (1 + \dots) \quad (15)$$

The ... are derivatives of φ and $\bar{\varphi}$ multiplied by positive integer powers of $(z - w)$ and $(\bar{z} - \bar{w})$, and so can be neglected as z gets near to w . Operators are mutually local if the OPE is invariant under 2π rotations, i.e. sending $(z - w) \rightarrow e^{2\pi i}(z - w)$ and $(\bar{z} - \bar{w}) \rightarrow e^{-2\pi i}(\bar{z} - \bar{w})$. To obtain a mutually local set of vertex operators in the free boson theory, (15) means that

$$\alpha\alpha' - \beta\beta' = 2gn$$

for some integer n . For electric charges $\alpha = \beta = e$, this means that the magnetic charges must be $\alpha' = -\beta' = gm/2e$ for some integer m . This is Dirac quantization of monopole charge in two dimensions!

By using the XY model, we can now understand the meaning of the magnetic operators. Recall that because of the periodicity of θ , electric operators are $V_{e,e}$ for integer e . This means that magnetic operators are $V_{mg,-mg}$ for some integer m . The physical meaning of the magnetic operators is now easy to see by considering the OPE of the simplest electric and magnetic operators. The leading singularity is

$$V_{1,1}(z, \bar{z})V_{g,-g}(0, 0) \propto \sqrt{\frac{z}{\bar{z}}}.$$

Now rotate the location of the electric charge at z around the magnetic charge at the origin, i.e. send $z \rightarrow e^{i\gamma}z$ (and $\bar{z} \rightarrow e^{-i\gamma}\bar{z}$) and rotate γ from 0 to 2π . The phase of the operator product then rotates in the same way from 0 to 2π . Since $V_{1,1} = e^{i\phi(z, \bar{z})}$, what this means is that the value of ϕ shifts by 2π around a magnetic charge with $m = 1$. A magnetic charge is a vortex! It creates a cut in the field ϕ . This is allowed, because the underlying microscopic variable θ is defined only mod 2π .

Many other examples can be found of lattice models described by the Coulomb gas in the continuum limit. These are described beautifully in the review article by Nienhuis. In most (all?) of these, the field can be taken to be periodic. (In the six-vertex model, the simplest definition is not periodic, but rather integer instead. A periodic field is then found by Fourier transformation in field space, the opposite of what we have done with electric and magnetic charges!) Thus the vertex operators allowed depend on what the periodicity of the field is. We can always normalize the field (i.e. define the stiffness) so that the periodicity is 2π , as in the example above. String theorists use a different convention – they typically set $g = 2$, and then let $\phi \sim \phi + 2\pi R$ and call R the “compactification radius” (in string theory the bosonic field is a space-time coordinate of the string, with our 2d space the string worldsheet). In our convention $R = 1$, and a set local operators are $V_{e+mg, e-mg}$ with integer e and m . They have conformal dimensions

$$(h_{em}, \bar{h}_{em}) = \left(\frac{(e + mg)^2}{4g}, \frac{(e - mg)^2}{4g} \right), \quad (16)$$

so that each has scaling dimension $x_{em} = \frac{e^2}{2g} + \frac{gm^2}{2}$ and conformal spin em . To compare these formulas to those in Ginsparg’s lecture notes, set $g = 2R^2$; the electric/magnetic duality $g \rightarrow 1/g$ corresponds to the $R \rightarrow 1/(2R)$ “S-duality” familiar to string theorists.

This result is spectacular. All the scaling dimensions (up to an integer shift coming from derivatives of the bosons) are given by this formula. The only unknown in the XY model is how the stiffness g is related to the bare lattice coupling J/T ; in the XXZ chain it is related to the anisotropy Δ . If this is fixed by knowing the dimension of one operator, then the rest are determined uniquely! In fact, in the XXZ model, the integrability of the six-vertex (derived by Lieb) extended to the eight-vertex model by Baxter allows one to compute one of the exponents exactly. One finds

$$\Delta = \cos(\pi g) .$$

with $0 < g \leq 1$ and $-1 \leq \Delta < 1$. Note in the ferromagnetic case $\Delta = 1$ is the nonsensical limit $g = 0$, and so is no longer described by this Lorentz-invariant field theory; as is well known the dispersion relation is $E \propto k^2$.

Notice that there is no real value of g corresponding to $\Delta < -1$. The reason is that there at $\Delta = -1$ there occurs a transition to a \mathbb{Z}_2 ordered phase with a gap, so the free massless boson description no longer remains valid (we will see momentarily that for Δ just a little bit smaller than -1 , the scaling limit is given by the sine-Gordon field theory). This transition was known in the XXZ chain in the '60s I think (remind me to ask Elliott Lieb when he first knew about it!) as well as in the 1d classical Ising model with long-range interactions, but was first understood in field-theory language by Kosterlitz and Thouless from their studies of the XY model. The existence of the KT transition in the XY model easily follows combining the from the central idea of effective field theory:

Allow in the action any term allowed by the symmetries. For determining the phase transitions, you can neglect any *irrelevant* terms.

Operators in field theory in two dimensions are irrelevant if their scaling dimension is greater than 2. In the free-boson field theory, operators have dimension depending on the stiffness g , which when describing the continuum limit of the XY model depends on the coupling J/T such that large J/T correspond to large g . Thus if as J/T is varied, an operator turns from irrelevant to relevant, there is a phase transition.

In the XY model, there are two types of potentially relevant operators: electric and magnetic charges. (Operators $\partial_z \varphi$ and $\partial_{\bar{z}} \bar{\varphi}$ have dimension 1, independent of g , and any allowed terms including them allowed by symmetries save the original action are irrelevant.) Symmetry, however, forbids adding electric charges to the action. The reason is that this operators $\cos(e\phi)$ are not invariant under shifts of ϕ , a symmetry of the action coming from the original XY symmetry under shifts of θ . Thus these operators cannot cause a phase transition. However, the magnetic operators are allowed: they create vortices, and these are allowed in the XY model because the underlying lattice makes them well defined. These operators have scaling dimension

$$x_m = \frac{gm^2}{2} .$$

When g is large (large J/T), these are all irrelevant. However, when $g = 4$, the operator $V_{g,-g}$ with $m = 1$ becomes relevant. This is the source of the KT transition! The precise relation of J/T to g can only be computed perturbatively (in the vortex fugacity). The physics is clear, however. At large J/T (low temperatures), vortex/antivortex pairs are tightly bound, i.e. the partition function is dominated by flat configurations in ϕ , with small local fluctuations. As the temperature is increased (or the coupling lowered), these become looser-bound, i.e. the fluctuations around the flat configuration gets bigger. At the KT transition, the vortex-antivortex pairs unbind.

There is an analogous transition in the XXZ chain. However, here it is the magnetic operators that are forbidden due to the $U(1)$ symmetry, while the electrical one causes a KT transition at $\Delta = 1$. The model is critical for $|\Delta| \leq 1$ but at the antiferromagnetic Heisenberg point $\Delta = -1$, a KT transition to a \mathbb{Z}_2 ordered phase occurs (although here by an electric operator instead of a magnetic one).

There is much more that can be done with a free boson. Since one knows the entire spectrum for any g , one can compute the explicit partition functions on a cylinder or torus. These are exceptionally beautiful mathematically, because they can be written in terms of Jacobi theta functions that satisfy many remarkable identities (this is the branch number theory Ramanujan became famous doing, an area of math I didn't even mention in the introduction!). They are exceptionally useful physically as well. To give a completely biased example, in a paper posted last week by Stephan et al, it was shown how to rewrite the entanglement entropy of the quantum dimer model in terms of these partition functions.

3 The ubiquitous c

I stole the title of this section from a talk given by John Cardy, available at arXiv:1008.2331. In this talk he explains how the “conformal anomaly” or “central charge” c is not just universal, but appears in a remarkable number of different physical quantities. I will explain what it is, and how it is indeed fundamental to conformal field theory.

3.1 The stress tensor

The stress tensor $T^{\mu\nu}$ is defined as the response of the action to doing a general coordinate transformation (or “strain”) $r^\mu \rightarrow r^\mu + \alpha^\mu(r)$:

$$dS = -\frac{1}{2\pi} \int T_{\mu\nu} \partial^\mu \alpha^\nu(r) d^2 r . \quad (17)$$

This is the Euclidean version of the energy-momentum tensor of 1+1 dimensional space-time. If you like metrics, this definition is equivalent to

$$T^{\mu\nu} = \frac{\partial S}{\partial g_{\mu\nu}} .$$

Any time we know the action explicitly, we can of course express T and \bar{T} in terms of the fields. However, one does not need an explicit expression in terms of some fields for it to do CFT. In fact, the logic is essentially the reverse. The stress-energy tensor in a CFT satisfies a huge array of constraint, and the point of CFT is to understand how all of these fit together. One can completely characterize all the states and the corresponding correlators by knowing some of these properties (such as the c). In other words, one does not need to have an explicit expression for the action to understand what happens when one varies it!

In any theory, the stress tensor obeys the conservation law $\partial^\mu T_{\mu\nu} = 0$. In any two-dimensional theory these two equations read in complex coordinates

$$\partial_{\bar{z}}T + \partial_z\Theta = 0; \quad \partial_zT + \partial_{\bar{z}}\Theta = 0. \quad (18)$$

where $T \equiv T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$ and $\Theta = T_{z\bar{z}} = T_{\bar{z}z}$. Doing a scale transformation gives a variation of the action proportional to Θ , the trace of the stress tensor. Since at a critical point, the action is invariant under scale transformations, this means that $\Theta = 0$ for a CFT, and that T depends only on z , and \bar{T} on \bar{z} .

By construction, in CFT the operator T has conformal dimensions $(2, 0)$ while \bar{T} has dimensions $(0, 2)$. This means that the two-point function

$$\langle T(z)T(0) \rangle = \frac{c/2}{z^4} \quad (19)$$

Note that the normalization of T is fixed by its definition, so for a given theory, c is a uniquely defined quantity. The coefficient c is called the *central charge*, or *conformal anomaly*. Just knowing this number gives a huge amount of information about the theory, as I will explain in the following.

Because the stress-energy tensor is the generator of all conformal transformations, it plays a central role in CFT. For example, one can determine the scaling dimension of operators just by knowing the OPE of the operator with the energy momentum tensor. Although the following may seem a little abstract, I will explain later on how this fact gives a very effective way of determining scaling dimensions numerically. It follows by considering the coordinate transformation $z \rightarrow z + \alpha z$ for $|r| < \epsilon_1$, r^μ invariant for $|r| > \epsilon_2$, while for $\epsilon_1 \leq |r| \leq \epsilon_2$ α^μ is any smooth function interpolating between the two. Using this in (17) gives a vanishing contribution for $|r| < \epsilon_1$ because it is a conformal transformation in this region (recall that $\partial^z = g^{z\bar{z}}\partial_{\bar{z}}$). Obviously it vanishes for $|r| > \epsilon_2$. In between, one integrates by parts. The bulk term vanishes because $\partial^\mu T_{\mu\nu} = 0$, and the surface term at $|r| = \epsilon_2$ to give

$$\delta S = -\frac{1}{2\pi} \int_{|r|=\epsilon_1} T_{\mu\nu} \alpha^\nu dS^\mu .$$

where S^μ is the normal vector to the surface. Rewriting with the specific form of α^μ at ϵ_1 gives

$$\delta S = \frac{\alpha}{2\pi i} \int T(z)zdz - \frac{\bar{\alpha}}{2\pi i} \int \bar{T}(\bar{z})\bar{z}d\bar{z} .$$

Now look at some correlation function of an operator $\mathcal{O}(0,0)$ of dimensions (h, \bar{h}) at the origin with other operators outside the disc of radius ϵ_1 . We're doing just a coordinate transformation, so the correlators with the original and transformed actions are related by

$$\langle \mathcal{O}(0,0) \dots \rangle_S = (1 + \alpha)^h (1 + \bar{\alpha})^{\bar{h}} \langle \mathcal{O}(0,0) \dots \rangle_{S+\delta S} .$$

Using the explicit δS gives for small α

$$\int dz z T(z) \mathcal{O}(0,0) = h \mathcal{O}(0,0)$$

because the foregoing applies for all correlators. Thus the OPE of T and \mathcal{O} must include a term

$$T(z) \mathcal{O}(0,0) = \dots + \frac{h}{z^2} \mathcal{O}(0,0) + \dots .$$

Repeating the above argument for a translation gives a similar term, yielding

$$T(z) \mathcal{O}(0,0) = \dots + \frac{h}{z^2} \mathcal{O}(0,0) + \frac{1}{z} \partial_z \mathcal{O}(0,0) + \dots . \quad (20)$$

Operators where these are the most singular terms in the OPE are called *primary fields*. This is not true for T , because of (19). The OPE of T with itself is

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2}{z^2} T(0) + \frac{1}{z} T'(0) + \dots \quad (21)$$

where the remaining terms are not singular.

One can use this OPE show that T does not transform under conformal transformations as simply as (4). This is why c is called the conformal ‘‘anomaly’’. There are fancy ways to understand this; recall that T generates conformal transformations, so the TT OPE is related to how a conformal transformation acts on a conformal generator. The answer is that under a conformal transformation,

$$T(z) = T(f(z))(f'(z))^2 - \frac{c}{12} \{f, z\}, \quad (22)$$

where $\{f, z\} \equiv (f''' f' - \frac{3}{2} f''^2) / f'^2$ is called the Schwarzian derivative.

The free boson gives some intuition where this comes from. Like all anomalies, it has to do with the short distance behavior. Naively applying the definition of T to the free-boson action gives

$$T = -g(\partial_z \varphi)^2, \quad \bar{T} = -g(\partial_{\bar{z}} \bar{\varphi})^2 . \quad (23)$$

This isn't quite right because when the operators $J = \partial_z \varphi$ are brought to the same point to make T , they have a singularity. To define something finite, one can *point split*, i.e., let

$$T \equiv J(z + \delta/2) J(z - \delta/2) - \frac{1}{2\delta^2} .$$

Basically, this means just ignore the singularity. Doing this gives $c = 1$:

$$\langle T(z)T(0) \rangle = 2 \left(\frac{1}{2g} \right)^2 g^2 \frac{1}{z^4} = \frac{1/2}{z^4} ,$$

where the extra factor in 2 comes from the two contractions in Wick's theorem. Thus $c = 1$ for the free boson! But the point splitting explains where the Schwartzian term comes from. The subtracted term is not invariant under conformal transformations. Rather than paraphrase or plagiarize any further, I will just quote Cardy's "ubiquitous c lecture:

This is a classic example of the appearance of an anomaly in quantum field theory, when a symmetry (in this case, conformal symmetry) is not respected by the necessary regularization procedure. In fact the form of the last term on the rhs, although complicated, is completely fixed by the requirement that it hold under a general iterated sequence of conformal mappings. The only arbitrariness is in the coefficient c , which is thereby a fixed parameter characterizing the particular CFT or universality class.

3.2 The central charge and universal energy coefficients

To make this look more like the kinds of systems we deal with in condensed matter, it's better to work on cylinder, instead of the plane. One of the many beautiful things about CFT is that we have the tools to do this. A infinitely-long cylinder of circumference β is obtained from the punctured plane by the conformal transformation $v + iw = \frac{\beta}{2\pi} \ln z$, where $-\infty < v < \infty$ and $0 \leq w < \beta$. One of the nice things about working in the 2d formulation is that one is free to choose which direction is space and which is (Euclidean) time. The most natural thing to do here is to think of w as the spatial coordinate around the cylinder (thus space here has periodic boundary conditions), while the coordinate v down the cylinder is Euclidean time. If instead you wish to make w time and v space, then you are faced with understanding what the meaning of periodic time is. This was answered Feynman: if you wish to equilibrium quantum statistical mechanics at inverse temperature β , then notice that the partition function is

$$Z = \text{tr } e^{-\beta \hat{H}} ,$$

where the hat means you can now think of this as an operator acting on Hilbert space. (If you want to stay classical, then think of it as acting on the boundary conditions at the end of the cylinder.) This partition function is exactly the path integral picture in Euclidean time with period β , i.e. exactly what we compute on the cylinder of circumference β .

The Hamiltonian of course is the generator of time translations, and so is related to the energy-momentum tensor. The stress tensor we have discussed is simply the Euclidean version of this tensor. Let τ be the Euclidean time coordinate, which can be

v or w depending on which way you want. The Hamiltonian is then the response to the coordinate transformation $\tau \rightarrow \tau' = \tau + \alpha\Theta(\tau - \tau_2)$, where Θ is the usual step function (you can smooth it slightly if you're worried). This gives

$$\widehat{H} = \frac{1}{2\pi} \int \widehat{T}_{\tau\tau} d\tau \quad (24)$$

The fact that there is a funny term in the OPE of T with itself has a remarkable consequence for the energy. As seen in (22), T does not just rescale under conformal transformations, but shifts as well. Thus even though the expectation value of $\langle T \rangle$ vanishes on the plane (since there is no dimensional part), when we transform to the cylinder, it can pick up an extra piece. In the v space, w time picture (i.e. where the partition function describes a 1+1d quantum system in equilibrium at non-zero temperature), this means that the energy density u at finite temperature is not zero. The energy density is then

$$\langle u \rangle = \frac{1}{2\pi} \langle T_{\tau\tau} \rangle = \frac{1}{2\pi} (\langle T \rangle_{\text{cylinder}} + \langle \overline{T} \rangle_{\text{cylinder}}); \quad (25)$$

note that indeed the energy density is of dimension 2 like T and \overline{T} . Plugging in the explicit $f = (L/2\pi) \ln z$ gives

$$u = \frac{\pi c}{6\beta^2} \quad (26)$$

in units where $\hbar = 1$ and $\beta = 1/(k_B T)$ (with T in the latter the temperature, not the energy-momentum tensor). The Stefan-Boltzmann law in 2d has a universal coefficient!

Going to the v time, w space picture gives a different interpretation of this result. Then it is the universal part of the *Casimir energy* $E_0(\beta)$, i.e. the energy of the ground state in a finite-size system. The point is that for a cylinder of circumference β and some very long length L , the partition function is dominated by this ground-state energy:

$$Z \sim e^{-LE_0(\beta)}$$

plus corrections exponentially small in L (even though a CFT is gapless, putting it in finite size still gives a finite difference between the excited-state energies and E_0). On general grounds, for β

$$E_0(\beta) = e_1\beta + e_2 + e_3/\beta + \dots \quad (27)$$

The first two pieces are the non-universal ‘‘bulk’’ and the boundary terms respectively (the latter usually vanishes for periodic boundary conditions). These must be non-universal, since the coefficients are dimensional. The third coefficient, however, is dimensionless, so it can be universal. A rerun of the energy-density calculation adapted to this picture gives

$$e_3 = \begin{cases} -\frac{\pi c}{6} & \text{periodic boundary conditions} \\ -\frac{\pi c}{24} & \text{on a strip} \end{cases} \quad (28)$$

This relation is exceptionally useful to test which CFT a given Hamiltonian may correspond to (or for that matter, if even is a CFT). Just put your Hamiltonian on the computer for varying sizes, and compute the ground state energy. Then fit it to (27) and extract c !

In fact, one can go further numerically than just the central charge. The dimensions of the operators are also determined by the finite-size spectrum. The dimension x_A describes the behavior of the system under scale transformations. But under the conformal transformation from the plane to the cylinder, scale transformations around the origin turn into translations. Thus the Hamiltonian eigenvalues (i.e. the finite-size energies) are given by the same kind of formula: each level obeys

$$E = E_0 + \frac{2\pi}{\beta}(h_A + \bar{h}_A)$$

for some operator A .