Problem sheet - HT tutorial 1

Self-assessed questions

Q1. Some basic manipulations of angular momentum operators

The angular momentum operators L_x, L_y and L_z are defined by $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ where $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$; for example,

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

(a) Write down the corresponding expressions for L_x and L_y. Check that cyclic interchange of x, y, z also cyclicly interchanges L_x, L_y and L_z

(b) Show that, for any differentiable function $f(x, y, z)$,

$$(L_x L_y - L_y L_x)f(x, y, z) = i\hbar L_z f(x, y, z).$$

Since the above relation is true for any f, it is usually written as an operator equation

$$[L_x, L_y] = L_x L_y - L_y L_x = i\hbar L_z.$$

Use cyclic interchange to infer that also

$$[L_y, L_z] = i\hbar L_x \quad \text{and} \quad [L_z, L_x] = i\hbar L_y.$$

(c) Changing to spherical polar coordinates show that

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

[Hint: it is easier to show that the RHS expression is equal to the LHS one, rather than the other way round; use the chain rule

$$\frac{\partial f}{\partial\phi} = \frac{\partial x}{\partial\phi}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial f}{\partial z}$$

for any f.] Expressions for L_x and L_y can also be worked out but are less pretty.

(d) Would it make any difference if we defined $\boldsymbol{L} = -\boldsymbol{p} \times \boldsymbol{r}$?

[JMR QM HT Q1.2]

Q2. Raising and lowering operators for angular momentum

In a way similar to the harmonic oscillator we can take an abstract algebraic approach to angular momentum without recourse to differential operators in Q1 above. As a first step towards this approach look up the definition of the operators $L_\pm$ in terms of L_x and L_y to prove the following relations

$$\begin{aligned}[L_+, L_-] &= 2\hbar L_z, \\ [L_z, L_\pm] &= \pm\hbar L_\pm.\end{aligned}$$

The operators $L_\pm$ are called raising and lower operators due to their action on an arbitrary angular momentum eigenstate labelled as $|l, m\rangle$. Write down their action on this state up to a constant $C_+(l, m)$ and $C_-(l, m)$, respectively.

Q3. Simple eigenfunctions of angular momentum

Consider the three functions $\cos\theta$, $\sin\theta\, e^{i\phi}$ and $\sin\theta\, e^{-i\phi}$.

(a) Verify by brute force that each of them are eigenfunctions of $\boldsymbol{L}^2$ and of L_z and find the corresponding eigenvalues.

(b) Find normalisation constants N such that

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} N^2|\psi(\theta,\phi)|^2\sin\theta\, d\theta\, d\phi = 1$$

for each of them. (This means that they are normalised over the full 4π solid angle subtended at the centre of the unit sphere). Show also that they are orthogonal to each other.

(c) The normalised eigenfunctions of $\boldsymbol{L}^2$ and L_z with eigenvalues $\ell(\ell+1)\hbar^2$ and $m\hbar$ respectively are called $Y_{\ell m}(\theta,\phi)$. So

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{11}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}, \quad Y_{1-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}.$$

Using $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that these functions can also be written as

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\frac{z}{r}, \quad Y_{11} = \sqrt{\frac{3}{8\pi}}\frac{(x+iy)}{r}, \quad Y_{1-1} = \sqrt{\frac{3}{8\pi}}\frac{(x-iy)}{r}.$$

(d) Sketch $|Y_{10}|^2, |Y_{11}|^2$ and $|Y_{1-1}|^2$ (a cross section in the $x-z$ plane will do - why?)

N.B. It pays to memorize these three functions, which all have $\ell = 1$, and $m = 0$, $m = 1$, $m = -1$ respectively.

[JMR QM HT Q1.3]

Q4. Measuring angular momentum

A particle is in the state with wavefunction

$$\psi = \frac{1}{\sqrt{2}}(Y_{11}(\theta,\phi) + Y_{1-1}(\theta,\phi))$$

(a) What value is obtained if $\boldsymbol{L}^2$ is measured?

(b) Does the particle have a definite value of L_z?

(c) What are the probabilities of getting results $\hbar, -\hbar$ and 0 for L_z? Are any other L_z results possible?

(d) Calculate

$$\langle\psi \mid L_z \mid \psi\rangle = \iint_{\text{unit sphere}} \psi^*(\theta,\phi)L_z\psi(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

and explain the answer.

(e) Suppose that when L_z is measured the result $\hbar$ is obtained. What is the wavefunction afterwards?

[JMR QM HT Q1.4]

Q5. Total angular momentum

Consider a system with a state of fixed total angular momentum $l = 2$. What are the eigenvalues of the following operators

(a) L_z,

(b) $\frac{3}{5}L_x - \frac{4}{5}L_y$,

(c) $2L_x - 6L_y + 3L_z$.

As a hint for (b) and (c) you may want to consider what the eigenvalues are for the angular momentum along an arbitrary axis defined by a unit vector $\vec{n}$?

Main questions

Q1. The central potential and angular momentum

The 3-D time-independent Schrödinger equation for a particle of mass m in a spherically symmetric potential (i.e. one that only depends on r, not on θ and ϕ) is

$$\left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(r)\right)\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}).$$

(a) By writing $\boldsymbol{\nabla}^2$ in terms of r, θ, ϕ show that this can be written as

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} + \frac{\boldsymbol{L}^2}{2mr^2}\psi + V\psi = E\psi$$

where

$$\boldsymbol{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right).$$

(b) Writing $\psi(\boldsymbol{r}) = R(r)Y(\theta,\phi)$ show that Y must satisfy $\boldsymbol{L}^2Y = cY$ where c is a constant, and where

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}r^2\frac{dR}{dr} + \left(V + \frac{c}{2mr^2}\right)R = ER.$$

(c) Writing $R = f(r)/r$, show that $f(r)$ satisfies the equation

$$-\frac{\hbar^2}{2m}\frac{d^2f}{dr^2} + \left(V + \frac{c}{2mr^2}\right)f = Ef.$$

(d) †This is of exactly the same form as a 1-D time independent Schrödinger equation, with V replaced by $V + c/2mr^2$. So, can we conclude that the spectrum in the original 3-D problem is the same as for this 1-D problem?

[JMR QM HT Q1.1]

Q2. The total angular momentum operator

The square of the total angular momentum is defined as

$$\boldsymbol{L}^2 = L_x^2 + L_y^2 + L_z^2.$$

(a) Using the result

$$[A, BC] = B[A, C] + [A, B]C,$$

and the commutation rules for the angular momentum operators show that

$$[L_x, \boldsymbol{L}^2] = [L_y, \boldsymbol{L}^2] = [L_z, \boldsymbol{L}^2] = 0.$$

(b) Explain why it is *not* possible, in general, to have states for which more than one component of $\boldsymbol{L}$ has a definite value. Also, explain why it *is* possible to have states with simultaneously definite values of $\boldsymbol{L}^2$ and of one component of $\boldsymbol{L}$. (Often the component chosen by convention is L_z.)

(c) Discuss the special case $\psi(x, y, z) = \psi(r)$. What are the values of L_x, L_y and L_z?

[JMR QM HT Q1.5]

Q3. A particle in a magnetic field

The Hamiltonian for a particle of charge q, mass m_0, in a constant magnetic field $\boldsymbol{B} = (0, 0, B)$ is (approximately)

$$H = -\frac{\hbar^2 \boldsymbol{\nabla}^2}{2m_0} - \frac{qB}{2m_0} L_z.$$

(a) The eigenvalues of which of the following are good quantum numbers: $\boldsymbol{L}^2$, L_x, L_y, L_z?

(b) Show (using the Ehrenfest Theorem result from MT) that the expectation values of L_x, L_y and L_z in a general state $|\psi\rangle$ satisfy the equations

$$\begin{aligned} \frac{d}{dt}\langle\psi|L_x|\psi\rangle &= \frac{qB}{2m_0}\langle\psi|L_y|\psi\rangle \\ \frac{d}{dt}\langle\psi|L_y|\psi\rangle &= -\frac{qB}{2m_0}\langle\psi|L_x|\psi\rangle \\ \frac{d}{dt}\langle\psi|L_z|\psi\rangle &= 0. \end{aligned}$$

Check that these are the same as the three components of the vector equation

$$\frac{d}{dt}\langle\psi|\boldsymbol{L}|\psi\rangle = \frac{q}{2m_0}\langle\psi|\boldsymbol{L}|\psi\rangle \times \boldsymbol{B}.$$

(c) The magnetic moment operator $\boldsymbol{\mu}$ is defined by $\boldsymbol{\mu} = \frac{q}{2m_0}\boldsymbol{L}$. Deduce that

$$\frac{d}{dt}\langle\psi|\boldsymbol{\mu}|\psi\rangle = \frac{q}{2m_0}\langle\psi|\boldsymbol{\mu}|\psi\rangle \times \boldsymbol{B}. \tag{14}$$

(d) You've seen the equation of motion (14) before in the Prelims vector work. Show that (i) the length of the vector $\langle\psi|\boldsymbol{\mu}|\psi\rangle$ is constant in time, and (ii) $\langle\psi|\boldsymbol{\mu}|\psi\rangle.\boldsymbol{B}$ is constant in time. Hence show that $\langle\psi|\boldsymbol{\mu}|\psi\rangle$ precesses around the direction of $\boldsymbol{B}$ with angular frequency $\omega = \frac{qB}{2m_0}$. Calculate ω for the electron in a field of 1 tesla.

[JMR QM HT Q1.7]

Q4. Analysing a Hamiltonian and its implications for angular momentum

A one particle system has the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 - \frac{e^2}{4\pi\epsilon_0 r} - e\mathcal{E}x.$$

(a) What is the physical origin of the last term in H ?

(b) Calculate the commutators $[L_x, x]$, $[L_y, x]$, $[L_z, x]$.

(c) Which of the observables represented by the operators $\boldsymbol{L}^2$, L_x, L_y, L_z are constants of the motion assuming (i) $\mathcal{E} = 0$ (ii) $\mathcal{E} \neq 0$?

[JMR QM HT Q1.8]