Problem sheet - HT tutorial 2

Self-assessed questions

Q1. Describing spin-$\frac{1}{2}$ angular momentum

The Pauli sigma matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Verify that the operators

$$S_x = \frac{\hbar}{2}\sigma_x, \quad S_y = \frac{\hbar}{2}\sigma_y, \quad S_z = \frac{\hbar}{2}\sigma_z$$

satisfy the commutation rules for angular momentum and that

$$\boldsymbol{S}^2 = S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}\hbar^2 I,$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) Find the eigenvalues (which are real), and corresponding normalised eigenvectors (which may be complex), of S_x, S_y, and S_z.

If you've done this right you should find that the matrices therefore have all the properties required to represent an angular momentum-like degree of freedom, with total angular momentum quantum number $j = \frac{1}{2}\hbar$. So just explain how it all fits with the answers you've got.

[JMR QM HT Q3.1]

Q2. Measurement of spin angular momentum

Measuring spin-$\frac{1}{2}$ Let $|\uparrow z\rangle$ and $|\downarrow z\rangle$ denote the eigenstates of $\boldsymbol{S}^2$ and S_z

$$\begin{aligned} \boldsymbol{S}^2|\uparrow z\rangle &= \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2|\uparrow z\rangle, \quad S_z|\uparrow z\rangle = +\tfrac{1}{2}\hbar|\uparrow z\rangle \\ \boldsymbol{S}^2|\downarrow z\rangle &= \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2|\downarrow z\rangle, \quad S_z|\downarrow z\rangle = -\tfrac{1}{2}\hbar|\downarrow z\rangle \end{aligned}$$

and similarly $|\uparrow x\rangle$ and $|\downarrow x\rangle$ denote the eigenstates of $\boldsymbol{S}^2$ and S_x.

(a) express $|\uparrow x\rangle$ and $|\downarrow x\rangle$ as linear combinations of $|\uparrow z\rangle$ and $|\downarrow z\rangle$ and vice versa.

(b) A system is prepared in the state $|\uparrow x\rangle$. Then a measurement of S_z is made. What are the possible outcomes and their probabilities?

(c) Suppose that the outcome is in fact $\frac{1}{2}\hbar$. What is now the state of the system? If we now measure S_x what do we find?

[JMR QM HT Q3.2]

Q3. Measurement of the spin angular momentum along some axis

Consider a spin-$\frac{1}{2}$ particle whose spin state was

$$|\psi\rangle = \frac{1}{\sqrt{5}}\left(2|\uparrow_z\rangle + |\downarrow_z\rangle\right),$$

where $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ are the $\pm\frac{1}{2}\hbar$ eigenstates of S_z, respectively. Suppose we now measure the projection of the particle's angular momentum along an axis defined by the unit vector $\mathbf{n} = \frac{1}{5}(3,4,0)$. What is probability that a result of $-\frac{1}{2}\hbar$ is obtained?

Q4. Analysis of the Stern-Gerlach measurement

Stern-Gerlach experiment A spin-$\frac{1}{2}$ particle of charge e has an intrinsic magnetic dipole moment $\boldsymbol{\mu} = g_s \frac{e}{2m}\boldsymbol{S}$. The combination $\mu_B = \frac{e\hbar}{2m}$ is called the *Bohr magneton* and has value $9.3 \times 10^{-24}\,\mathrm{JT}^{-1}$. The number g_s is not determined by non-relativistic QM; Dirac's relativistic wave equation tells us that $g_s = 2$. In fact, it is not exactly 2 but the deviation can be calculated in Quantum Electrodynamics which tells us that we expect $g_s/2 = 1.001\,159\,652\,153(28)$. The best experimental measurement is $g_s/2 = 1.001\,159\,652\,188(4)$. See V.W. Hughes and T. Kinoshita, Rev. Mod. Phys. **71** S133 (1999).

(a) What is the net *force* on a magnetic dipole in a uniform $\boldsymbol{B}$ field?

(b) Explain why there *is* a net force in a non-uniform $\boldsymbol{B}$ field. (Because it's more familiar you may find it easier to explain why there is a net force on an electric dipole in a non-uniform $\boldsymbol{E}$ field.)

(c) The only contribution to the magnetic moment of a silver atom comes from the valence electron so its magnetic moment behaves just like the electron's. Silver atoms with a kinetic energy of about 3×10^{-20}J each travel a distance of 0.03 m through a non-uniform magnetic field of gradient $2.3 \times 10^3\,\mathrm{Tm}^{-1}$. Explain why the beam splits into two and calculate the separation of the two beams at a distance of 0.25 m downstream from the magnet.

[JMR QM HT Q3.3]

Q5. Expectation values of an electron

An electron in the Coulomb field of a proton is in a state described by the wave function

$$\psi(\mathbf{r}) = \frac{1}{6}\left[4\psi_{100}(\mathbf{r}) + 3\psi_{211}(\mathbf{r}) - \psi_{210}(\mathbf{r}) + \sqrt{10}\psi_{21-1}(\mathbf{r})\right],$$

where $\psi_{nlm}(\mathbf{r})$ are the eigenstates of hydrogen. (a) Confirm that $\psi(\mathbf{r})$ is normalized properly. (b) What is the expectation value of the energy? (c) What is the expectation value of L^2? (d) What is the expectation value of L_z?

Main questions

Q1. The spherical box potential

The free particle stationary state Schrödinger equation in 3-D, separated in polar coordinates $\psi(r,\theta,\phi) = R_\ell(r)Y_{\ell m}(\theta,\phi)$, leads to the radial equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dR_\ell}{dr}\right) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}R_\ell = ER_\ell$$

(a) Verify that

$$R_0(r) = A_0\frac{\sin kr}{r}$$
$$R_1(r) = \frac{A_1}{r}\left(\frac{\sin kr}{kr} - \cos kr\right)$$

where A_0 and A_1 are constants, are solutions of the R_ℓ equation for $\ell = 0$ and $\ell = 1$ respectively.

(b) The spherical box potential is defined by $V(r) = 0$ for $r \leq a$, $V(r) = \infty$ for $r > a$. This will force the boundary condition $R_\ell(r = a) = 0$. Give approximate numerical values, in units of $\frac{\hbar^2}{2ma^2}$, for the lowest *two* energy eigenvalues in each of the cases (a) $\ell = 0$ (b) $\ell = 1$. [$\ell = 0$ is easy; for $\ell = 1$ you need to solve the equation $\tan x = x$.]

(c) ††What is wrong with the solution $R_0(r) = B_0\frac{\cos kr}{r}$ in the $l = 0$ case? N.B. This has nothing to do with normalization of the wavefunction. This function is normalizable in three dimensions.

[JMR QM HT Q2.1]

Q2. The Coulomb potential

(a) What quantum numbers are needed to describe the state of an electron (neglecting spin) in hydrogen? Why is it possible to obtain states with simultaneously well defined eigenvalues of H, $\boldsymbol{L}^2$ and of L_z? Draw a diagram of the spectrum up to and including second excited states and label it carefully with all the quantum numbers of the states.

(b) The ground state wavefunction in hydrogen is proportional to $e^{-r/a}$, where $a = 4\pi\epsilon_0\hbar^2/me^2$. Check that this satisfies the time independent Schrödinger equation and normalise it (N.B. get the volume element right!).

(c) $|\psi(x,y,z)|^2 dxdydz$ is the probability for finding a particle between (x,y,z) and $(x+dx, y+dy, z+dz)$. In r,θ,ϕ this becomes $|\psi(r,\theta,\phi)|^2 r^2 \sin\theta\, d\theta\, d\phi dr$. So the probability of finding the particle between r and $r+dr$, without caring about its angular position, is

$$P(r)\,dr = \left[\int_0^\pi d\theta \int_0^{2\pi} d\phi\, |\psi(r,\theta,\phi)|^2 \sin\theta\right] r^2\,dr.$$

Writing $\psi = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$ show that

$$P(r)\,dr = r^2|R_{n\ell}(r)|^2\,dr.$$

(d) Sketch $P(r)$ for the ground state wavefunction. Find the value of r for which it is a maximum and calculate the average value of r in this state.

[JMR QM HT Q2.3]

Q3. The 3D harmonic potential

A particle moves in 3-D subject to the potential $V = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$.

(a) Writing $\psi(x, y, z) = X(x)Y(y)Z(z)$ show that the time-independent Schrödinger equation can be solved in terms of the solutions for the 1-D oscillator.

(b) Assuming the 1-D results, show that the energy eigenvalues are

$$E(n_x, n_y, n_z) = \hbar\omega\left(\frac{3}{2} + n_x + n_y + n_z\right)$$

where the n's are integers ≥ 0. What is the degeneracy of the levels with (i) $E = 3\hbar\omega/2$, (ii) $E = 5\hbar\omega/2$, and (iii) $E = 7\hbar\omega/2$? Compare the energy spectrum and degeneracies for the Coulomb problem and the 3-D oscillator problem.

(c) †To save you looking them up the ground and first excited states for the 1-D oscillator have wavefunctions

$$\begin{aligned}\phi_0(x) &= A_0 e^{-x^2/2a^2}\\ \phi_1(x) &= A_1 x e^{-x^2/2a^2}.\end{aligned}$$

Write down the wavefunctions for the ground and first excited states of the 3-D oscillator.

(d) †Rewrite the potential in polar coordinates. Are the eigenvalues of $\boldsymbol{L}^2$ and L_z good quantum numbers for this Hamiltonian?

(e) †Using (3) rewrite the 3-D oscillator wavefunctions you found in part (b) in terms of the spherical harmonics and functions of r. What are the $\boldsymbol{L}^2$ eigenvalues of your wavefunctions? What are the L_z eigenvalues of your wavefunctions?

[JMR QM HT Q2.4]

Q4. Mathematical tricks with Pauli matrices

The Pauli matrices we can be formed into a 3D vector operator $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. Given two scalar 3D vectors **a** and **b** prove that

$$(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\sigma \cdot (\mathbf{a} \times \mathbf{b}).$$

Use this result to show that for any vector **a** with magnitude a we have

$$\exp(i\sigma \cdot \mathbf{a}) = \cos(a) + i\left(\sigma \cdot \frac{\mathbf{a}}{a}\right)\sin(a).$$

Q5. Hamiltonians of rotating systems

The Hamiltonian for an axially symmetric rotator is

$$H = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_3},$$

where I_1 and I_3 are moments of inetria. (a) What are the eigenvalues of H? (b) Sketch the spectrum assuming that $I_1 > I_3$. (c) What is the spectrum in the limit that I_1 is much larger than I_3? Suppose we have another system described by the Hamiltonian

$$H = \frac{L^2}{2I} + \alpha L_z,$$

where I is a moment of inertia and α is a coupling constant. (d) What are the eigenstates and associated energy eigenvalues of this system?