Problem sheet - HT vacation

Main questions

Q1. Proving some basic properties of the exchange operator

Fun with exchange operators Let $\mathcal{X}$ be the exchange operator. Prove that:
(a) $\mathcal{X}^2 = 1$
(b) $\mathcal{X}$ has eigenvalues ± 1.
(c) If $\mathcal{X}Q\mathcal{X} = Q$ for operator Q, then $[\mathcal{X}, Q] = 0$.
(d) If $\mathcal{X}Q\mathcal{X} = W$, then $\mathcal{X}Q^2\mathcal{X} = W^2$.
(e) If $[\mathcal{X}, H] = 0$ then non-degenerate eigenstates of H must also be eigenstates of $\mathcal{X}$.
(f) If $[\mathcal{X}, H] = 0$ then $\langle + | H | - \rangle = 0$, where

$$\mathcal{X} |+\rangle = |+\rangle, \qquad \mathcal{X} |-\rangle = -|-\rangle$$

[Hint: if in doubt about operator manipulations, allow the operator product or sum to act on a state, and then if the result doesn't depend on the state, it must be a property of the operators themselves.]

[JW QM HT Q6.1]

Q2. Proving some more properties

More fun, and we will get to a nice result. Prove that
(a) $\mathcal{X}x_1\mathcal{X} = x_2$
(b) $\mathcal{X}p_1\mathcal{X} = p_2$ [Hint: do the differentiation, but think carefully. If in doubt, try the wavefunction $ax_1^2 + bx_2^5$ just to get the hang of things.]
(c) $\mathcal{X}V(x_1, x_2)\mathcal{X} = V(x_2, x_1)$ [Hint: argue that V can always be expanded as a power series in powers of x_1 and x_2, and just treat a general term $x_1^n x_2^m$ from such a series.]
(d) $[\mathcal{X}, K] = 0$ where $K = p_1^2/2m + p_2^2/2m$ is the combined kinetic energy of a pair of identical particles. [use part (b), and 1(d) and 1(c)]
(e) If $V(x_1, x_2) = V(x_2, x_1)$ then $[\mathcal{X}, H] = 0$. [use parts (c) and (d) and 1(c)]
Since potential energy will never depend on particle labelling, (c.f. question 5.5) and using the extension of these results to include spin as well, the conclusion from part (e) is that exchange symmetry is always a constant of the motion. Also, using 1(e), non-degenerate energy eigenstates of identical particles always have definite exchange symmetry (and you may like to show further that the degenerate energy eigenstates can always be combined in such a way as to ensure they have definite exchange symmetry).

[JW QM HT Q6.2]

Q3. One last property

Prove that the exchange operator χ is hermitian.

Q4. Implications of exchange symmetry for two particle ground states

Consider two non-interacting electrons in an infinite potential box. What is the ground state if the two electrons are (a) in the same spin state, or (b) in a different spin state? (c) Which of these state has the lowest energy? (d) What is the first excited state if the electrons have the same spin state? In all cases ensure that any states you construct possess the correct exchange symmetry.

Q5. Ground states for systems with three identical particle systems

Practice on exchange symmetry A one-dimensional harmonic potential well has the form $V(x) = (1/2)m\omega^2x^2$. The lowest three energy eigenstates are $g(x)$, $f(x)$ and $h(x)$ (to keep the notation uncluttered it will be convenient to use g, f, h rather than $\psi_n(x)$). A convenient notation for fermionic spin states is $\uparrow$, $\downarrow$ for $|s = 1/2, m_s = \pm 1/2\rangle$ (spin half) and $\Uparrow$, $\uparrow$, $\downarrow$, $\Downarrow$ for $|s = 3/2, m_s = 3/2 \cdots -3/2\rangle$ (spin 3/2). Suppose three identical particles are in the well. Write down a possible form for the ground state, and hence deduce the ground state energy, when
(a) the particles each have spin zero
(b) the particles each have spin half
(c) the particles each have spin 3/2
(in all cases assume the particles do not interact with one another) [Hint: (b) and (c) require careful thought. Begin by listing some low-lying *single*-particle states having the form of a product "(spatial part) $\otimes$ (spin part)". Then use a determinant to help you write down a state which is antisymmetric w.r.t. exchange of any pair. For (b) and (c) the ground state of the 3-particle system is degenerate, so there is more than one correct answer.]

[JW QM HT Q6.4]

Q6. Examining the difference between bosons, fermions and distinguishable particles

More practice on exchange symmetry Imagine a situation in which there are 3 particles and only 3 states a, b, c available to them. Show that the total number of allowed, distinct configurations for this system is
(a) 27 if the particles are non-identical
(b) 10 if they are bosons
(c) 1 if they are fermions
(d) write down the state in the case of 3 fermions

[JW QM HT Q6.5]