Problem sheet - MT tutorial 1

Self-assessed questions[1]

1. a) A simple vector space

   Show that the space of 2×2 matrices is a linear vector space. What is its dimension? Give a basis for this space.

   [MM Q1.1]

1. b) What happens we specify i) that the trace of the matrix is equal to 1, or ii) that the matrix is hermitian? Is this a vector space? How many real-valued parameters are required to specify the matrix in each case?

2. Another vector space

   What is the dimension of the space of $n \times n$ matrices all of whose components are zero except possibly the diagonal components.

   [MM Q1.3]

3. Vector components of functions

   What are the coordinates of the function $f(t) = 3\sin(t) + 5\cos(t)$ with respect to the basis $\{\sin(t), \cos(t)\}$?

   [MM Q1.6]

4. Practise with complex vectors

   Find real values α and β such that the complex vectors $\mathbf{u} = \alpha \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$ and $\mathbf{v} = \beta \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$ are normalised. What is the value of the scalar product $\mathbf{u}^{\dagger}.\mathbf{v}$? Prove that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Are there any further linearly independent two-dimensional complex vectors? If so, find the necessary vectors to make an orthogonal basis. Express the vector $\begin{pmatrix} 1 \\ i \end{pmatrix}$ as a linear combination of the basis vectors.

   [MM Q1.10]

5. Forming orthogonal sets

   Construct a third vector which is orthogonal to the following pairs and normalise all three vectors

   (a) $(1,2,3)^T, \quad (-1,-1,1)^T$ (b) $(1+i\sqrt{3}, 2, 1-i\sqrt{3})^T, \quad (1,-1,1)^T$
   (c)* $(1-i, 1, 3i)^T, \quad (1+2i, 2, 1)^T$

   [MM Q1.11]

[1]The format of this problem sheet is that the first section of questions **must** be attempted and handed in, but will **not** be marked. Instead you will receive an answer sheet for you to self-assess your effort.

6. Orthogonalising with Gramm-Schmidt

Using the Schmidt procedure construct an orthonormal set of vectors from the following:

$$\vec{x}_1 = (0,0,1,1)^T \ , \ \vec{x}_2 = (1,0,-1,0)^T \ , \ \vec{x}_3 = (1,2,0,2)^T \ , \ \vec{x}_4 = (2,1,1,1)^T .$$

[MM Q1.12]

7. Adjoint vector space

V' is the adjoint space of the vector space V. What objects comprise V'?

[JB QM1 Q1.3]

8. Find a *bra* from a *ket*

Given that $|\psi\rangle = \mathrm{e}^{\mathrm{i}\pi/5}|a\rangle + \mathrm{e}^{\mathrm{i}\pi/4}|b\rangle$, express $\langle\psi|$ as a linear combination of $\langle a|$ and $\langle b|$.

[JB QM1 Q1.4]

9. The importance of $\hbar$

Planck's constant $\hbar$ is equal to 1.0×10^{-34} Js, to two significant figures. A system (e.g. a mechanical watch) has moving parts of size d and mass m, and the movement occurs on a characteristic timescale τ.

(i) Construct a quantity, call it S, having the same dimensions as $\hbar$ from d, m and τ (such quantities are said to have the dimensions of *action*). The rule of thumb is that if S has numerical value much bigger than $\hbar$ then quantum effects are negligible.

(ii) Evaluate your S and compare it to $\hbar$ for
a) The final stage of a turbofan engine (the thing that makes an aircraft fly; typically the final stage turbine rotates at an incredible 30,000 rpm!).
b) The motion of a mechanical wristwatch.
c) A bacteria "swimming".

(iii) Now take d to be a typical atomic size, and m to be the electron mass. Find τ such that your action quantity is equal to $\hbar$ in this case. Defining an average velocity by $v = d/\tau$, calculate the corresponding kinetic energy of the electron, in eV.

[JMR QM MT Q1.5]

10. The de Broglie wavelength

Microscopes using waves with wavelength λ can resolve objects roughly as small as λ but no smaller. Determine the kinetic energy of electrons in an electron microscope needed to resolve

(i) a DNA molecule (10^{-8}m)

(ii) a proton (10^{-15}m).

[Use the de Broglie relation $\lambda = h/p$; in each case consider whether you should use the non-relativistic expression $T = p^2/2m$ for the kinetic energy T, or the relativistic one $T + mc^2 = [m^2c^4 + c^2p^2]^{1/2}$.

[JMR QM MT Q1.6]

11. Energy and momentum

Einstein brilliantly hypothesised the existence of quanta (photons) of light together with the energy-frequency relation $E = hf$. In a theory consistent with special relativity this should be generalized to a relation between the energy-momentum four-vector $(E, \mathbf{p})$ and another four-vector with first component hf. The relation is

$$(E, \mathbf{p}) = (hf, h\mathbf{k}) \tag{1}$$

where $\mathbf{k}$ is the wavevector of the electromagnetic radiation. Note that the magnitude of the momentum part of the generalized energy-frequency relation can be rewritten as

$$\lambda = \frac{h}{|\mathbf{p}|} \tag{2}$$

which motivated de Broglie in his definition of 'matter waves'.

(i) Verify that the relation (1) is consistent with the constant velocity of light being c. [Hint: the relativistic energy-momentum relation for a particle of rest mass m_0 is $E^2 = (m_0c^2)^2 + (pc)^2$. What rest mass must the photon have given that the relativistic expression for the velocity is $v = dE/dp$?]

[JMR QM MT Q1.2]

12. The Compton effect (book work question)

Thus consider a photon of initial momentum $\mathbf{p}$ incident upon an electron at *rest*. After the collision let the photon and electon momenta be $\mathbf{p}'$ and $\mathbf{q}$ respectively.

(i) Write down the equations for conservation of momentum and energy and show that the equation for the conservation of momentum may be written as

$$\mathbf{q}^2 = \left(\frac{hf}{c}\right)^2 + \left(\frac{hf'}{c}\right)^2 - 2\frac{hf}{c}\frac{hf'}{c}\cos(\theta) \tag{3}$$

where f and f' are the frequencies of the initial and scattered photons respectively, and θ is the angle between them.

(ii) Show that the equation for the conservation of energy may be written as

$$q^2c^2 = (hf - hf')^2 + 2m_ec^2(hf - hf') \tag{4}$$

where m_e is the electron mass, and combine with the above equation to show that

$$f' = \frac{f}{1 + (hf/m_ec^2)(1 - \cos\theta)}. \tag{5}$$

From this form we see that for initial photon energies $hf \ll m_ec^2$ the scattering does not, to a good approximation, change the frequency of the radiation. In this regime the scattering is well described by the classical electromagnetic 'Thomson scattering' process. Show that an alternate form of the previous equation is

$$\lambda' - \lambda = \frac{h}{m_ec}(1 - \cos\theta), \tag{6}$$

where λ' and λ are the wavelengths of the initial and final photons. These formulae agree very well with the measurements of the shift in wavelength or frequency of the second component of the scattered radiation. [Roughly speaking the unmodified component is due to scattering by the entire atom; more precisely the electrons that are tightly bound to the atoms in the material of the crystal.] Is the energy of the scattered photon always reduced, increased, or neither? Physically why?

(iii) The quantity $h/(m_ec)$ has the dimensions of a length, and is known as the Compton length of the electron. Evaluate this length. Similarly we may define the Compton wavelength λ_c of any *massive* particle (or bound system) of mass M to be $\lambda_c = h/(Mc)$. Evaluate this length for carbon, the atomic constituent of graphite. Given that Compton used X-rays of wavelength 0.71 Angstrom, what are, at $\theta = 90^o$, the relative shifts $(\lambda' - \lambda)/\lambda$ in the wavelength of the incoming X-rays in the two cases of scattering off a single electron, and scattering off a carbon atom?

[JMR QM MT Q1.3]

Main questions[2]

Q1. Properties of adjoint vectors or *bras*

What properties characterise the bra $\langle a|$ that is associated with the ket $|a\rangle$?

[JB QM1 Q1.5]

Q2. Probabilities from complex probability amplitudes

An electron can be in one of two potential wells that are so close that it can "tunnel" from one to the other. Its state vector can be written

$$|\psi\rangle = a|A\rangle + b|B\rangle,$$

where $|A\rangle$ is the state of being in the first well and $|B\rangle$ is the state of being in the second well and all kets are correctly normalised. What is the probability of finding the particle in the first well given that: (a) $a = \mathrm{i}/2$; (b) $b = \mathrm{e}^{\mathrm{i}\pi}$; (c) $b = \frac{1}{3} + \mathrm{i}/\sqrt{2}$?

[JB QM1 Q1.6]

Q3. Similarly for a more complicated state

An electron can "tunnel" between potential wells that form a chain, so its state vector can be written

$$|\psi\rangle = \sum_{-\infty}^{\infty} a_n|n\rangle,$$

where $|n\rangle$ is the state of being in the nth well, where n increases from left to right. Let

$$a_n = \frac{1}{\sqrt{2}}\left(\frac{-\mathrm{i}}{3}\right)^{|n|/2} \mathrm{e}^{\mathrm{i}n\pi}.$$

a. What is the probability of finding the electron in the nth well?
b. What is the probability of finding the electron in well 0 or anywhere to the right of it?

[JB QM1 Q1.7]

Q4. Continuous vectors for function expansions

Consider the vector space of continuous, complex-valued functions on the interval $[-\pi, \pi]$. Show that

$$\langle \mathbf{f}|\mathbf{g}\rangle = \int_{-\pi}^{\pi} dt\ f^*(t)\ g(t) \qquad (1)$$

defines a scalar product on this space. Are the following functions orthogonal with respect to this scalar product? (a) $\sin(t),\ \cos(t)$ (b) $\exp(int),\ \exp(ikt)$ n,k, integers (c) $t^2,\ t^4$

[MM Q1.13]

[2] These problems must be attempted and handed in. They will be marked.

Q5. A first play with Pauli matrices

Consider the three matrices $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Which of the matrices are symmetric, which are hermitian? By calculating the commutators of these matrices show that they can be written as $[\sigma^a, \sigma^b] = 2i\epsilon_{abc}\sigma^c$, where ϵ_{abc} is the epsilon tensor and on the right hand side the summation convention is employed (i.e. the index c is summed over). Write $\exp(i\alpha\sigma^y)$ (α is a real number) as a 2×2 matrix. What does it represent? Show that $\exp(i\alpha\sigma^y)$ is unitary without writing it explicitly as a 2×2 matrix.

[MM Q1.16]

Q6. Implications of the de Broglie wavelength

(i) Calculate the de Broglie wavelength of a non-interacting gas of particles of mass m at temperature T. Assume that T is low enough so that the non-relativistic form of kinetic energy applies, and recall that the mean linear KE of a particle at temperature T is $3k_BT/2$ where k_B is Boltzmann's constant. Note that this wavelength λ_{TdB} (known as the 'thermal de Broglie wavelength') grows as T decreases.

(ii) Evaluate λ_{TdB} for Helium gas at room temperature (300K). Estimate the mean spacing between He atoms in such a gas at atmospheric pressure. Do you expect quantum effects to be important for this gas?

(iii) Estimate the mean interparticle spacing for liquid Helium near its boiling point (4.2K with density 125 kg/m^3). What is the thermal de Broglie wavelength? Do you expect quantum effects to be important for liquid Helium?

[JMR QM MT Q1.7]