Problem sheet - MT tutorial 2

Self-assessed questions

Q1. Multiplication of rotations

(a) Multiply together the two matrices $A = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ and $B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and interpret the result in terms of 2D rotations.

(b) Evaluate the product $A^T A$ and interpret the result. Evaluate $A^T B$ and interpret the result.

(c) Find the eigenvalues and eigenvectors of A. (Remember that they do not have to be real. Why not?)

[MM Q2.2]

Q2. Non-commuting rotations

Write down the matrix R_1 for a three dimensional rotation through $\pi/4$ about the z-axis and the the matrix R_2 for a rotation through $\pi/4$ about the x-axis. Calculate $Q_1 = R_1 R_2$ and $Q_2 = R_2 R_1$; explain geometrically why they are different.

[MM Q2.7]

Q3. Constructing a real symmetric matrix

Construct a real symmetric matrix whose eigenvalues are 2, 1 and -2, and whose corresponding normalized eigenvectors are $\frac{1}{\sqrt{2}}(0,1,1)^T$, $\frac{1}{\sqrt{2}}(0,1,-1)^T$ and $(1,0,0)^T$.

[MM Q2.5]

Q4. Diagonalising Hermitian matrices

By finding the eigenvectors of the Hermitian matrix $H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}$ construct a unitary matrix U such that $U^\dagger \; H \; U = D$, where D is a real diagonal matrix.

[MM Q2.8]

Q5. Revision on probability

Assuming the birthdays of the population are equally distributed throughout the year, how many people, taken at random, need to gather together in a room to give a greater than 75 percent chance of at least two of them sharing a common birthday?

[SB PS Q2]

Q6. The Poisson distribution

Consider the Poisson probability distribution:

$$P(n|\mu) = \frac{\mu^n e^{-\mu}}{n!}$$

(a) Show that this is a well-behaved probability in that :
$\sum_{n=0}^{\infty} P(n|\mu) = 1$ and $\int_0^{\infty} P(n|\mu) d\mu = 1$.

(b) Show that the mean value of the probability distribution $\langle n \rangle = \mu$.

[SB PS Q3]

Q7. Continuous probability distributions

For a certain continuous variable x, the probability that it has a value lying between x and $x + dx$ is $\rho(x)dx$. The possible values of x range from a to b.

(i) What conditions must $\rho(x)$ satisfy?

(ii) Define the average value $\langle f(x) \rangle$ of a function of x, $f(x)$.

(iii) The *variance* of the distribution is σ^2, defined by $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$. Show that $\sigma^2 = \langle x^2 \rangle - (\langle x \rangle)^2$. (A customary measure of the "spread" of a distribution is σ, which by the above result is equal to $[\langle x^2 \rangle - (\langle x \rangle)^2]^{1/2}$. Frequently this may be written as Δx.)

[JMR QM MT Q2.3]

Main questions

Q1. Rotation matrices

Which of these matrices represents a rotation?

$$\begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1/4 & 3/4 & -\sqrt{3/8} \\ 3/4 & 1/4 & \sqrt{3/8} \\ \sqrt{3/8} & -\sqrt{3/8} & -1/2 \end{pmatrix}$$

Find the angle and axis of the rotation. What does the other matrix represent?

[MM Q2.1]

Q2. Diagonalising Pauli matrices

(a) Find the eigenvalues of the Pauli matrix $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Normalize the two corresponding eigenvectors, $\vec{u}_1$, $\vec{u}_2$, so that $\vec{u}_1^\dagger \cdot \vec{u}_1 = \vec{u}_2^\dagger \cdot \vec{u}_2 = 1$. Check that $\vec{u}_1^\dagger \cdot \vec{u}_2 = 0$. Form the matrix $U = (\,\vec{u}_1 \quad \vec{u}_2\,)$ and verify that $U^\dagger U = I$. Evaluate $U^\dagger\, \sigma^y\, U$. What have you learned from this calculation?

(b) A general 2-component complex vector $\vec{v} = (c_1, c_2)^T$ is expanded as a linear combination of the eigenvectors $\vec{u}_1$ and $\vec{u}_2$ via

$$\vec{v} = \alpha\, \vec{u}_1 + \beta\, \vec{u}_2, \tag{1}$$

where α and β are complex numbers. Determine α and β in terms of c_1 and c_2 in two ways: (i) By equating corresponding components of (1), (ii) By showing that $\alpha = \vec{u}_1^\dagger \cdot \vec{v}$, $\beta = \vec{u}_2^\dagger \cdot \vec{v}$ and evaluating these products.

[MM Q2.3]

Q3. Diagonalisation of non-Hermitian matrices

What are the eigenvalues and eigenvectors of the matrix $\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$? Can σ^+ be diagonalized?

[MM Q2.11]

Q4. Using a diagonalised form of a matrix

Find the eigenvalues and eigenvectors of the matrix $F = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$. Hence, proving the validity of the method you use, find the values of the elements of the matrix F^n where n is a positive integer.

[MM Q2.6]

Q5. The exponential probability distribution

This question is about a continuous probability distribution known as the exponential distribution. Let x be a continuous random variable which can take any value x 0. It is said to be exponentially distributed if it takes values between x and $x + dx$ with probability

$$P(x)dx = Ae^{-x/\lambda}dx$$

where λ and A are constants.

(a) Find the value of A that makes $P(x)$ a well-defined continuous probability distribution so that $\int_0^\infty P(x)dx = 1$

(b) Show that the mean value of the probability distribution is $\langle x \rangle = \int_0^\infty xP(x)dx = \lambda$.

(c) Find the variance of this probability distribution.

Both the exponential distribution and the Poisson distribution are used to describe similar processes, but for the exponential distribution x is the the actual time between successive radioactive decays, successive molecular collisions, or successive horse-kicking incidents (rather than x being simply the number of such events in a specified interval).

[SB PS Q4]

Q6. Infinite-sized vector spaces and commutation relations

By taking the trace of both sides prove that there are no finite dimensional matrix representations of the momentum operator p and the position operator x which satisfy $[p, x] = -i\hbar$. Why does this argument fail if the matrices are infinite-dimensional (as Heisenberg's were)?

[MM Q2.9]

Q7. From vectors to wavefunctions

How is a wave-function $\psi(x)$ written in Dirac's notation? What's the physical significance of the complex number $\psi(x)$ for given x?

[JB QM1 Q1.8]

Q8. Probabilistic interpretation of the wavefunction

Give as many "considerations" as you can why a general wavefunction ψ does **not** have suitable properties to be interpreted as a probability density, but the square modulus $|\psi|^2$ does.

[JMR QM MT Q2.2]

Q9. Einstein-de Broglie waves and the time-dependent Schrödinger equation (TDSE)

A plane wave solution of the time-dependent Schrödinger equation in one dimension is given by

$$\psi(x,t) = Ae^{ikx-i\omega t}. \tag{10}$$

(i) By substituting this solution into the TDSE for the case that $V(x) = 0$ find the relation between ω and k for such waves.

(ii) The "phase velocity" v_p is defined to be ω/k and the "group velocity" v_g is defined to be $d\omega/dk$. Find expressions for v_p and v_g in terms of the "particle velocity" defined by $v = p/m$. Are the results what you expect, and why?

[JMR QM MT Q2.1]

Q10. Separation of variables of the TDSE

Consider solutions in which the x- and t- dependence is *separated* ie we write $\psi(x,t) = \phi(x) \times T(t)$.

(i) Show that

$$\frac{1}{T(t)} i\hbar \frac{dT(t)}{dt} = \frac{1}{\phi(x)} \left(-\frac{\hbar^2}{2m} \frac{d^2\phi(x)}{dx^2} \right) + V(x) \tag{11}$$

and explain why each side of this equation must equal the same constant "A".

(ii) Solve the T-equation for $T(t)$ given that $T(0) = 1$. Show that if A is real, $\mid \psi(x,t) \mid^2$ is independent of t. What is the frequency ω of the wave in terms of A? Assuming the Einstein relation $E = \hbar\omega$, find E in terms of A, and obtain an expression for the average value of x. Such solutions are called *stationary state solutions*: why? Do all wavefunctions have to satisfy the TISE?

(iii) Suppose V depends on t as well as on x: $V(x,t)$. Will such a separation of the x and t variables be possible, in general? Can you invent a $V(x,t)$ for which it would be mathematically possible (even if not physically sensible)?

(iv) Returning to the case $V(x)$, suppose A is in fact complex, $A = E - i\Gamma/2$. Show that the total (integrated over x) probability decays exponentially with a half-life of $(\hbar \ln 2)/\Gamma$. Suggest a physical problem in which such a solution might be useful.

[JMR QM MT Q2.4]