Problem sheet - MT tutorial 4

Self-assessed questions

Q1. Methods of characteristics

A function $u(r,t)$ satisfies the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where c is a constant. By introducing the new function $v(r,t) = ru(r,t)$, and writing $\xi = r + ct, \eta = r - ct$, reduce this equation to

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0.$$

Hence show that the general solution $u(r,t)$ has the form $u(r,t) = \frac{1}{r}[f(r+ct)+g(r-ct)]$, where f and g are arbitrary (twice-differentiable) functions. What does the solution represent physically if t is the time and r is the 3-D polar variable?

[MM Q4.2]

Q2. Modes of a 1D string

The transverse displacement $y(x,t)$ (assumed small) of a string stretched between the points $x = 0$ and $x = a$ satisfies the equation

$$\frac{\partial^2 y}{c^2 \partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

Find the solution satisfying each of the following initial ($t = 0$) conditions:-

(i) $y(x,0) = L\sin(\pi x/a)$, $\frac{\partial y}{\partial t}(x,0) = 0$;

(ii) $y(x,0) = 0$, $\frac{\partial y}{\partial t}(x,0) = V\sin(2\pi x/a)$;

(iii) $y(x,0) = L\sin(\pi x/a)$, $\frac{\partial y}{\partial t}(x,0) = V\sin(2\pi x/a)$;

(iv) $y(x,0) = 2L\sin(\frac{3\pi x}{2a})\cos(\frac{\pi x}{2a})$, $\frac{\partial y}{\partial t}(x,0) = 0$.

Sketch the displacement of the string in part (iv) at time $t = a/2c$.

[MM Q4.3]

Q3. Commuting operators

Show that if there is a complete set of mutual eigenkets of the Hermitian operators A and B, then $[A,B] = 0$.

[JB QM Q1.25]

Q4. Eigenstates of observables

Let Q be the operator of an observable and let $|\psi\rangle$ be the state of our system.

a. What are the physical interpretations of $\langle\psi|Q|\psi\rangle$ and $|\langle q_n|\psi\rangle|^2$, where $|q_n\rangle$ is the n^{th} eigenket of the observable Q?

b. What is the operator $\sum_n |q_n\rangle\langle q_n|$, where the sum is over all eigenkets of Q?

c. If $u_n(x)$ is the wavefunction of the state $|q_n\rangle$, write dow an integral that evaluates to $\langle q_n|\psi\rangle$.

[JB QM Q1.10]

Q5. Taking hermitian conjugates

Given that $(AB)^\dagger = B^\dagger A^\dagger$, show that

$$(ABCD)^\dagger = D^\dagger C^\dagger B^\dagger A^\dagger.$$

[JB QM Q1.26]

Q6. Expanding commutators

Prove that

$$[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC.$$

Explain the similarity with the rule for differentiating a product.

[JB QM Q1.27]

Q7. Diagonalizing Hermitian operators (book work question)

In Dirac's notation if Q is a Hermitian operator then $|v\rangle = Q|u\rangle$ implies $\langle v| = \langle u|Q$. Eigenstates of Q are labelled by their eigenvalues $q_1, \dots$ and satisfy $Q|q_n\rangle = q_n|q_n\rangle$.

Show that $\langle u|Q|v\rangle = (\langle v|Q|u\rangle)^*$ and hence that the eigenvalues of Q must be real.

Show that when $q_n \neq q_m$ then $\langle q_n|q_m\rangle = 0$.

Suppose that $|a\rangle$ and $|b\rangle$ are eigenstates of Q with the *same* eigenvalue; then the previous proof of orthogonality fails. However it is always possible to construct linear combinations of $|a\rangle$ and $|b\rangle$, let's call them $|\tilde{a}\rangle$ and $|\tilde{b}\rangle$, which *are* orthogonal. Do it.

[JMR QM Q4.8]

Main questions

Q1. Mathematica question

Go through the Mathematica notebook and familiarise yourself with basic Mathematica input. The most straight-forward way to produce all of the output is to chose "Evaluate Notebook" from the "Evaluate" menu.

Briefly describe the principle of the time evolution calculation in the last section *Functions of operators → Time evolution → Calculate time evolution using stationary states:* describing in your answer what is happening in each step using Dirac notation.

Q2. Ehrenfest's theorem

Prove that

$$\frac{d}{dt}\langle\psi|A|\psi\rangle = \frac{i}{\hbar}\langle\psi|[H,A]|\psi\rangle \tag{26}$$

where A is any operator (not explicitly depending on t) representing an observable dynamical quantity. You can do this either by writing out $\langle\psi|A|\psi\rangle$ as an integral, and differentiating with respect to time or directly in the Dirac notation by using

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$
$$-i\hbar\frac{\partial}{\partial t}\langle\psi| = \langle\psi|H$$

What is the corresponding result if the operator A does depend explicitly on t?

[JMR QM Q4.4]

Q3. Commuting observables

What does it mean to say that two operators commute? What is the significance of two observables having mutually commuting operators?

Given that the commutator $[P,Q] \neq 0$ for some observables P and Q, does it follow that for all $|\psi\rangle \neq 0$ we have $[P,Q]|\psi\rangle \neq 0$?

[JB QM Q1.11]

Q4. A three level system

A three-state system has a complete orthonormal set of states $|1\rangle, |2\rangle, |3\rangle$. With respect to this basis the operators H and B have matrices

$$H = \hbar\omega\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \qquad B = b\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

where ω and b are real constants.

a. Are H and B Hermitian?
b. Do the eigenvectors of either H or B define a unique basis of eigenvectors?
c. Show that H and B commute. Give a basis of eigenvectors common to H and B.

[JB QM Q1.22]

Q5. A damped thermal wave

The temperature T in a one-dimensional bar whose sides are perfectly insulated obeys the heat flow equation

$$\frac{\partial T(x,t)}{\partial t} = K\frac{\partial^2 T(x,t)}{\partial x^2}$$

where K is a constant. The bar extends from $x = 0$ to $x = \infty$. The temperature at the end $x = 0$ oscillates in time according to $T(x = 0, t) = T_0 \cos \omega t$. By looking for solutions that are separated in x and t ($T = X(x)F(t)$) find the solution for all $x \geq 0$, and t, which matches the boundary condition at $x = 0$. Sketch T versus x for $\omega t = \pi/2$, given that $\frac{\omega}{2K} = \frac{1}{a^2}$.

[MM Q4.4]

Q6. The 2D Laplace equation in polars

Laplace's equation in two dimensions may be written, using plane polar coordinates r, θ, as

$$r\frac{\partial}{\partial r}\left(r\frac{\partial V(r,\theta)}{\partial r}\right) + \frac{\partial^2 V(r,\theta)}{\partial \theta^2} = 0.$$

Find all separable solutions of this equation which have the form $V(r, \theta) = R(r)S(\theta)$, which are single valued for all r, θ. What property of the equation makes any linear combination of such solutions also a solution?

A continuous potential $V(r, \theta)$ satisfies Laplace's equation everywhere except on the concentric circles $r = a, r = b$ where $b > a$.

(i) Given that $V(r = a, \theta) = V_0(1 + \cos\theta)$, and that V is finite as $r \to 0$, find V in the region $r \leq a$.

(ii) Given, separately, that $V(r = b, \theta) = 2V_0 \sin^2\theta$, and V is finite as $r \to \infty$, find V for $r \geq b$.

[MM Q4.5]

Q7. The modes of a square membrane

The equation governing the motion of a square vibrating membrane has the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

with $u = 0$ on the boundary.

Show that the normal modes are given by

$$u(x, y, t) = A_{m,n} \sin(m\pi x/L) \sin(n\pi y/L) \cos(\omega_{m,n}t)$$

where $\omega_{m,n}$ is the frequency of the mode and L is the length of a side of the square. Express $\omega_{m,n}$ in terms of m and n and give any restrictions which apply to m and n.

Show that the second lowest frequency is a factor of $\sqrt{\frac{5}{2}}$ larger than the frequency of the lowest mode, that two modes have this frequency and that by combining them it is possible to have a node along either diagonal of the square. Deduce the fundamental mode for a triangular membrane obtained by constraining the membrane along a diagonal.

[MM Q4.6]