Problem sheet - MT tutorial 5

Self-assessed questions

Q1. Periodic extensions of functions

Sketch graphs of the following functions in the range $-2\pi < x < 2\pi$ given that all the functions are periodic with period 2π:

(a) $f(x) = |x|, \quad -\pi < x < \pi;$

(b) $f(x) = x, \quad -\pi < x < \pi;$

(c) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

[MM Q5.1]

Q2. Odd and even functions

State whether the following functions are even, odd or neither even nor odd:

(a) $\sin(x)$ (b) $\cos(x)$ (c) $\sin^2(x)$ (d) $\cos^2(x)$ (e) $\sin(2x)$ (f) $\cos(2x)$

(g) $x\cos(x)$ (h) $\sin(x) + \cos(x)$ (i) $x + \sin(x)$ (j) e^x

[MM Q5.2]

Q3. Energy over time

A system has a time-independent Hamiltonian that has spectrum $\{E_n\}$. Prove that the probability P_k that a measurement of energy will yield the value E_k is is time-independent.

[JB QM Q1.15]

Q4. Basic properties of the Fourier transform

Show that

(a) The Fourier transform of $f(ax)$ is $\frac{1}{a}\tilde{f}(k/a)$;

(b) The Fourier transform of $f(a+x)$ is $e^{ika}\tilde{f}(k)$;

(c) The Fourier transform of $e^{iqx}f(x)$ is $\tilde{f}(k-q)$;

(d) The Fourier transform of $\frac{df(x)}{dx}$ is $ika\tilde{f}(k)$;

(e) The Fourier transform of $xf(x)$ is $i\frac{d\tilde{f}(k)}{dk}$.

[MM Q5.10]

Q5. Incompatible observables

The operators A and B do not commute. The eigenstates of A are $|0\rangle$ and $|1\rangle$ and satisfy $A|a\rangle = a|a\rangle$. The eigenstates of B are

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

with eigenvalues ± 1 respectively.

A system starts in the state $|0\rangle$. A measurement of B yields the value $+1$. What is the probability of this and what is now the state of the system?

Now a measurement of A is made. What are the possible outcomes and what state will the system be in afterwards?

Suppose the measurements of A and B are made in the opposite order. Discuss what happens.

Suppose alternating measurements of A and B are made *ad infinitum*. Discuss what happens.

[JMR QM Q5.6]

Q6. Fourier transform of an ODE

By taking the Fourier transform of the differential equation

$$\frac{d^2\Phi}{dx^2} - K^2\Phi = f(x) \ ,$$

show that its solution can be written as

$$\Phi(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \ e^{ikx} \ \frac{\tilde{f}(k)}{k^2 + K^2}.$$

[MM Q5.9]

Q7. Superpositions of a particle in a well

A particle is confined in a potential well such that its allowed energies are $E_n = n^2\mathcal{E}$, where $n = 1, 2, \ldots$ is an integer and $\mathcal{E}$ a positive constant. The corresponding energy eigenstates are $|1\rangle, |2\rangle, \ldots, |n\rangle, \ldots$. At $t = 0$ the particle is in the state

$$|\psi(0)\rangle = 0.2|1\rangle + 0.3|2\rangle + 0.4|3\rangle + 0.843|4\rangle.$$

a. What is the probability, if the energy is measured at $t = 0$ of finding a number smaller than $6\mathcal{E}$?
b. What is the mean value and what is the rms deviation of the energy of the particle in the state $|\psi(0)\rangle$?
c. Calculate the state vector $|\psi\rangle$ at time t. Do the results found in (a) and (b) for time t remain valid for arbitrary time t?
d. When the energy is measured it turns out to be $16\mathcal{E}$. After the measurement, what is the state of the system? What result is obtained if the energy is measured again?

Show that for properly normalised ψ, $\sum_r P(q_r|\psi) = 1$. Why is this significant? Show further that the expectation of Q is $\langle Q\rangle \equiv \int_{-\infty}^{\infty} \psi^* \hat{Q}\psi \, dx$.

[JB QM Q1.16]

Main questions

Q1. Superposition of energy eigenstates (note that the *certain instant* the question refers to is $t = 0$)

A particle in the infinite-sided box has the wavefunction (16). At a certain instant, its energy is measured and found to have the value $h^2/2ma^2$. What is the probability of finding the particle in the region $0 \leq x \leq a/2$ (i) before the energy measurement? (ii) after it? Explain the answers qualitatively, with the aid of a sketch.

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + \phi_2(x)\right) \tag{16}$$

[JMR QM Q5.1]

Q2. Collapsing superpositions

The eigenstates of two commuting operators A and B are denoted $|a, b\rangle$ and satisfy the eigenvalue equations $A|a, b\rangle = a|a, b\rangle$ and $B|a, b\rangle = b|a, b\rangle$. A system is set up in the state

$$|\psi\rangle = N\left(|1, 2\rangle + |2, 2\rangle + |1, 3\rangle\right)$$

What is the value of the normalization constant N?

A measurement of the value of A yields the result 1. What is the probability of this happening? What is the new state $|\psi'\rangle$ of the system?

Then a measurement of the value of B yields the result 2. What is the probability of this happening? What is the new state $|\psi''\rangle$ of the system?

Given that the system starts in the state $|\psi\rangle$ and then A is measured and then B is measured what is the probability that it ends up in the state $|\psi''\rangle$?

Repeat the above but measure B first and then A. Comment on your results.

[JMR QM Q5.5]

Option section: Mathematica version Either complete Q5, Q6 and Q7 in the form here, using Mathematica to compute the integrals, and then complete the additional Mathematica problem on time evolution with Fourier Transforms (recommended) or alternatively, complete the alternative versions on the next page without Mathematica.

Q5. Fourier series expansions (I strongly suggest that for this question and the next 2 questions you use Mathematica or Wolfram Alpha do some of the trig integrals you will encounter and to check your answers, i.e. plot the first 10 terms of the series and see if it looks right):

Find the Fourier series for the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}.$$

[MM Q5.3]

Q6. Fourier sine and cosine series expansions

Find (a) the Fourier sine series and (b) the Fourier cosine series for the function

$$f(x) = x \sin x, \qquad 0 < x < \pi.$$

In order to determine the Fourier sine series, you need to extend the function to the interval $[-\pi, 0]$ by requiring that $f(-x) = -f(x)$. Now determine the Fourier sine series for the resulting odd function on the interval $[-\pi, \pi]$. This Fourier series involves only sines as the extended function is odd. The Fourier sine series equals $f(x)$ on the original interval $[0, \pi]$.

[MM Q5.4]

Q7. Another example (with Mathematica)

Find (a) the Fourier sine series and (b) the Fourier cosine series for the function $f(x) = x^2$ in the interval $0 < x < 2$.

[MM Q5.5]

Mathematica additional component: Complete the problems in PotentialWellquestions.nb. Either submit a commented version (i.e., send me the notebook by e-mail or print it and hand it in, but before you do add text lines in the notebook that explain each step of what you did), or write a series of bullet points explaining each step in your handwritten submission.

Option section: Non-Mathematica version. Complete the alternative versions of Q5 and Q6 from the 2023 Mathematical Methods problem sets, with the additional mathematical components, as well as Q7.

Q5. Fourier series expansions

a) Find the Fourier series for the function $f : [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \leq x < 0 \\ \sin x & \text{for} \quad 0 \leq x \leq \pi \end{cases}$$

(b) Find the Fourier series for the functions $f : [-\pi, \pi] \to \mathbb{R}$ defined by $f(x) = x^2$.

(c) Use the result from part (b) to sum the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$.

[MM 2023 P2.1]

Q6. Fourier sine and cosine series expansions

(a) For the functions $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = x \sin x$ find the cosine Fourier series.

(b) For the function from part (a), find the sine Fourier series.

(c) For the function $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = x$ compute the cosine Fourier series.

(d) Compute the sine Fourier series for the function from part (c). Comment on the convergence properties of this series and its cosine Fourier series counterpart. Sum the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

[MM 2023 P2.2]

Q7. Another example (without Mathematica)

Find (a) the Fourier sine series and (b) the Fourier cosine series for the function $f(x) = x^2$ in the interval $0 < x < 2$.

[MM Q5.5]