Problem sheet - MT tutorial 6

Main questions

Q1. Dispersion of a Gaussian wavepacket At time $t = 0$, a wavefunction is given by

$$\psi(x,t=0) = A\, e^{ik_0 x} e^{-x^2/2a^2} .$$

a) Determine the normalisation constant A from $\int_{-\infty}^{\infty} |\psi(x,0)|^2 = 1$

b) Sketch and discuss the form of $|\psi(x,0)|^2$.

c) We will now express $\psi(x,0)$ as a superposition of plane waves. Calculate the Fourier transform $a(k)$ of $\psi(x,t=0)$, showing that

$$a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0) e^{-ikx} dx \quad (1)$$

$$\propto a e^{-a^2(k-k0)^2/2} \quad (2)$$

Hints: i) The integral is solved by *completing the square*, ii) remember that $I = \int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$.

d) What is the approximate relation between the width of the wave packet in configuration / position space and its width in k space? How does this relate to the position-momentum uncertainty relation (with $p = \hbar k$)?

e) The general form of a wave function, now including time dependence, is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k)\, e^{i(kx-\omega t)} dk$$

Using the *dispersion relation* for de Broglie waves,

$$\omega(k) = \frac{\hbar k^2}{2m},$$

and the result for $a(k)$, calculate $a(k,t)$.

f) Use this to compute the function $\psi(x,t)$ for any time t. Note: This integral is somewhat challenging. Show that

$$|\psi(x,t)|^2 \propto \exp\left(-\frac{[x-(hk_0/m)t]^2}{a^2[1+(\hbar t/ma^2)^2]}\right)$$

g) Consider this form, especially answering the following questions regarding the mean position and spread of the wavefunction:

(a) Why do you expect this change in the mean position? What does it correspond to physically, and what is the role of k_0? Does this make sense given the form of the momentum space wavefunction?

(b) Why does the spread of the wavefunction in position behave as it does in time? Is there a corresponding change in the spread of the wavefunction in momentum ? What does this mean for the uncertainty relation $\Delta x \Delta p \geq \hbar/2$?

Q2. Consider a harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$A = \sqrt{\frac{m\omega}{2}}x + i\frac{p}{\sqrt{2m\omega}}$$
$$A^\dagger = \sqrt{\frac{m\omega}{2}}x - i\frac{p}{\sqrt{2m\omega}}$$

i) Show that $[A, A^\dagger] = \hbar$ and $H = \omega A^\dagger A + \hbar\omega/2$.

ii) Show that $[H, A] = -\hbar\omega A$ and $[H, A^\dagger] = \hbar\omega A^\dagger$

iii) Assume that $H|\psi\rangle = E|\psi\rangle$; show that $|\psi'\rangle = A|\psi\rangle$ satisfies $H|\psi'\rangle = (E - \hbar\omega)|\psi'\rangle$. Deduce that there must be a state $|0\rangle$ satisfying $A|0\rangle = 0$ and give its energy.

iv) Show that $|\psi'\rangle = A^\dagger|\psi\rangle$ satisfies $H|\psi'\rangle = (E + \hbar\omega)|\psi'\rangle$. Now you can deduce the spectrum E_n and how the corresponding states $|n\rangle$ are related to $|0\rangle$.

v) It's easy to compute the correct normalization too. Being careful we have

$$|n+1\rangle = C_n A^\dagger |n\rangle$$

where the states are all normalised and C_n is a constant. Show that

$$1 = \langle n+1|n+1\rangle = |C_n|^2\hbar(n+1).$$

This tells you C_n; find the constant N_n such that

$$|n\rangle = N_n (A^\dagger)^n |0\rangle$$

is correctly normalized.

[JMR QM Q6.4]

Q3. Central elements of the Harmonic oscillator

3.1 After choosing units in which everything, including $\hbar = 1$, the Hamiltonian of a harmonic oscillator may be written $\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2)$, where $[\hat{x}, \hat{p}] = \mathrm{i}$. Show that if $|\psi\rangle$ is a ket that satisfies $H|\psi\rangle = E|\psi\rangle$, then

$$\frac{1}{2}(\hat{p}^2 + \hat{x}^2)(\hat{x} \mp \mathrm{i}\hat{p})|\psi\rangle = (E \pm 1)(\hat{x} \mp \mathrm{i}\hat{p})|\psi\rangle.$$

Explain how this algebra enables one to determine the energy eigenvalues of a harmonic oscillator.

3.2 Given that $\hat{a}|n\rangle = \alpha|n-1\rangle$ and $E_n = (n + \frac{1}{2})\hbar\omega$, where the annihilation operator of the harmonic oscillator is

$$\hat{a} \equiv \frac{m\omega\hat{x} + \mathrm{i}\hat{p}}{\sqrt{2m\hbar\omega}},$$

show that $\alpha = \sqrt{n}$. Hint: consider $|\hat{a}|n\rangle|^2$.

3.3 The pendulum of a grandfather clock has a period of 1 s and makes excursions of 3 cm either side of dead centre. Given that the bob weighs 0.2 kg, around what value of n would you expect its non-negligible quantum amplitudes to cluster?

3.4 Show that the minimum value of $E(p, x) \equiv p^2/2m + \frac{1}{2}m\omega^2 x^2$ with respect to the real numbers p, x when they are constrained to satisfy $xp = \frac{1}{2}\hbar$, is $\frac{1}{2}\hbar\omega$. Explain the physical significance of this result.

3.5 How many nodes are there in the wavefunction $\langle x|n\rangle$ of the nth excited state of a harmonic oscillator?

3.6 Show that for a harmonic oscillator that wavefunction of the second excited state is $\langle x|2\rangle = \text{constant} \times (x^2/\ell^2 - 1)\mathrm{e}^{-x^2/4\ell^2}$, where $\ell \equiv \sqrt{\hbar/2m\omega}$ and find the normalising constant.

[FE QM Problem sheet 3]

Q4. Coherent states in a 1D harmonic oscillator:

We define *coherent states* as

$$|\varphi_\alpha\rangle := e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\phi_n\rangle$$

where $\hat{H} = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$, $\hat{H}|\phi_n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|\phi_n\rangle$.

a) Show that these coherent states are eigenstates of the lowering operator,

$$\hat{a}|\varphi_\alpha\rangle = \alpha|\varphi_\alpha\rangle$$

with complex eigenvalue α (Note that a is not Hermitian).

b) Show by calculating that these states are normalised, $||\varphi_\alpha|| = \sqrt{\langle\varphi_\alpha|\varphi_\alpha\rangle} = 1$.

c) Show that the probability when measuring the energy that we find an eigenvalue $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ is given by the Poisson distribution

$$p_n = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}.$$

d) Based on (c) or otherwise show that these states have the following mean and standard deviation of the energy:

$$\langle H\rangle_\alpha \equiv \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right),$$
$$\Delta H_\alpha = \hbar\omega|\alpha|,$$

so that $\Delta H_\alpha / \langle H\rangle_\alpha \approx 1/|\alpha|$ for $|\alpha| \gg 1$. In this sense, they behave as quasi-classical states.

e) Show that these states have the following position and momentum expectation values and standard deviations:

$$\langle X\rangle_\alpha = \sqrt{\frac{2\hbar}{m\omega}}\, Re[\alpha], \ \langle P\rangle_\alpha = \sqrt{2m\hbar\omega}\, Im[\alpha],$$

and

$$\Delta X_\alpha = \sqrt{\frac{\hbar}{2m\omega}}, \ \Delta P_\alpha = \sqrt{\frac{m\hbar\omega}{2}}.$$

f) Compute $\Delta X\, \Delta P$. Is $|\varphi_\alpha\rangle$ a state of minimum uncertainty?

Q5. Time evolution of a coherent state of the 1D harmonic oscillator: a harmonic oscillator is prepared at time $t = 0$ in the coherent state $|\psi(t = 0)\rangle = |\varphi_{\alpha_0}(x)\rangle$. Consider the time evolution under the Hamiltonian $\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + 1/2)$, and show that

a) The time evolution of the state is given by

$$|\psi(t)\rangle = \hat{U}(t)|\psi(t = 0)\rangle = e^{-i\hat{H}t/\hbar}|\varphi_{\alpha_0}(x)\rangle = e^{-i\omega t/2}|\varphi_{\alpha_t}\rangle,$$

where $\alpha_t := \alpha_0 e^{-i\omega t}$, so that a coherent state remains a coherent state in time evolution under the harmonic oscillator hamiltonian $\hat{H}$.

b) The time dependence of the position and momentum obey the same equations as in classical mechanics.

c) For the standard deviations, $\Delta X_t = \Delta X_0$, $\Delta P_t = \Delta P_0$.

Q6. Consider the 1D harmonic oscillator with ground state oscillator length x_0.

a) Construct a linear combination of the ground and first excited states (determining coefficients c_0 and c_1),

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

such that $\langle\hat{x}\rangle$ is maximised.

b) Taking the state from (a) as the state at $t = 0$, $|\psi(t = 0)\rangle$, determine the time evolution of the expectation value $\langle\hat{x}\rangle(t)$ in terms of x_0.