The Manin-Drinfeld theorem

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Introduction

A theorem due to Manin and Drinfeld states that any degree 0 divisor on a modular curve which is supported on the cusps is a torsion element in the Jacobian. This is equivalent to the statement that a certain mixed Hodge structure in the cohomology of the modular curve splits. In this short note, we give a minimalist introduction to mixed Hodge structures and a proof of the Manin-Drinfeld theorem, loosely following [Elk90]. Finally, we briefly summarize some extensions and recent developments.

Contents

1 Hodge structures

1.1 Pure and mixed Hodge structures

We begin by recalling some Hodge theory:

Definition 1.1. Let k be an integer. A **(pure) Hodge structure of weight** k on a $\mathbb{Z}$-module H is a decomposition $H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$ such that $\overline{H^{p,q}} = H^{q,p}$.

Example 1.2. A basic example is the *Hodge Tate structure* on $\mathbb{Z}(k) := (2\pi i)^k\mathbb{Z}$, with $\mathbb{Z}(k)\otimes\mathbb{C} = [\mathbb{Z}(k)\otimes\mathbb{C}]^{-k,-k}$ of weight $-2k$. More generally, if H is any Hodge structure, then $H(k) := H\otimes\mathbb{Z}(k)$ is a Hodge structure called the *k-th Tate twist of H*.

Example 1.3. The motivating example of a Hodge structure arises from Hodge theory; if M is a d-dimensional compact Kähler manifold, then for each $1 \le k \le 2d$ the singular cohomology group $H^k(M;\mathbb{Z})$ has a decomposition $H^k(M;\mathbb{Z}) \cong \bigoplus_{p+q=k}\mathcal{H}^{p,q}$ where $\mathcal{H}^{p,q}$ is the $\mathbb{C}$-vector space of harmonic (p,q)-forms.

In particular, if X is a smooth complex projective variety, then by the GAGA principle we have natural Hodge structure on the algebraic de Rham cohomology of X.

It is natural to ask whether there is a weaker notion of a Hodge structure which holds for the cohomology of a possibly singular or non-projective variety. This naturally leads to the notion of a mixed Hodge structure:

Definition 1.4. A **mixed Hodge structure** on a $\mathbb{Z}$-module H consists of:

(i) a decreasing filtration[1] $F^\bullet$ on $H\otimes\mathbb{C}$, called the **Hodge filtration**,
(ii) an increasing filtration $W_\bullet$ on $H\otimes\mathbb{Q}$, called the **weight filtration**,

such that $\mathrm{gr}_m W_\bullet := W_m/W_{m-1}$ is a Hodge structure of weight m, and

$$F^p(\mathrm{gr}_m W_\bullet) = (W_m\cap F^p)/(W_{m-1}\cap F^p). \tag{1.1}$$

Example 1.5. Fix an integer k, and suppose H has a mixed Hodge structure with

$$W_m = \begin{cases} H_{\mathbb{Q}} & \text{if} \quad m\ge k,\\ 0 & \text{otherwise.}\end{cases} \tag{1.2}$$

Then this naturally defines a pure Hodge structure of weight k on H. Conversely, if H is a pure Hodge structure, then we can define the weight filtration on H by eq. (1.2) and the Hodge filtration by $F^pH\otimes\mathbb{C} = \bigoplus_{n\ge p}H^{n,k-n}$. In the case of singular cohomology on a complex Kähler manifold, F^p has the natural interpretation of "at least p dz's". In summary, pure Hodge structures are special cases of mixed Hodge structures.

Definition 1.6. A **morphism of mixed Hodge structures** $H\to H'$ is a $\mathbb{Z}$-linear map $\phi\colon H\to H'$ which preserves the filtrations; in other words, we have

$$\phi_{\mathbb{Q}}(W_mH)\subset W_mH' \quad \text{and} \quad \phi_{\mathbb{C}}(F^pH)\subset F^pH', \tag{1.3}$$

where $\phi_{\mathbb{Q}}$ and $\phi_{\mathbb{C}}$ are the induced maps on the filtrations.

[1] Recall that a *decreasing filtration* $F^\bullet$ is a nested sequence of subobjects (for example subspaces of a vector space) $\ldots\subset F^{p+1}A\subset F^pA\subset\ldots$, while an *increasing filtration* $W_\bullet$ on an object A is a nested sequence of subobjects $\ldots\subset W_mA\subset W_{m+1}A\subset\ldots$. The *associated graded* of a filtration $F^\bullet$ (resp. $W_\bullet$) is the object $\mathrm{gr}\,F^\bullet = \bigoplus_{p\in\mathbb{Z}}\mathrm{gr}_p F^\bullet$ (resp. $\mathrm{gr}\,W_\bullet = \bigoplus_{m\in\mathbb{Z}}\mathrm{gr}_m W_\bullet$) where $\mathrm{gr}_p F^\bullet = F^pA/F^{p+1}A$ (resp. $\mathrm{gr}_m W_\bullet = W_mA/W_{m-1}A$)

The category of mixed Hodge structures, denoted by MHS, is very well-behaved; for example, the kernel and cokernel of any morphism of mixed Hodge structures inherit canonical mixed Hodge structures. More generally, we have the following:

Theorem 1.7 ([PS08, §I.3.1])**.**

(i) *The category* MHS *is abelian.*

(ii) *The category* MHS *is closed under extensions: if H' and H'' are mixed Hodge structures and H any $\mathbb{Z}$-module such that*

$$0 \to H' \to H \to H'' \to 0 \tag{1.4}$$

is an exact sequence of $\mathbb{Z}$-modules, then there is a canonical mixed Hodge structure on H such that eq. (1.9) *is an exact sequence in* MHS.

(iii) *The category* MHS *admits tensor products and internal Homs.*

1.2 The mixed Hodge structure on a curve

To preserve generality, we pass for a moment to the language of schemes. The heart of Deligne's work [Del71], "Hodge II", is the following theorem:

Theorem 1.8 ([Del71, Thm. 3.2.5(iii)])**.** *Let Y be a smooth, separated and quasi-projective scheme over $\mathbb{C}$. For each $n \in \mathbb{N}$, $H^n(Y(\mathbb{C});\mathbb{Z})$ admits a mixed Hodge structure which is functorial in Y.*

The proverbial margin is too small for a full proof, but we will describe the situation when Y is a smooth, affine, quasi-projective curve over $\mathbb{C}$ with a fixed smooth projective completion X, and $S := X \setminus Y$.

For any variety V we can define the *algebraic de Rham cohomology* as follows: if $\mathscr{F}^\bullet$ is any complex of sheaves on V, then since the category of sheaves on V has enough injectives we can find a quasi-isomorphism $\mathscr{F}^\bullet \to I^\bullet$ where $I^\bullet$ is a complex of injective sheaves. The *hypercohomology groups of the complex $\mathscr{F}^\bullet$* are the cohomology groups of the complex $\Gamma(V, I^\bullet)$, where $\Gamma(V,-)$ is the global sections functor.

Now let Ω^1_V be the sheaf of differentials on V, and consider the complex of sheaves $\Omega^\bullet_V$ where $\Omega^n_V := \bigwedge^n \Omega^1_V$.

Definition 1.9. The **algebraic de Rham cohomology groups of** V, $H^n_{\mathrm{dR}}(\Omega^\bullet_V)$ is the hypercohomology of $\Omega^\bullet_V$.

In particular, if V is affine, then by Serre's criterion for affineness any quasi-coherent sheaf on V is acyclic, so $H^n_{\mathrm{dR}}(\Omega^\bullet_V) = H^n(\Gamma(V, \Omega^\bullet_V))$, that is, algebraic de Rham cohomology coincides with sheaf cohomology of the sheaf of algebraic differentials on V.

There is a natural filtration on these groups; let $\sigma_{\geq i}$ be the truncation functor which sends a complex $A^\bullet$ to the corresponding complex $\sigma_{\geq i}A^\bullet$ whose n-th term is A^n if $n \geq i$, and 0 otherwise. This is easily checked to be exact, and we get a filtration on $H^\bullet_{\mathrm{dR}}$ by

$$F^i H^n_{\mathrm{dR}} = \mathrm{Im}(\mathbb{H}^n(\sigma_{\geq i}\Omega^\bullet_V) \to \mathbb{H}^n(\Omega^\bullet_V)). \tag{1.5}$$

We now narrow our focus to the curves X and Y defined above. We have an exact sequence of filtered complexes, the "residue exact sequence",

$$0 \to \Omega^\bullet_X \to \Omega^\bullet_X(\log S) \xrightarrow{\mathrm{Res}} i_*\Omega^{\bullet-1}_S \to 0. \tag{1.6}$$

Here $\Omega^\bullet_X(\log S)$ is the complex of algebraic differentials on X with at most simple poles at S, and the map Res sends a differential to the locally constant function specfified by its residues at points of S. Since we are dealing with curves, $\Omega^p_X = 0$ for $p \geq 2$, so eq. (1.6) simply the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r, "\mathrm{Res}"] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_X \arrow[r] & \Omega^1_X(\log S) \arrow[r, "\mathrm{Res}"] & i_*\mathcal{O}_S \arrow[r] & 0
\end{tikzcd} \tag{1.7}$$

Taking the long exact sequence in hypercohomology corresponding to eq. (1.6) gives

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}^0(\Omega^\bullet_X) \arrow[r] & \mathbb{H}^0(\Omega^\bullet_X(\log S)) \arrow[r] & \mathbb{H}^{-1}(\Omega^\bullet_S) \arrow[dll, out=0, in=180] \\
 & \mathbb{H}^1(\Omega^\bullet_X) \arrow[r] & \mathbb{H}^1(\Omega^\bullet_X(\log S)) \arrow[r] & \mathbb{H}^0(\Omega^\bullet_S) \arrow[dll, out=0, in=180] \\
 & \mathbb{H}^2(\Omega^\bullet_X) \arrow[r] & \dots &
\end{tikzcd} \tag{1.8}$$

Note that $\mathbb{H}^{-1}(\Omega^\bullet_S) = 0$, so the interesting part of the sequence starts at the second line. To relate this to Y, we use the following theorem, which in rough terms states that only simple poles are required to compute cohomology:

Theorem 1.10 ([Del71, 3.2.2])**.** *For all $n \geq 0$, we have canonical isomorphisms $\mathbb{H}^n(\Omega^\bullet_X(\log S)) \cong \mathbb{H}^n(\Omega^\bullet_Y)$.*

On the other hand, since Y is affine by assumption, $\mathbb{H}^n(\Omega^\bullet_Y) \cong H^n(Y(\mathbb{C}), \Omega^1_Y)$. In particular, the filtration on $\mathbb{H}^n(\Omega^\bullet_X(\log S))$ gives rise to a filtration $F^\bullet H^n(Y(\mathbb{C}), \Omega^1_Y)$.

Next we turn our attention to Betti (or singular) cohomology; the inclusion $Y(\mathbb{C}) \hookrightarrow X(\mathbb{C})$ determines the following exact sequence in *relative homology*, as described in [Hat02, §2.1]:

$$\begin{tikzcd}
0 \arrow[r] & 0 = H_2(Y(\mathbb{C});\mathbb{Z}) \arrow[r] & H_2(X(\mathbb{C});\mathbb{Z}) \arrow[r] & H_2(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Z}) \arrow[dll] \\
& H_1(Y(\mathbb{C});\mathbb{Z}) \arrow[r] & H_1(X(\mathbb{C});\mathbb{Z}) \arrow[r] & H_1(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Z}) = 0
\end{tikzcd} \tag{1.9}$$

Next, we dualise; by Poincaré duality, we can fix an isomorphism of $\mathbb{Z}$-modules $H_2(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Z})^\vee \xrightarrow{\sim} \frac{1}{2\pi i}H^0(S(\mathbb{C});\mathbb{Z})$, where the seemingly arbitrary choice of normalisation $\frac{1}{2\pi i}$ will be justified soon. Viewing $H^0(S(\mathbb{C});\mathbb{Z})$ as functions on the finite set of points S, we can natural identify it with $\mathrm{Div}_S(X)$, the set of divisors of X which take the value 0 on points not in S. Similarly, if we fix an isomorphism $H_2(X(\mathbb{C});\mathbb{Z})^\vee \cong \frac{1}{2\pi i}\mathbb{Z}$ the map $H_2(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Z})^\vee \to H_2(X(\mathbb{C});\mathbb{Z})^\vee$ is precisely the degree map, $\frac{1}{2\pi i}\sum_{s\in S} n_s \cdot s \mapsto \frac{1}{2\pi i}\sum_{s\in S} n_s$. In summary, the exact cohomology sequence dual to eq. (1.9) is isomorphic to

$$0 \to H^1(X(\mathbb{C});\mathbb{Z}) \to H^1(Y(\mathbb{C});\mathbb{Z}) \to \frac{1}{2\pi i}\mathrm{Div}_S(X) \xrightarrow{\deg} \frac{1}{2\pi i}\mathbb{Z} \to 0. \tag{1.10}$$

There is a natural morphism from de Rham cohomology to Betti cohomology; any $\omega \in \Omega^n_X$ gives rise to a singular n-cochain $\gamma \mapsto \int_\gamma \omega$. This allows us to compare the exact sequences (1.9) and (1.10) (tensored with $\mathbb{C}$): we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_X(X) \arrow[r] \arrow[d] & \Omega^1_X(\log S)(X) \arrow[r, "\mathrm{Res}"] \arrow[d, "\phi"] \arrow[dr, dashed] & H^0(S,\Omega^1_X) \arrow[r] \arrow[d] & H^1(X,\Omega^1_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(X(\mathbb{C});\mathbb{C}) \arrow[r] & H^1(Y(\mathbb{C});\mathbb{C}) \arrow[r] & \frac{1}{2\pi i}\mathrm{Div}_S(X)\otimes\mathbb{C} \arrow[r, "\deg"] & \mathbb{C} \arrow[r] & 0
\end{tikzcd} \tag{1.11}$$

where we have used that $H^1(X,\Omega^1_X(\log S)) = 0$: indeed, $\Omega^1_X(\log S) \cong \Omega^1_X \otimes \mathcal{O}_S$, so by Serre duality

$$H^1(X,\Omega^1_X(\log S)) = H^1(X,\Omega^1_X \otimes \mathcal{O}_S) = H^0(X,\mathcal{O}(-S))^\vee = 0. \tag{1.12}$$

The dashed arrow in eq. (1.11) sends a meromorphic differential ω to $\frac{1}{2\pi i}\sum_{s\in S}\left(\int_{\gamma_s}\omega\right)\cdot s$, where γ_s is the class in $H_1(Y(\mathbb{C});\mathbb{C})$ of a sufficiently small loop around s.

Next we define the filtrations on $H^1(Y)$: starting from eq. (1.10), we interpret $\frac{1}{2\pi i}\mathrm{Div}_S(X)$ as the Hodge Tate twist of $H^0(S;\mathbb{Z})$. Then we have a diagram of $\mathbb{Z}$-modules,

$$0 \to H^1(X(\mathbb{C});\mathbb{Z}) \to H^1(Y(\mathbb{C});\mathbb{Z}) \to H^0(S;\mathbb{Z})[-1] \to H^2(X;\mathbb{Z}) \to 0. \tag{1.13}$$

Now let $U := \ker\deg$. Since $H^0(S;\mathbb{Z})[-1] \to H^2(X;\mathbb{Z})$ is a morphism of mixed Hodge structures, theorem 1.7 implies that U has a mixed Hodge structure of type $(1,1)$. Explicitly, $U_\mathbb{Z} = \frac{1}{2\pi i}\mathrm{Div}^0_S(X)$, the set of integral divisors $\sum_{s\in S} n_s \cdot s$

with $\sum_{s\in S} n_s = 0$, where $W_2 U = \frac{1}{2\pi i}\operatorname{Div}_S^0(X)\otimes\mathbb{Q}$ and $F^1 U \frac{1}{2\pi i}\operatorname{Div}_S^0(X)\otimes\mathbb{C}$. There is a corresponding diagram of $\mathbb{Z}$-modules

$$0 \to H^1(X(\mathbb{C});\mathbb{Z}) \to H^1(Y(\mathbb{C});\mathbb{Z}) \to U \to 0, \tag{1.14}$$

and this allows us to define a natural weight filtration on $H^1(Y(\mathbb{C});\mathbb{Z})$:

$W_0 H^1(Y) = 0$	$F^0 H^1(Y) = H^1(Y;\mathbb{C})$
$W_1 H^1(Y) = H^1(X;\mathbb{Q})$	$F^1 H^1(Y) = \operatorname{Im}\phi$
$W_2 H^1(Y) = H^1(Y;\mathbb{Q})$	$F^2 H^1(Y) = 0$

Here ϕ is as in eq. (1.11), $\omega \mapsto ([\gamma] \mapsto \int_{[\gamma]}\omega)$, and $F^\bullet$ is precisely the weight filtration on $H^1_{\mathrm{dR}}(\Omega^\bullet(\log S))$ under the isomorphism in theorem 1.10.

2 The Manin-Drinfeld theorem

If $N \in \mathbb{N}$, we denote by $\Gamma(N)$ the subgroup of $\mathrm{Sl}_2(\mathbb{Z})$ which are entry-wise congruent modulo N to the identity matrix. A subgroup Γ of $\mathrm{Sl}_2(\mathbb{Z})$ is said to be a *congruence subgroup* if $\Gamma \supset \Gamma(N)$. Any such Γ inherits a left action on the upper-half plane $\mathfrak{h}$, and we set $Y(\Gamma) := \Gamma \setminus \mathfrak{h}$ and denote by $X(\Gamma)$ its smooth projective completion. By the GAGA principle, the analytification of $X(\Gamma)$ is precisely the compact Riemann surface arising from $\Gamma \setminus \mathfrak{h}$.

The set $S := X(\Gamma) \setminus Y(\Gamma)$ is precisely the set of cusps of Γ, that is, the image under the quotient map of the Γ-orbits of $\mathbb{P}^1(\mathbb{Q})$. Note that ∞ is always a cusp; therefore we have a canonical embedding of $X(\Gamma)$ into its Jacobian $J(\Gamma)$ by $x \mapsto [x] - [\infty]$. Here we identify $J(\Gamma)$ with $\operatorname{Pic}^0(X(\Gamma))$, the group of degree 0 divisors modulo *principal divisors*, that is, any divisor D of the form $D = (f) = \sum_{P\in X} \operatorname{ord}_P f \cdot P$ where f is a meromorphic function on $X(\Gamma)$.

Theorem 2.1 (Manin-Drinfeld). *If D is a divisor of degree 0 with support in S, then D has finite order in $J(\Gamma)$.*

Equivalently, the image of S in $J(\Gamma)$ generates a finite subgroup.

Example 2.2. Fix p prime and set $\Gamma = \Gamma_0(p)$. The cusps of $X_0(p) := X(\Gamma_0(p))$ are the two points P_0 and P_∞, corresponding to the cusps 0 and ∞. Setting

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1+q^n)^{24}, \qquad q = e^{2\pi i\tau}, \tag{2.1}$$

the meromorphic function $\Delta(\tau)/\Delta(p\tau)$ has divisor $(p-1)(P_0 - P_\infty)$; see [Apo90, Chap. 4.7] for a proof. Thus the class of $P_0 - P_\infty$ in $J(\Gamma_0(p))$ is torsion.

Remark. The Manin-Mumford conjecture (now Raynaud's theorem) states that there are only finitely many torsion points in the image of a curve X in its Jacobian. The Manin-Drinfeld theorem then implies that we cannot expect the set to be empty in general.

2.1 Hodge-theoretic interpretation

We want to show the following criterion:

Proposition 2.3. *Let X, Y and $S := X \setminus Y$ be as in section 1.2. Then the subgroup generated by the image $\mathrm{Div}^0_S(X)$ in $\mathrm{Pic}^0(X)$ is finite if and only if the mixed Hodge structure on $H^1(Y(\Gamma);\mathbb{Z})$ splits over $\mathbb{Q}$.*

Let $R \subset \mathbb{C}$ be a subring. The statement that $H^1(Y(\Gamma);\mathbb{Z})$ splits over R simply means that we can find a morphism of Hodge structures η which fits into the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^1(X;R) \arrow[r] & H^1(Y;R) \arrow[r] & U \otimes R \arrow[r] \arrow[l, bend right, dashed, "\eta"'] & 0 \end{tikzcd} \tag{2.2}$$

In other words, we want η to preserve the Hodge filtration (and the weight filtration, but this is automatic). We are looking for a map

$$\frac{1}{2\pi i}\mathrm{Div}^0_S \otimes R \to H^1(Y;R) \cap \mathrm{Im}\,\phi, \qquad \frac{1}{2\pi i} D \mapsto \frac{1}{2\pi i}\nu_D \tag{2.3}$$

such that $\mathrm{Res}(\nu_D) = D$. The statement that $\frac{1}{2\pi i}\nu_D \in H^1(Y;R) = \mathrm{Hom}(H_1(Y;\mathbb{Z}),R)$ simply means that

$$\frac{1}{2\pi i}\int_{[\gamma]} \nu_D \in R \qquad \text{for all } [\gamma] \in H_1(Y;\mathbb{Z}). \tag{2.4}$$

From the natural decomposition $H_1(Y;\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \oplus \bigoplus_{s\in S}\mathbb{Z}\gamma_s$, where γ_s is a sufficiently small loop around s, the following lemma is immediate:

Lemma 2.4. *$\frac{1}{2\pi i}\nu \in \Omega^1_X(\log S)(X)$ defines a class $\phi\left(\frac{1}{2\pi i}\nu\right) \in H^1(Y;R)$ if and only if the following hold:*

(i) $\int_{[\gamma]} \nu \in 2\pi i R$ *for all* $\gamma \in H_1(X;R)$,

(ii) $\mathrm{Res}\,\nu \in \mathrm{Div}^0_S(X) \otimes R$.

A classical result, see for example [Lan82, Prop. IV.5.3], states that the map

$$\Omega^1(X) \to H^1(X;\mathbb{R}), \qquad \omega \mapsto \left([\gamma] \mapsto \mathrm{Re}\int_\gamma \omega\right) \tag{2.5}$$

is an $\mathbb{R}$-linear isomorphism. Both spaces both have dimension equal to the genus of X, and one can show that the map is injective.

Lemma 2.5. *When $R = \mathbb{R}$ there exists a unique splitting η in eq. (2.2).*

Proof. Fix $\frac{1}{2\pi i}D \in \frac{1}{2\pi i}\operatorname{Div}^0_S(X) \otimes \mathbb{R}$. By exactness of eq. (1.11), there exists some $\tilde{\nu} \in \Omega^1(X, \log S)$ such that $\operatorname{Res}\tilde{\nu} = D$; by lemma 2.4 it suffices to modify $\tilde{\nu}$ to some ν satisfying $\int_\gamma \nu \in 2\pi i\mathbb{R}$ for any $[\gamma] \in H_1(Y;\mathbb{Z})$. By the result just cited, there is a unique element $\omega \in \Omega^1(X)$ such that $\operatorname{Re}\int_\gamma \omega = \operatorname{Re}\int_\gamma \tilde{\nu}$. Then $\nu := \omega - \tilde{\nu} \in \Omega^1(X, \log S)$ satisfies $\int_\gamma \nu \in i\mathbb{R} = 2\pi i\mathbb{R}$ and $\operatorname{Res}\nu = D$ since ω is holomorphic, and by uniqueness of ω is independent of choice of $\tilde{\nu}$. Setting $\eta(D) := \nu$ then gives the desired map. □

By uniqueness, any splitting defined over $\mathbb{Q}$ necessarily lifts to the splitting above, and by construction and lemma 2.4 η descends to $\mathbb{Q}$ if and only if for all D, $\int_\gamma \eta(D) \in 2\pi i\mathbb{Q}$.

Lemma 2.6. *Fix $\omega \in \Omega^1(X, \log S)$. Then $\int \omega \in \mathbb{Z}$ for all $[\gamma] \in H_1(X;\mathbb{Z})$ if and only if there exists a meromorphic function f on X with $\operatorname{Div} f \in \operatorname{Div}^0_S X$ such that $\omega = \frac{1}{2\pi i}\frac{df}{f}$.*

Proof. Suppose first that there exists a meromorphic function $f\colon X \to \mathbb{C} \sqcup \{\infty\}$ as above. Then f restricts to a holomorphic function $Y \to \mathbb{C}$ which induces a morphism $f^*\colon H^1(\mathbb{C}^\times;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$. Then $\omega = f^*(\frac{dz}{z})$, and

$$\int_\gamma \omega = \int_\gamma f^*\left(\frac{1}{2\pi i}\frac{dz}{z}\right) = \frac{1}{2\pi i}\int_{f^*\gamma}\frac{dz}{z}, \tag{2.6}$$

which lies in $\mathbb{Z}$ by Cauchy's integral formula.

Conversely, let $\rho\colon \widetilde{Y} \to Y$ be the universal covering map and $\Gamma := \operatorname{Aut}(\widetilde{Y})$. Then $\pi_1(\widetilde{Y}) = 0$, hence $(\pi_1(\widetilde{Y})^{\mathrm{ab}})^\vee = H^1(\widetilde{Y};\mathbb{Z}) = 0$, so $\rho^*\omega = dh$ for some meromorphic function h on $\widetilde{Y}$. Because $\rho^*\omega$ is invariant under Γ, this satisfies $h \circ \gamma - h \in \mathbb{Z}$ for any $\gamma \in \Gamma$, and so $\exp(2\pi i h)$ is invariant under Γ. Since $Y = \Gamma \setminus \widetilde{Y}$, $\exp(h)$ descends to a meromorphic function f on Y. We then compute

$$\rho^* df = d(\rho^* f) = d(\exp(2\pi i h)) = 2\pi i \exp(2\pi i h)dh = \rho^*(2\pi i f\omega), \tag{2.7}$$

so $\omega = \frac{1}{2\pi i}\frac{df}{f}$ as required. □

Note that $\operatorname{Div} f = \sum_{s\in S}\operatorname{Res}_s \omega \cdot s = \operatorname{Res}\omega$. In particular, for $\omega = \eta(D)$ we have $\operatorname{Div} f = \operatorname{Res}\eta(D) = D$, which precisely means that $[D] \in \operatorname{Pic}^0(X)$ is trivial. In summary, we get a string of equivalent statements,

- the mixed Hodge structure of $H^1(Y;\mathbb{Z})$ splits over $\mathbb{Q}$,
- η is defined over $\mathbb{Q}$,
- $\int_\gamma \eta(D) \in 2\pi i\mathbb{Q}$ for all $[\gamma] \in H_1(X;\mathbb{Z})$,

- $n \cdot \int_\gamma \eta(D) = \int_\gamma \eta(nD) \in 2\pi i\mathbb{Z}$ for some $n \in \mathbb{N}$ and for all $[\gamma] \in H_1(X;\mathbb{Z})$,
- $n[D] = [nD] = [0] \in \mathrm{Pic}^0(X)$ for some $n \in \mathbb{N}$.

This proves proposition 2.3.

2.2 A special case

Now we specialise to the case where $\Gamma = \Gamma(N)$. Let $M_2(N)$ denote the space of modular forms of weight 2 on $\Gamma(N)$, and $S_2(N)$ the cuspidal subspace, that is, the modular forms which vanish at the cusps of $X(\Gamma(N))$. We have natural identifications $M_2(N) \cong H^0(X, \Omega^1(\log S))$ and $S_2(N) \cong H^0(X, \Omega^1)$ by $f \mapsto f dz$. For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{R})$ with positive determinant, we define the *slash operator* by

$$f|_\alpha(z) = \det(\alpha) \cdot (cz+d)^{-2} f\left(\frac{az+b}{cz+d}\right). \tag{2.8}$$

The *Petersson inner product* on $S_2(N)$ is defined by

$$\langle f, g\rangle := \int_{\Gamma(N)\backslash\mathfrak{h}} f(z)\overline{g(z)}\frac{dxdy}{y^2}, \qquad z = x+iy. \tag{2.9}$$

This gives rise to a norm $\|f\| := \langle f, f\rangle^{1/2}$ on $S_2(N)$, and one checks that $\|f|_\alpha\| = \|f\|$ for any α as above.

We will define *Hecke correspondences* on the cohomology of modular curves; for further details, see [DI95, I.3.2] or [Ste82, I.1.2]. Fix a prime $p \equiv 1 \pmod N$, and let $\alpha_p := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. An easy computation shows that

$$\Gamma' := \Gamma(N) \cap (\alpha_p^{-1}\Gamma(N)\alpha_p) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) : c \equiv 0 \pmod p\right\}. \tag{2.10}$$

The double coset $M := \Gamma(N)\alpha_p\Gamma(N)$ is the set of matrices of determinant p congruent to the identity matrix modulo N. Setting $\alpha_j := \begin{pmatrix} 1 & Nj \\ 0 & p \end{pmatrix}$ for $j = 0, \dots, p-1$ we get a decomposition $M = \bigcup_{j=0}^p \Gamma(N)\alpha_j$. Then $\Gamma(N) = \bigcup_{j=0}^p \Gamma'\beta_j$ where $\beta_j := \alpha_p^{-1}\alpha_j$. We have natural maps

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\mathrm{Pr}_{\Gamma'}} & X(\Gamma') \\ \| & & {\scriptstyle\psi_1}\downarrow\;\downarrow{\scriptstyle\psi_2} \\ \mathfrak{h} & \xrightarrow{\mathrm{Pr}_{\Gamma}} & X(\Gamma) \end{array} \tag{2.11}$$

with the property that $\mathrm{Pr}_\Gamma = \mathrm{Pr}_{\Gamma'} \circ \psi_1$ and $\mathrm{Pr}_\Gamma \circ \alpha_p = \mathrm{Pr}_{\Gamma'} \circ \psi_2$. This induces an endomorphism $T_p = \psi_{1*} \circ \psi_2^*$ on $H^1(X;\mathbb{Z})$. This is called a *Hecke correspondence*.

Lemma 2.7. *The Hecke correspondence on $H^1(X;\mathbb{Z})$ respects the Hodge structure.*

Proof. This is a general fact about correspondences on curves, and the proof is not terribly enlightening. See [Elk90, N° II] for the details. □

In particular, the Hecke correspondence acts on $H^1(X;\Omega(\log S))$, and under the isomorphism with $M_k(\Gamma)$, the action coincides with that of the usual Hecke operators:

$$T_p(f) = \sum_{j=0}^{p} f|_{\alpha_p\beta_j} = \sum_{j=0}^{p} f|_{\alpha_j}. \tag{2.12}$$

Recall that for any cusp s of $X(N)$, we can define a *q-expansion of f at s* by $f = \sum_{n=0}^{\infty} a(n)q^{n/N}$ where $q = e^{2\pi i z}$. Then a straightforward computation shows

$$T_p(f)(z) = p^{-1}\sum_{n=0}^{\infty} a(n)q^{n/(Np)}. \tag{2.13}$$

We have a crude bound on the eigenvalues of T_p: if f is a cusp form with $T_p(f) = \lambda f$ for some $\lambda \in \mathbb{C}^\times$, then by the triangle inequality,

$$|\lambda|\|f\| = \|T_p(f)\| \le \sum_{j=0}^{p} \|f|_{\alpha_j}\| = (p+1)\|f\|, \tag{2.14}$$

with equality if and only if f is an eigenform of all the slash operators $|_{\alpha_j}$. But from the computation

$$f|_{\alpha_p} = p\sum_{n=0}^{\infty} a(n)q^{pn/N} \tag{2.15}$$

one finds that this is equivalent to $f = 0$. On the other hand, the constant term of $f|_{\alpha_j}$ equals $pa(0)$ if $j = p$, and $a(0)/p$ otherwise. Therefore, the constant term of $T_p(f)$ is $(p+1)a(0)$. Since the Hecke action commutes with conjugation by $\mathrm{Sl}_2(\mathbb{Z})$, for any cusp s we have $T_pf(s) = (p+1)f(s)$. This shows that every eigenvalue of T_p acting on the quotient $M_k(N)/S_k(N)$ is $p+1$.

Now let $P \in \mathbb{Z}[T]$ be the characteristic polynomial of T_p acting on $H^1(X(N);\mathbb{Q})$. Then $P(T_p)$ acting on $H^1(Y(N);\mathbb{Q})$ is an endomorphism of Hodge structures whose image is an orthogonal complement to $H^1(X(N);\mathbb{Q})$ in $H^1(Y(N);\mathbb{Q})$, and so the Hodge structure of $H^1(X(N))$ splits, and theorem 2.1 holds for $\Gamma = \Gamma(N)$.

2.3 Conclusion

Finally, if $\Gamma \supset \Gamma(N)$ is an arbitrary congruence subgroup, then we have a natural covering map $\pi\colon X(N) \to X(\Gamma)$ and a corresponding group homomorphism $\pi_*\colon \mathrm{Div}^0 X(N) \to \mathrm{Div}^0 X(\Gamma)$, which preserves cusps because π does. Therefore, if $D \in \mathrm{Div}^0_S X(\Gamma)$, then $\pi^* D$ is a divisor supported at the cusps of $X(N)$, hence torsion. This proves theorem 2.1 in full generality. □

3 Applications and Extensions

The proof of Manin-Drinfeld given in the preceding section cannot be expected to hold for non-congruence subgroups of $\mathrm{Sl}_2(\mathbb{Z})$, since they do not admit Hecke correspondences. In fact, the statement is false for modular curves parameterising Fermat curves, see [KL81, §8.3].

Example 3.1 ([Sch86])**.** Belyi's theorem states that any smooth compact connected algebraic curve C defined over $\overline{Q}$ admits a map to $\mathbb{P}^1_{\mathbb{C}}$ branched at 3 points. Removing the branched points gives an unramified cover$Y(2)$, so by the Galois correspondence for Riemann surfaces there exists a subgroup $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ such that C is isomorphic to the compactification of $\Gamma \setminus \mathfrak{h}$; the branch points are mapped to the cusp. If we choose $C \to \mathbb{P}^1_{\mathbb{C}}$ such that the branch points have infinite order in the Jacobian – the proof of Belyi's theorem implies we can choose the branch points freely – then Γ gives a counterexample to Manin-Drinfeld.[2]

Recall from section 2 that every meromorphic differential ω on $X(\Gamma)$ with poles of order at most 1 determines a cuspidal divisor. Conversely, $D \in \mathrm{Div}^0_S$ determines a unique meromorphic differential $\omega_D \in H^0(X, \Omega^1(\log S))$ satisfying $(\omega_D) = D$ and $\int_\partial \omega_D \in i\mathbb{R}$ for any $\partial \in H_1(X(\Gamma);\mathbb{Z})$ disjoint from S.

The meromorphic differentials ω_D can be determined explicitly. Let κ be a cusp of $X(\Gamma)$ and Γ_κ the stabiliser of κ in Γ. Fix $\sigma \in \mathrm{Sl}_2(\mathbb{Z})$ such that $\sigma(\infty) = \kappa$ and

$$\sigma^{-1}\Gamma_\kappa\sigma = \left\{\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}\right\}. \tag{3.1}$$

If $z = x + iy \in \mathfrak{h}$ and $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$, then we define the *real-analytic Eisenstein series*

$$\begin{aligned} E_\kappa(z,s) &= \sum_{\gamma \in \Gamma_\kappa \setminus \Gamma} \mathrm{Im}(\sigma^{-1}\gamma(z))^s \\ &= \sum_{\gamma \in \Gamma_\kappa \setminus \Gamma} \frac{y^2}{|cz+d|^{2s}}, \qquad \sigma^{-1}\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \end{aligned} \tag{3.2}$$

and set $G_{2,\kappa}(z) = \lim_{s \to 1^+}\left(2i\frac{\partial}{\partial z}E_\kappa(z,s)\right)$. This is a real analytic automorphic form of weight 2 on Γ, and a generalisation of *Hecke's modified Eisenstein series*. Given a divisor $D = \sum_{\kappa \in S} n_\kappa \kappa$, let $G_{2,\kappa} := \sum_{\kappa \in S} n_\kappa G_{2,\kappa}$. Then we have

$$\omega_D = 2\pi i G_{2,D}(z)dz. \tag{3.3}$$

[2] The statement that any curve as above is uniformised by a modular curve might seem surprising at first, since modular curves intuitively are "arithmetic". However, the density of *congruence subgroups* in the collection of all subgroups of $\mathrm{Sl}_2(\mathbb{Z})$ is 0, a statement made precise in section 2 of Pete Clark's notes. Non-congruence subgroups generally don't have the same arithmetical significance as congruence subgroups; for example they don't necessarily admit many correspondences.

In light of the proof of lemma 2.6, this amounts to showing that the residual divisor of $G_{2,D}(z)$ equals D, and that $G_{2,D}(z)$ is holomorphic away from the support of D.

Theorem 3.2 ([Sch86, Thm. 3])**.** *The image of D in the Jacobian of $X(\Gamma)$ is torsion if and only if all the Fourier coefficients of $G_{2,D}$ in the expansion at ∞ are algebraic.*

3.1 Computing Ramanujan sums

In the note [MR87], we start off with the explicit Fourier expansion

$$G_{2,\kappa}(z) = \mathbb{1}_{\kappa=\infty} - \pi y - 4\pi^2 \sum_{m=1}^{\infty} m A_{\kappa,m} q^m, \qquad q = e^{2\pi i z}, \tag{3.4}$$

where $\mathbb{1}_{\kappa=\infty} = 1$ if $\kappa = \infty$ and 0 otherwise, C is a constant independent of κ, and

$$A_{\kappa,m} = \lim_{s\to 1^+} \sum_{c>0} \frac{1}{c^{2s}} \left(\sum_{0\le d<Nc} \exp(2\pi i m d/c) \right), \qquad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_\kappa^{-1}\Gamma. \tag{3.5}$$

To get rid of the constant terms, we consider the differences $A_{\kappa,m} - A_{\infty,m}$. Then the following is immediate from theorem 3.2:

Proposition 3.3 ([MR87, Prop. p.254])**.** *theorem 2.1 holds if and only if for all cusps κ of Γ and all $m \in \mathbb{N}$, we have that $\pi^2(A_{\kappa,m} - A_\infty) \in \overline{\mathbb{Q}}$.*

The inner sums in eq. (3.5) can be viewed as "generalised Ramanujan sums" in the following sense: if we take $\Gamma = \Gamma(N)$ and $\kappa = 0$, then $\sigma_\kappa = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$\sigma_\kappa^{-1}\Gamma = \left\{ \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) : c \equiv \pm 1 \bmod N,\ d \equiv 0 \bmod N,\ (c,d) = 1 \right\}. \tag{3.6}$$

Therefore,

$$A_{\kappa,m} = \sum_{\substack{c>0 \\ c\equiv \pm 1 \bmod N}} \frac{1}{c^2} \sum_{\substack{0\le d<Nc \\ (c,d)=1 \\ d\equiv 0 \bmod N}} \exp(2\pi i m d/(cN)), \tag{3.7}$$

and the inner sums bear resemblance to, and can be evaluated using, Ramanujan sums of the form

$$\sum_{\substack{0\le h<k \\ (h,k)=1}} \exp(2\pi i m h/k). \tag{3.8}$$

The strategy for proving proposition 3.3 is to reduce to the case of $\Gamma = \Gamma(N)$ as in section 2.3, and then to rewrite the sums $A_{\kappa,m}$ first in terms of the Möbius function, and then in terms of Dirichlet series. The result is an expression consisting of finite sums of roots of unity, finite sums of Dirichlet characters, and known rational values of Dirichlet L-functions, proving algebraicity.

An interesting corollary is that the finiteness of the image of the cuspidal divisor group in the Jacobian of a Fermat curve, known by a result of Rohrlich, implies the rationality of the Fourier coefficients of the associated Eisenstein series.

3.2 Rationality of Rademacher symbols

The recent paper [Bur21] continues this line of investigation, now with Γ a general non-cocompact Fuchsian group, meaning a discrete subgroup of $\mathrm{Sl}_2(\mathbb{Z})$ with non-compact fundamental domain. We can attach Eisenstein series to any cusp κ as above, and the key result is that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ which is hyperbolic (i.e. $|a+d| > 2$) and has positive trace, we have ([Bur21, Lemma 3.1])

$$\int_{\gamma} G_{\kappa}(z)\,dz = \Psi_{\kappa}(\gamma), \tag{3.9}$$

where $\int_{\gamma}$ is the integral along the geodesic connecting a fixed $z_0 \in \mathfrak{h}$ and $\gamma(z_0)$, and Ψ_{κ} is the *Rademacher symbol* associated to the cusp κ. Its definition is somewhat complicated, so we limit our attention to the very special case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ define the *Dedekind symbol*

$$\Phi(\gamma) = \begin{cases} \frac{b}{d} & \text{if } c = 0, \\ \frac{a+d}{c} - 12\,\mathrm{sign}(c) \cdot s(a,c) & \text{if } c \neq 0, \end{cases} \tag{3.10}$$

where $s(a,c)$ denotes the Dedekind sum, see [Apo90, §3.3]. Then the Rademacher symbol associated to the cusp ∞ is given by $\Psi_{\infty}(\gamma) = \Phi(\gamma) - 3\,\mathrm{sign}(c(a+d))$, and this is invariant under conjugation. From the preceding section and eq. (3.9) it is clear that the Manin-Drinfeld theorem is equivalent to the rationality of the Rademacher symbols $\Psi_{\kappa}(\gamma)$ for all hyperbolic elements $\gamma \in \Gamma$ of positive trace.

This immediately extends the Manin-Drinfeld to classes of non-congruence subgroups known to have rational Dedekind-Rademacher symbols; in particular, this includes non-cocompact Fuchsian groups of genus 0, and the *Helling groups* $X(N)^+$, which are congruence subgroups enlarged with all their respective Atkin-Lehner involutions.

3.3 Concluding remarks

There are many other topics related to the Manin-Drinfeld theorem worth pursuing, including:

- Generalisations, for example the extensions to imaginary quadratic fields and CM-fields by Kurčanov, [Kur78, Kur80].

- Understanding better why the naïve generalisation of Manin-Drinfeld fails to hold for Hilbert modular surfaces, as explained in the end of section 7.6 of [Fra98].
- The proof of Manin-Drinfeld using modular symbols, described in [Lan87, §4.2].
- In [BM16], the authors rephrase Manin-Drinfeld as saying that so-called *Eisenstein cycles* define a rational subspace of $H_1(X(\Gamma);\mathbb{Q})$. By writing these as rational combinations of modular symbols, they obtain an "explicit" version of Manin-Drinfeld for $\Gamma = \Gamma(N)$ where N is an odd prime.

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