The Étale Topology of Schemes

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Introduction

The theory of schemes is an integral part of the revolution in algebraic geometry that took place in the previous century, and it has since spread throughout neighbouring areas of mathematics, most prominently to number theory. Its origins are easy to trace: in the first half of the previous century it was clear that the foundations of algebraic geometry were lacking in rigour. The unfortunate consequence of this was that several results, particularly in the Italian school of algebraic geometry led by Castelnuovo, Enriques and Severi, turned out to be imprecise or even false. As a response to this, Oscar Zariski and Pierre Samuel wrote their influential volumes on commutative algebra ([ZCS75], [ZS60]) to cover the prerequisite material for a textbook on algebraic geometry which never materialised. André Weil, inspired by his own endeavours on geometry over finite fields, wrote the tome [Wei62]. However, the establishment of the foundations of algebraic geometry is usually attributed to Grothendieck, who developed the language of *schemes*, which partly through lack of user-friendly exposition – [Mum13] being an obvious exception – quickly obtained the reputation of somewhat of an arcane art.

The rough idea of scheme theory is that starting from the geometric point of view leads to trouble. For example, the Hilbert Nullstellensatz is a fundamental result in complex algebraic geometry which breaks down in spectacular ways over non-algebraically closed fields. However, starting from commutative algebra, we can recover geometric information from purely algebraic definitions. By analogy, defining a manifold in terms of a choice of embedding into $\mathbb{R}^N$ gives too much auxiliary information, and it is difficult to distinguish intrinsic and extrinsic information.

Whereas a manifold locally looks like Euclidean space, a scheme is locally identified with a topological space called the *prime spectrum* of a given ring. This contains all the information of the ring, and in fact lets us recover the ring completely. The topology on this space is very different from the Euclidean topology, but we can nevertheless study "functions" on open sets just as with manifolds. The natural way of structuring this information is through *sheaves*, which we discuss in Chapter 1.

Let us give an example to illustrate why the prime spectrum of a ring is a natural starting point: let $R = \mathbb{C}[x]$, and observe that we have a natural identification of the set of maximal ideals of R, denoted $\mathrm{mSpec}(R)$, with the space $\mathbb{C}$, by $(x - a) \leftrightarrow a$. The problem with $\mathrm{mSpec}(R)$ is that given a map of rings $\phi\colon R \to S$, the ideal $\phi^{-1}(\mathfrak{m})$ is not necessarily maximal, so a ring morphism does not give rise to a function of the associated maximal spectra. However, if we consider more generally prime ideals, we do indeed have an associated map, and in fact a functor of categories. Moreover, the additional ideal 0 in this case turns out to provide valuable geometric information.

However, this topology turns out to be insufficient in several ways. To compensate, we add more open sets which allow us transfer ideas from the Euclidean setting. In particular, we can define the *étale fundamental group*, the highlight of this thesis, which simultaneously generalises the covering space theory of Riemann surfaces, and the Galois theory of field extensions.

On references: very little content in this text is original, because the project was intended as an exploratory one. However, the figures in the text are all drawn by myself in TikZ. Chapter 1 uses predominantly [LE06] with [KS13] for some details. Chapter 2 uses [LE06], [EH06] and to some extent [GW10], as well as [Mum13]. Chapter 3 includes some proofs from [Mil00] on the étale site, but primarily [Sza09] with supplements from [Len08]. For appendix A on commutative algebra, the standard references are [AM94] and [MR89]. The historical claims in the introduction are backed up by Prof. N. Katz' review of [Del74].

Although the notes are reasonably self-contained, it is assumed that the reader is familiar with the language of categories, and some Galois theory. Readable sources of these are [Lei16] and [Nag77], respectively. For context, a little familiarity with the theory of covering spaces from, say, [Hat02] is useful but not necessary.

I am grateful to my supervisor, Dr. Clark Barwick, both for guidance on choice of topic, and for useful comments on drafts.

Notation and conventions

Throughout, R will be a commutative ring with 1. If $f \in R$, (f) denotes the ideal generated by f in R. If R is a ring, then $I \leq R$ means that I is an ideal of R, and $R^\times$ denotes the set of units in R. Given left R-modules M and N, $\mathrm{Hom}_R(M, N)$ is shorthand for $\mathrm{Hom}_{\mathbf{Mod}_R}(M, N)$. Unless explicitly stated, all modules will be left modules. If H is a normal subgroup of G, we write $H \trianglelefteq G$. We use **bold type** for letters denoting categories. To denote a set consisting of a single point where the choice of point is unimportant, we use $*$. By (attempted) convention, greek letters denote morphisms of rings while roman letters denote those of topological spaces, sheaves and schemes.

Chapter 1

Sheaves

A sheaf is in very rough terms a tool for structuring local data on a topological space. For example, on a manifold we might be interested in studying the continuous or smooth functions defined on a given subset. We can regard these as groups "lying over" the open sets, related by restriction maps which are in a natural way group homomorphisms. Like so many other places in mathematics, this is best formulated using the language of categories:

1.1 Basic concepts

Definition 1.1.1. Let X be a topological space, and $\mathbf{U}(X)$ be the following category: the objects of $\mathbf{U}(X)$ are open sets of X, and morphisms are given by

$$\mathrm{Hom}_{\mathbf{U}(X)}(U, V) = \begin{cases} \{i_{U,V}\} & \text{if} \quad U \subseteq V, \\ \varnothing & \text{otherwise,} \end{cases} \tag{1.1}$$

where $i_{U,V} \colon U \to V$ denotes the inclusion map of U into V. A **presheaf** $\mathscr{F}$ on X with values in $\mathbf{C}$ is a contravariant functor $\mathscr{F} \colon \mathbf{U}(X)^{\mathrm{op}} \to \mathbf{C}$.

To unpack slightly, this means the following: given open sets $U \subseteq V \subseteq W$ with inclusions $i_{U,W} = i_{V,W} \circ i_{U,V}$, we have corresponding maps denoted $\rho_{V,U} := \mathscr{F}(i_{U,V})$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathscr{F}(W) \arrow[r, "\rho_{W,V}"] \arrow[rr, bend left, "\rho_{W,U}"] & \mathscr{F}(V) \arrow[r, "\rho_{V,U}"] & \mathscr{F}(U), \end{tikzcd} \tag{1.2}$$

or in other words, $\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$. It is important to note that the order of composition is reversed because $\mathscr{F}$ is contravariant.

Although it is not strictly necessary, we will henceforth assume that $\mathbf{C}$ admits a forgetful functor into **Set**, so that we can talk about elements. We also assume that for any collection of objects in $\mathbf{C}$, their categorical product and coproduct are also objects in $\mathbf{C}$.

Example 1.1.2. As a first example, let us consider the Sierpinski topological space, that is, the set $\{0, 1\}$ where the subsets $\varnothing$, $\{1\}$ and $\{0, 1\}$ are open. The situation looks as follows:

$$
\begin{tikzcd}
\{0,1\} & & \mathscr{F}(\{0,1\}) \arrow[dr] \arrow[dd] & \\
& \{1\} \arrow[ul] \arrow[r, dashed, "\mathscr{F}"] & {} & \mathscr{F}(\{1\}) \arrow[dl] \\
\varnothing \arrow[uu] \arrow[ur] & & \mathscr{F}(\varnothing) &
\end{tikzcd}
$$

On the left, we see the lattice of open sets, and on the right the corresponding lattice after applying $\mathscr{F}$. Note that the arrows are reversed, since $\mathscr{F}$ is contravariant, and there are no requirements on what the objects $\mathscr{F}(U)$ should be.

Example 1.1.3. Let X be a topological space, and $\mathscr{F}$ be the functor assigning to each open $U \subseteq U$ the ring $C(U)$ of continuous real-valued functions on U. For $U \subset V$, we define the restriction map $\mathscr{F}(i_{U,V}) = \rho_{V,U} \colon C(V) \to C(U)$ to be simply the restriction of $f \colon V \to \mathbb{R}$ to U, explicitly $\rho_{V,U}(f) = f|_U$. This is a presheaf of rings on X.

Definition 1.1.4. A **section** over U is an element of $\mathscr{F}(U)$. The sections over X are called **global sections**.

Given a section $s \in \mathscr{F}(V)$ and a restriction map $\rho_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$, we are prone to write $s|_U := \rho_{V,U}(s)$.

Example 1.1.5 (The constant pre-sheaf)**.** Let S be a set, and let $\mathscr{F}$ be the presheaf on X defined by $\mathscr{F}(U) = S$. This is called a *constant pre-sheaf*.

Definition 1.1.6. For any $x \in X$, the **stalk** at x of $\mathscr{F}$ is the direct limit $\mathscr{F}_x := \varinjlim_U F(U)$ where U runs over open neighbourhoods of x and the transition maps are the restriction maps. For $s \in \mathscr{F}(U)$, we denote by s_x the image of s in $\mathscr{F}_x$.

In particular, if $\mathscr{F}$ is a sheaf of sets on X, then $\mathscr{F}_x$ is the set of all $s \in \bigsqcup_{U \ni x} \mathscr{F}(U)/\sim$ where $s \sim t$ if there exists a neighbourhood U of x such that for all $V \subseteq U$, $s|_V = t|_V$.

The reason we are not content with presheaves (as the prefix suggests), is that the notion of isomorphisms of presheaves is too strong. We want sheaves to track local data, so it is desirable for two sheaves to be equal if they are equal at all the stalks. On the other hand, presheaves can be isomorphic at the stalks without being globally isomorphic.[1]

Example 1.1.7. Let $X = U \sqcup V$ for some sets U and V where open sets are $\{\varnothing, U, V, X\}$, let $\mathscr{F}$ be the constant presheaf of groups $\mathscr{F}(W) = \mathbb{Z}$ for all $W \subseteq X$, and $\mathscr{G}$ the locally constant presheaf determined by $\mathscr{G}(U) = \mathscr{G}(V) = \mathbb{Z}$ and $\mathscr{G}(X) = \mathbb{Z} \times \mathbb{Z}$ with $(m,n)|_U = m$, $(m,n)|_V = n$. For $x \in U$, we see that

$$\mathscr{F}_x = \mathscr{F}(U) \sqcup \mathscr{F}(X)/\sim = \{m \in \mathbb{Z} \sqcup \mathbb{Z} \colon m = n \text{ iff } m|_U = n|_U\} \cong \mathscr{F}(U) \tag{1.3}$$

and similarly for $\mathscr{F}(V)$. On the other hand, $\mathscr{G}_x = \{s \in \mathbb{Z} \sqcup \mathbb{Z} \times \mathbb{Z} \colon s = t \text{ iff } s|_U = t|_U\} \cong \mathscr{G}(U)$, hence the presheaves are equal on the stalks, but not globally.

[1] This reflects the fact that **Sh** can be viewed as a localisation of the category of presheaves, informally by imposing that morphisms which are isomorphisms at stalks be invertible.

Definition 1.1.8. A **sheaf** is a presheaf satisfying the following, for any open $U \subseteq X$ and any open covering $U \subseteq \bigcup_i U_i$:

(i) If $s, t \in \mathscr{F}(U)$, and $s|_{U_i} = t|_{U_j}$ for all i, then $s = t$;
(ii) Given $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, there exists $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

The second condition states that "compatible" local sections can be glued in a natural way, and the first that the gluing is unique. The category of presheaves on X with values in $\mathbf{C}$, where arrows are given by natural transformations, is denoted by $\mathbf{pSh}(X, \mathbf{C})$. We are prone to write $\mathbf{pSh}(X)$ or simply $\mathbf{pSh}$ if there is no ambiguity. Then $\mathbf{Sh}(X, \mathbf{C}) = \mathbf{Sh}(X)$ is the full subcategory consisting of presheaves which are also sheaves.

The sheaf condition can be restated more compactly as an equaliser diagram

$$\mathscr{F}(U) \xrightarrow{d_0} \prod_i \mathscr{F}(U_i) \underset{d_2}{\overset{d_1}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(U_{ij}). \tag{1.4}$$

where $U_{ij} := U_i \cap U_j$, and $d_0 \colon s \mapsto (s|_{U_k})_k$, $d_1 \colon (s_i)_i \mapsto (s_i|_{U_{ij}})_{i,j}$ and $d_2 \colon (s_i)_i \mapsto (s_i|_{U_{ij}})_{j,i}$. Unpacking the definitions, this means that $\mathscr{F}(U)$ is in bijective correspondence with the elements $(s_i) \in \prod_i \mathscr{F}(U_i)$ satisfying $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, which is precisely definition 1.1.8.

Example 1.1.9. The presheaf of real-valued continuous functions on a topological space X considered in example 1.1.3 is a sheaf of rings; one readily sees that any collection of compatible sections $\{f_i \colon U_i \to \mathbb{R}\}$ glues to a unique continuous function $f \colon \bigcup_i U_i \to \mathbb{R}$ defined by $f(x) = f_i(x)$ for $x \in U_i$.

Example 1.1.10. Let $\mathscr{F}$ be the presheaf of bounded functions on $\mathbb{R}$, that is, the functor assigning to each open $U \subseteq \mathbb{R}$ the set of bounded functions on U. For $U \subseteq V$, we have a corresponding restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$ which sends a bounded function $f \colon V \to \mathbb{R}$ to $f|_U \colon U \to \mathbb{R}$ defined by $x \mapsto f(x)$. This is not a sheaf, because the function $f(x) = x$ is bounded on every bounded subset, but is not globally bounded.

Example 1.1.11 (The skyscraper sheaf)**.** Fix $x \in X$, and $A \in \mathbf{Ab}$. The *skyscraper sheaf* is the sheaf defined as follows:

$$\mathscr{F}(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise;} \end{cases} \tag{1.5}$$

and $\rho_{U,V} := \mathrm{Id}_A$ if $x \in U \cap V$, and 0_A otherwise. It is a good exercise to check that this is a sheaf; we omit the details because it will follow effortlessly from the construction in example 1.2.8.

Example 1.1.12. The constant presheaf $\mathscr{F}$ in 1.1.5 is not generally a sheaf. More precisely, $\mathscr{F}$ is a sheaf if and only if every open set in X is connected: if every such U is connected, then for any covering U_i we can glue the $s|_{U_i}$ to a section a constant section on U, which is necessarily unique. Conversely, if an open set U is not connected, say $U = V_1 \cup V_2$ for clopen sets V_1 and V_2, we can choose $s_1 \in \mathscr{F}(V_1)$ to equal 0 and $s_2 \in \mathscr{F}(V_2)$ to be 1. These clearly don't glue to a constant section on U.

Example 1.1.13. We can fix the problem in the previous example by requiring that $\mathscr{F}$ be *locally constant* as in example 1.1.7. In particular, we form a presheaf of locally constant functions $f \colon X \to S$, that is, functions constant on connected open sets. One easily checks that this is in fact a sheaf.

One might ask if we can always turn a presheaf into a sheaf as above, and this turns out to be true:

Theorem 1.1.14. *The forgetful functor* $F\colon \mathbf{Sh}(X) \to \mathbf{pSh}(X)$ *admits a left adjoint. In other words, there exists functor* $S\colon \mathbf{pSh}(X) \to \mathbf{Sh}$ *such that* $\mathrm{Hom}_{\mathbf{pSh}}(\mathscr{F}, F\mathscr{G}) \cong \mathrm{Hom}_{\mathbf{Sh}}(S\mathscr{F}, \mathscr{G})$.

Proof. [KS13], Prop. 2.2.3. □

Definition 1.1.15. The functor $S\colon \mathbf{pSh} \to \mathbf{Sh}$ defined above is called the **sheafification functor**.

1.2 Morphisms of sheaves

Recall that a morphism of sheaves $\mathscr{F}$ and $\mathscr{G}$ over X is a natural transformation $\phi\colon \mathscr{F} \to \mathscr{G}$. Explicitly, it consists of a map $\phi_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ for every open $U \subseteq X$, such that the diagram

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\phi_V} & \mathscr{G}(V) \\ \downarrow{\scriptstyle \rho_{V,U}} & & \downarrow{\scriptstyle \rho'_{V,U}} \\ \mathscr{F}(U) & \xrightarrow{\phi_U} & \mathscr{G}(U) \end{array} \tag{1.6}$$

where ρ and ρ' are restriction maps, commutes. We say that a morphism $\phi\colon \mathscr{F} \to \mathscr{G}$ is *injective at the sections* (resp. *surjective at the sections*) if the corresponding maps $\phi_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ are all injective (resp. surjective).

Let us check that the definition of a sheaf indeed gives the kind of "local identification" we wanted:

Lemma 1.2.1. *Let* $\phi\colon \mathscr{F} \to \mathscr{G}$ *be a morphism of sheaves on* X*. If for every* $x \in X$ *the induced morphism* $\phi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ *is injective, then* ϕ *is injective at the sections. If moreover the maps* ϕ_x *are surjective, then* ϕ *is also surjective at the sections.*

Proof. Fix $s, t \in \mathscr{F}(U)$ such that $\phi_U(s) = \phi_U(t)$, and note that $(\phi_U(s))_x = \phi_x(s_x) = \phi_x(t_x) = (\phi_U(t))_x$ for all $x \in U$. Thus s and t are equal on some sufficiently small neighbourhood U_0 of x, and since this holds for every x, by condition (i) in the definition of a sheaf, we get that $s = t$.

Now, suppose further the stalk maps ϕ_x are surjective, and let $t \in \mathscr{G}(U)$. We want to find $s \in \mathscr{G}(U)$ such that $\phi_U(s) = t$. Fix $x \in U$. Then t_x has a preimage in $\mathscr{F}_x$, say s_x, so there exists some $U_{i(x)} \ni x$ and $s_{i(x)} \in \mathscr{F}(U_{i(x)})$ for which $\phi_{U_{i(x)}}(s_{i(x)}) = t|_{U_0}$. This being the case for every $x \in U$, we obtain a collection of sections $s_{i(x)} \in \mathscr{F}(U_{i(x)})$ where $\{U_{i(x)}\}$ covers U. We moreover see that $s_{i(x)}|_{U_{i(x)} \cap U_{i(y)}} = s_{i(y)}|_{U_{i(x)} \cap U_{i(y)}}$ since ϕ_x is injective. Therefore, by gluing the $s_{i(x)}$, we obtain a section $s \in \mathscr{F}(U)$ satisfying $\phi_U(s) = t$. □

Do note that surjectivity of ϕ_x for all x need not imply that ϕ is surjective at the sections; we require injectivity as well.

Example 1.2.2. Let $X = \mathbb{C}^\times$, and let $\mathscr{F}$ be the sheaf of holomorphic functions on X and $\mathscr{G}$ the sheaf of invertible holomorphic functions. Then the map $\alpha\colon \mathscr{F} \to \mathscr{G}$ defined by $\alpha_U(f) = \exp f$ is surjective at the stalks, since every holomorphic function has a locally defined logarithm, but it is not surjective at the sections: for example, the identity function is not in $\alpha_U(X)$ since this would require a globally defined logarithm on $\mathbb{C}^\times$, which we recall from complex analysis does not exist.

Theorem 1.2.3. *A morphism of sheaves $\phi\colon \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if all the induced morphisms $\phi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ are isomorphisms.*

Proof. Note first that if ϕ is an isomorphism, then it follows immediately from definition that each induced map ϕ_x for $x \in X$ is an isomorphism. Conversely, if all ϕ_x are isomorphisms, then by lemma 1.2.1, each map $\phi_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism, hence invertible. These assemble to a natural transformation $\psi\colon \mathscr{G} \to \mathscr{F}$, as it is is straightforward to see that an appopriate version of eq. (1.6) commutes. Then ψ is inverse to ϕ, proving our claim. □

One might expect that since the data of a basis for a topology on X is sufficient to determine the topology, a sheaf is uniquely determined by its values at basic open sets. This is indeed the the case:

Proposition 1.2.4. *Let X be a topological space with a basis $\mathscr{B}$, and let $\mathbf{U}(\mathscr{B})$ be the full subcategory of $\mathbf{U}(X)$ with objects given by those in $\mathscr{B}$. Suppose $\mathscr{F}$ is a presheaf on $\mathscr{B}$ which satisfies for any covering $V = \bigcup_i V_i$ where $V, V_i \in \mathscr{B}$ the following:*

(i) *If $s, t \in \mathscr{F}(V)$ and $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$;*
(ii) *For $s_i \in \mathscr{F}(V_i)$ satisfying $s_i|_{V_{i,j}} = s_j|_{V_{i,j}}$ for all $V_{i,j} \subseteq V_i \cap V_j$ in $\mathscr{B}$, there exists $s \in \mathscr{F}(V)$ such that $s|_{V_i} = s_i$ for all i.*

Then $\mathscr{F}$ extends uniquely to a sheaf on X.

Note that this is essentially the sheaf condition, but with $U_i \cap U_j$ replaced with $V_{i,j}$ in (ii) since $U_i \cap U_j$ is generally not a basic open set.

Proof. Let U be an open set in X, and define

$$\mathscr{F}(U) = \varprojlim_{\mathscr{B} \ni V \subseteq U} \mathscr{F}(V). \tag{1.7}$$

Explicitly, a section $s \in \mathscr{F}(U)$ is given by tuples $(s_V)_{V \subseteq U}$ satisfying $s_V|_W = s_W$ for all $W \subset V$. To see that this is indeed a sheaf, let $\{U_i\}$ cover U, and fix $s, t \in \mathscr{F}(U)$ satisfying $s|_{U_i} = t|_{U_i}$ for each i. We need to show that $s|_V = t|_V$ for any basic $V \subset U$. Since $\{U_i\}$ also covers V, we can refine this to a cover of basic open sets $\{V_{ij}\}$ by writing $U_i = \bigcup_{i,j} V_{i,j}$. Since $s|_{U_i} = t|_{U_i}$ by assumption and by definition of the inverse limit, $s|_{V_{i,j}} = t|_{V_{i,j}}$ for all $V_{i,j}$. By (i), we therefore have that $s|_V = t|_V$. The uniqueness criterion follows from uniqueness of the s_V. Finally, the newly constructed sheaf is unique up to isomorphism because any collection of isomorphisms $\mathscr{F}(V) \to \mathscr{F}'(V)$ for each $V \in \mathscr{B}$ induces isomorphisms $\mathscr{F}(U) \to \mathscr{F}'(U)$ by the universal property of the inverse limit. □

Another useful operation is that of gluing together locally defined sheaves.

Proposition 1.2.5. *Let $\{U_i\}$ be an open covering of a topological space X, suppose $\mathscr{F}_i$ is a sheaf on U_i and $f_{ij}\colon \mathscr{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathscr{F}_j|_{U_i \cap U_j}$ are isomorphisms satisfying the* cocycle conditions *$f_{ii} = \mathrm{Id}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all i, j, k. Then there exists a unique sheaf $\mathscr{F}$ on X and isomorphisms $g_i\colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$ such that $g_j = f_{ij} \circ g_i$ on $U_i \cap U_j$.*

Proof. Define $\mathscr{F}$ by $\mathscr{F}(U) = \amalg_i \mathscr{F}_i(U \cap U_i)/\sim$ where $s_i \sim s_j$ if and only if $s_i|_V = s_j|_V$ where $V := U_i \cap U_j \cap U$ for all pairs i, j. The cocycle conditions ensure that $\sim$ is an equivalence relation: in particular, symmetry follows from noting that $f_{ii} = \mathrm{Id} = f_{ji} \circ f_{ij}$. One easily checks that the presheaf $\mathscr{F}$ is in fact a sheaf. The natural projections $\mathscr{F} \to \mathscr{F}_i$ induce maps $g_i \colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$, which are isomorphisms by construction; finally the sheaf $\mathscr{F}$ is uniquely determined by the data of $\mathscr{F}_i$ as a consequence of theorem 1.2.3. □

Definition 1.2.6. The sheaf $\mathscr{F}$ constructed above is called the **gluing of the** $\mathscr{F}_i$ via the isomorphisms f_{ij}.

Given a continuous map $f\colon X \to Y$, we can transfer a sheaf on X to a sheaf on Y via f in a natural way:

Definition 1.2.7. Let $f\colon X \to Y$ be a continuous map and $\mathscr{F}$ a sheaf on X. Define $f_*\mathscr{F}$ to be the sheaf on Y determined by $f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V))$. This is called the **pushforward of $\mathscr{F}$ by f**.

The map $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ is also called the *direct image functor*; it is indeed a functor if we define $f_*(\phi)\colon f_*\mathscr{F} \to f_*\mathscr{G}$ in the natural way. Explicitly, $(f_*\phi)_V$ sends $s \in \mathscr{F}(f^{-1}(V))$ to $\phi_{f^{-1}(V)}(s) \in \mathscr{G}(f^{-1}(V))$.

Example 1.2.8. Let $x \in X$, and $\iota\colon \{x\} \to X$ be the inclusion map. Since $\{x\}$ is a topological space endowed with the discrete topology, we can define a sheaf $\mathscr{F}$ of, say, abelian groups on $\{x\}$ by $\mathscr{F}(\{x\}) = A$, $\mathscr{F}(\varnothing) = 0$. Then $\iota^*\mathscr{F}$ is precisely the skyscraper sheaf on X defined in example 1.1.11!

It is also possible to go in the reverse direction, although it requires some more work.

Definition 1.2.9. Let $f\colon X \to Y$ be continuous, and let $\mathscr{G}$ be a sheaf on Y. Then $f^{-1}\mathscr{G}$, called the **pullback of $\mathscr{G}$ by f**, is the presheaf defined by

$$f^{-1}\mathscr{G}(U) := \varinjlim_{V \subseteq f(U)} \mathscr{G}(V). \tag{1.8}$$

It is a straightforward, albeit slightly tedious exercise to show that $f^{-1}\mathscr{G}$ is in fact a sheaf.

Example 1.2.10. We can use sheaves to give a sleek definition of a manifold: Fix a dimension n, let M be a topological space (Hausdorff, second countable if we want) along with a covering $\{U_i\}$, and let $\mathscr{F}$ be the sheaf of continuous functions on M. Then M is a manifold if (and only if) for each i, the subsheaf $(U_i, \mathscr{F}|_{U_i})$ is isomorphic to $\mathbb{R}^n$ with the sheaf of continuous functions. Equipping $\mathbb{R}^n$ instead with the sheaf of C^k-functions, we obtain a C^k-manifold.

Example 1.2.11. While on the topic of manifolds, let $E \xrightarrow{\pi} M$ be a vector bundle on a manifold M. Then the functor $F\colon U \mapsto \Gamma(E, U) := \{s \in C(U) : \pi \circ s = \mathrm{Id}_U\}$ assigning to each open set the *sections of π over U* defines a sheaf of vector spaces. This is in fact very closely related to how sheaves were originally defined, see [Zar56].

1.3 Sheaves of modules

Recall from differential geometry that when we defined various vector bundles on a smooth manifold M, they had a natural structure of $C^\infty(M)$-modules. For example, given a vector field $X: M \to TM$ and $f \in C^\infty(M)$, we obtain a new vector field fX defined by $f(m)X_m \in T_mM$. It seems reasonable that this should be a local notion, and should serve as sufficient motivation for the following definition:

Definition 1.3.1. Let X be topological space, and $\mathscr{F}$ a sheaf of rings on X. A **sheaf of modules $\mathscr{M}$ over $\mathscr{F}$** is a sheaf of abelian groups such that for any open $U \subseteq X$, $\mathscr{M}(U)$ is an $\mathscr{F}(U)$-module, and respecting the following commutative diagram:

$$\begin{tikzcd} \mathscr{F}(V) \times \mathscr{M}(V) \arrow[r] \arrow[d] & \mathscr{M}(V) \arrow[d] \\ \mathscr{F}(U) \times \mathscr{M}(U) \arrow[r] & \mathscr{M}(U) \end{tikzcd} \tag{1.9}$$

where the horizontal arrows are the multiplication maps, and the vertical ones are restrictions.

Informally, we require the multiplication map to commute with restriction in a natural way. We will see a number of examples in chapter 2, so for now we are content with the following:

Example 1.3.2. Let $E \xrightarrow{\pi} M$ be a vector bundle over a smooth manifold. Then the sheaf of smooth real-valued functions on M is a sheaf of modules over the sheaf of sections Γ defined in example 1.2.11. Indeed, for any open $U \subseteq M$, we know that $\Gamma(U)$ is a C^∞-module, and multiplication commutes with restriction because, for any open $U \subseteq V$, $\sigma \in \Gamma(V)$ and $f \in C^\infty(V)$, we have $(f\sigma)|_U = f_U\sigma_U$ since they are pointwise equal on U.

Example 1.3.3. For each open $U \subseteq X$, let I_U be an ideal of $\mathscr{F}(U)$, and suppose the functor $\mathscr{I}: U \mapsto \mathscr{I}(U) = I_U$ is a sheaf, with restriction maps inherited from $\mathscr{F}$. This is called a *sheaf of ideals*, and we claim that it is a sheaf of modules:

Indeed, for $U \subseteq V$, $v \in \mathscr{F}(V)$ and $x \in \mathscr{I}(V)$ we have $\rho_{V,U}(vx) = \rho_{V,U}(v)\rho_{V,U}(x) = \rho_{V,U}(v)\rho'_{V,U}(x)$ where $\rho'_{V,U}$ is the map $I_V \to I_U$ induced by $\rho_{V,U}$. On the other hand, $\rho_{V,U}(vx) = \rho'_{V,U}(vx)$ since $vx \in I_U$, so the diagram does indeed commute.

Example 1.3.4. Note that $\mathscr{I}$ need not be a sheaf in general; for example, the ring of continuous functions on an open subset of $\mathbb{R}$ has an ideal consisting of bounded functions. But we saw in example 1.1.10 that the subpresheaf of the sheaf of continuous functions consisting of bounded functions is not a sheaf.

We define morphisms of sheaves of modules in the natural way:

Definition 1.3.5. Fix a sheaf of rings $\mathscr{F}$, and let $\mathscr{M}$ and $\mathscr{N}$ be sheaves of $\mathscr{F}$-modules over a topological space X. A **morphism of sheaves of $\mathscr{F}$-modules** is a morphism of sheaves $\phi\colon \mathscr{M} \to \mathscr{N}$ such that for any open $U \subset X$, $s \in \mathscr{F}(U)$ and $m \in \mathscr{M}(U)$ we have $\phi_U(sm) = s\phi_U(m)$.

Example 1.3.6. The inclusion map of a sheaf of ideals forms a morphism of sheaves of modules.

1.4 Locally ringed topological spaces

Following [LE06] and [EH06]. A special class of sheaves of rings that are ubiquitous in algebraic geometry are those for which the stalks have the structure of *local rings*, that is, rings with a unique maximal ideal.

Definition 1.4.1. A **locally ringed topological space** (or a locally ringed space for short) consists of a topological space X along with a sheaf of rings $\mathscr{O}_X$ such that each stalk $\mathscr{O}_{X,x}$ is a local ring, i.e. has a unique maximal ideal $\mathfrak{m}_x$. The sheaf $\mathscr{O}_X$ is called the **structure sheaf** of $(X, \mathscr{O}_X)$, and we call $\kappa(x) := \mathscr{O}_{X,x}/\mathfrak{m}_x$ the **residue field** of X at x.

Example 1.4.2. The sheaf of differentiable functions on a smooth manifold M is a ringed topological space; the stalks $\mathscr{O}_{X,x}$ then consists of differentiable functions which are defined on some neighbourhood of x, and it is straightforward to show that the unique maximal ideal of $\mathscr{O}_{M,x}$ consists of functions that vanish at x. The residue field of M at x is then precisely $\{f/g : g(x) \neq 0\}$, the field of ratios of smooth functions where the denominators do not vanish at x.

Example 1.4.3. Let $U \subseteq X$ be an open subset of a locally ringed space. Then $(U, \mathscr{O}_X|_U)$ is itself a locally ringed space, since $(\mathscr{O}_X|_U)_x = \mathscr{O}_{X,x}$.

One might wonder what a morphism of locally ringed spaces should be. Let us first consider the special case of smooth manifolds as above. Let M and N be smooth manifolds, and $\phi\colon M \to N$ a continuous map. One easily shows that this is a smooth map if and only if for every open $V \subseteq N$ and $f\colon V \to \mathbb{R}$, the pullback $\phi^* f := f \circ \phi$ is a smooth function on the open set $\phi^{-1}(V)$. In the language of sheaves, we require that f induces a morphism of sheaves given by $\mathscr{F}_N(V) \to \phi_* \mathscr{F}_M(V)$. In a more general setting, the continuous map ϕ does not canonically induce a morphism of locally ringed spaces, so this datum, namely $\phi^\#\colon \mathscr{F}_N \to \phi_* F$, needs to be specified separately. Now ϕ and $\phi^\#$ ought to be compatible somehow, and in line with the philosophy of sheaves being determined by stalks, the right notion turns out to be that the induced map from $\phi^\#$ on the stalks preserve maximal ideals in the sense that $\phi_x^{\#-1}(\mathfrak{m}_x) = \mathfrak{m}_{\phi(x)}$.

Definition 1.4.4. A **morphism of locally ringed topological spaces** consists of a continuous map $\phi\colon X \to Y$ and a morphism of sheaves $\phi^\#\colon \mathscr{O}_Y \to \phi_* \mathscr{O}_X$ such that for any $x \in X$, $\phi_x^{\#-1}(\mathfrak{m}_x) = \mathfrak{m}_{\phi(x)}$.

The following gives a simple recipe for constructing such morphisms:

Proposition 1.4.5 (Gluing of morphisms). *Let X and Y be locally ringed topological spaces, suppose $\{U_i\}$ is a covering of X and $f_i\colon U_i \to X$ a collection of morphisms of locally ringed spaces such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all pairs (i, j). Then there exists a unique morphism of locally ringed spaces $f = (f, f^\#)\colon X \to Y$ such that $f|_{U_i} = f_i$.*

Proof. From topology we know that the continuous maps f_i assemble to a continuous map $f\colon X \to Y$. Since each U_i is naturally a locally ringed space, we have morphisms of sheaves $f_i^\#\colon \mathscr{O}_Y \to (f_i)_* \mathscr{O}_X|_{U_i}$. We claim that these assemble to a sheaf map $f^\#\colon \mathscr{O}_Y \to f_* \mathscr{O}_X$. Indeed, note that there is a natural identification $\mathscr{O}_X(f^{-1}(U)) = \prod_i \mathscr{O}_X(f_i^{-1}(U))/\sim$, where $s_i \sim s_j$ if $s_i|_{f_i^{-1}(U) \cap f_j^{-1}(U)} = s_j|_{f_i^{-1}(U) \cap f_j^{-1}(U)}$. Since the maps $\mathscr{O}_Y(U) \to \mathscr{O}_X(f_i^{-1}(U))$ assemble to a map $\mathscr{O}_Y(U) \to \prod_i \mathscr{O}_X(f_i^{-1}(U))$ by $s \mapsto (f_i^\#(s))_i$ which factors through the quotient, we have a unique map $f^\#\colon \mathscr{O}_Y \to f_* \mathscr{O}_X$, as required. Since the condition on the stalks is a local one, this is naturally satisfied by $f^\#$. □

Chapter 2

Schemes

After a crisis of foundations in the beginning of the 20th century, algebraic geometry was put on firm footing through its reformulation in terms of schemes. While the ideas can arguably be traced back to Chevalley (cf. [GC04]), the theory was brought to fruition by Grothendieck in his seminal multi-volume treatise *Éléments de géométrie algébrique*, or EGA for short.

One guiding intuition behind scheme theory is to "globalise" rings – this is the view taken in [Mum13] – by patching together spectra of rings. By analogy, in example 1.2.10 we defined a manifold as a topological space equipped with a sheaf locally isomorphic to that of functions on $\mathbb{R}^N$. The moral of the story is that the passage from rings to schemes is analogous to passing from Euclidean subsets to an abstract manifold.

2.1 The affine scheme

An affine scheme is simply the prime spectrum of a ring (cf. section A.1) made into a locally ringed space. More precisely:

Definition 2.1.1. Let R be a ring, and consider $X := \operatorname{Spec} R$ equipped with the Zariski topology. Let $\mathscr{O}_X$ be the presheaf generated by $\mathscr{O}_X(D(f)) = R_f$ for any principal open set $D(f) \subseteq \operatorname{Spec} R$. The pair $(X, \mathscr{O}_X)$ is called the **affine scheme over** R.

Let us check that this does indeed form a locally ringed space.

Proposition 2.1.2. *Let $(X, \mathscr{O}_X)$ be an affine scheme. Then $(X, \mathscr{O}_X)$ is a locally ringed topological space.*

Proof. We did not specify what the functor $\mathscr{O}_X$ does to maps, so we will do that now: Suppose $D(f) \subseteq D(g)$ for some $f, g \in R$. Then by proposition A.2.9, $f^n = gr$ for some $r \in R$, $n \in \mathbb{N}$, and so g is invertible in R_f. Then we have a map $R_g \to R_f$ determined by $ag^{-m} \mapsto ar^m f^{-nm}$, which is easily checked to be a ring homomorphism. We will take this as our restriction map. If $D(f) = D(g)$, the map is evidently an isomorphism.

In light of proposition 1.2.4, we know that $\mathscr{O}_X$ is a sheaf provided it satisfies the corresponding sheaf conditions (i) and (ii) on the collection of principal open sets $\{D(f) : f \in R\}$. Note also that it suffices to consider coverings of $X = \operatorname{Spec} R$, since any basic open set $U = D(f)$ can be viewed as $\operatorname{Spec} R_f$ in a natural way, and finite such since $\operatorname{Spec} R$ is quasi-compact (cf. proposition A.1.12)

So, let $\{D(f_i)\}$ be a covering of $\operatorname{Spec} R$, and fix $s \in R$ such that $s|_{D(f_i)} = 0$ for all i. Proving that $s = 0$ is now equivalent to (i) after replacing s with $x - y$. Then $f_i^m s = 0$ for some $m \in \mathbb{N}$ sufficiently large, and since $D(f_i) \subseteq D(f_i^m)$, we can write $1 = \sum_{i=1}^n r_i f_i^m$. Multiplying by s now gives $s = \sum_{i=1}^n r_i f_i^m s = 0$, as required.

As for (ii), suppose $s_i \in D(f_i)$ for each i, with $s_i|_{D(f_i f_j)} = s_j|_{D(f_i f_j)}$. We want to define some $s \in R$ by analogy with partitions of unity on a paracompact manifold. In R_{f_i}, we can write $s_i = a_i f_i^k$ where $k \in \mathbb{N}$ is independent of i. By assumption there exists a sufficiently large $m \in \mathbb{N}$ such that $(f_i f_j)^m (f_j^k a_i - f_i^k a_j) = 0$, or equivalently for $l = m + k$, $b_i := f_i^k a_i$, that $f_j^l b_i = f_i^l b_j$. As in the previous part, write $1 = \sum_{i=1}^n r_i f_i^l$, and define $s := \sum_{i=1}^n r_i b_i$. Now

$$f_j^l s = \sum_{i=1}^n r_i f_j^l b_i = \sum_{i=1}^n r_i f_i^l b_j = b_j, \tag{2.1}$$

which shows that $s|_{D(f_j)} = s_j$, since a_j has the same image as b_j in R_{f_j} for any j.

Finally, to show that $\mathscr{O}_{X,x}$ is a local ring for any $x \in X$, fix a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ corresponding to x. By definition,

$$\mathscr{O}_{X,x} = \varinjlim_{D(f)\ni x} \mathscr{O}_X(D(f)) \cong \varinjlim_{f \notin \mathfrak{p}} R_f = R_{\mathfrak{p}}, \tag{2.2}$$

and since $R_{\mathfrak{p}}$ is a local ring whose unique maximal ideal is precisely the image of $\mathfrak{p}$ in the localisation, this concludes our proof. □

Since $\mathscr{O}_X(D(1)) = R_1 = R$, we also have the following:

Corollary 2.1.3. *We have* $\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) = R$.

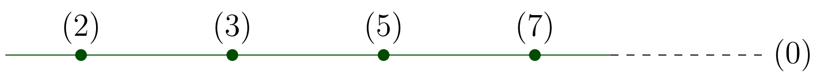
A *morphism of affine schemes* is simply a map of locally ringed topological spaces between two affine schemes. As such, affine schemes and their morphisms these assemble to a category denoted by $\mathbf{Sch}_{\mathrm{Aff}}$.

We next consider a few examples in order to get some intuition for this object.

Example 2.1.4. Let $X := \operatorname{Spec} k$, where k is a field. Then X consists of a single point, $*$, corresponding to the only prime ideal 0 of k, and by the preceding proposition $\mathscr{O}_{\operatorname{Spec} k}(*) = k$.

Example 2.1.5. Recall that a *discrete valuation ring* (DVR) is a principal ideal domain with a unique non-zero prime ideal. Let R be a DVR, and note that $X := \operatorname{Spec} R = \{0, \mathfrak{p}\}$, where $\mathfrak{p}$ is the unique prime ideal. The topology on X is then the Sierpinski topology, where $\mathfrak{p}$ is closed, while 0 is not.

Example 2.1.6. Letting $X := \operatorname{Spec} \mathbb{Z}$, we recall that as a set, $X = \{0, 2, 3, 5, \ldots\}$, and using lemma A.2.8 we readily see that $\kappa(p) = \mathbb{Z}_{(p)}/(p) = \mathbb{Z}/(p) \equiv \mathbb{F}_p$ for $p \neq 0$. For $p = 0$, we similarly see that $\kappa(0) = \mathbb{Q}$. We can visualise $\operatorname{Spec} \mathbb{Z}$ as the collection of primes in $\mathbb{Z}$, along with the "generic point" 0, which is dense, being contained in any open set.



It is worth keeping in mind that there is no "line" between two primes; however, the space is not topologically disconnected either, since any open set contains infinitely many primes.

Definition 2.1.7. Let R be a ring, and define **affine n-space** over R as $\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \ldots, x_n]$.

Example 2.1.8. Let k be a field, and $X := \mathbb{A}^1_k$ the *affine line*. Then each point in X corresponds to an irreducible polynomial $P \in \mathbb{A}^1_k$. In particular, if k is algebraically closed, then each irreducible polynomial is on the form $x - a$ for some $a \in k$, and via the identification $(x-a) \leftrightarrow a$ we obtain $\mathbb{A}^1_k$ that $\mathbb{A}^1_k$ as a set can is in bijection with $k \sqcup *$, where $*$ corresponds to the maximal ideal (0).

One might ask whether there are natural subobjects in $\mathbf{Sch}_{\mathrm{Aff}}$. Indeed, in the construction of the affine scheme we saw that showing the sheaf axioms for $\mathscr{O}_X(X)$ was just as easy as for $\mathscr{O}_X(D(f))$ for fixed $f \in R$. Indirectly, we proved the following:

Lemma 2.1.9. *Let $X = \operatorname{Spec} R$ be an affine scheme, and let $f \in R$. Then the open subset $D(f)$ equipped with the structure sheaf $\mathscr{O}_X|_{D(f)}$ is isomorphic to* $\operatorname{Spec} R_f$.

Since affine schemes are determined by the data of the underlying rings, the following proposition seems self-evident.

Proposition 2.1.10. *Let $\phi \colon R \to S$ be a ring homomorphism. Then ϕ induces a morphism of affine schemes $(\tilde{\phi}, \phi^\#) \colon$* $\operatorname{Spec} S \to \operatorname{Spec} R$.

Proof. Recall that $\tilde{\phi} \colon \operatorname{Spec} S \to \operatorname{Spec} R$ is a continuous map with respect to the Zariski topology; we ought to define a map $\phi^\# \colon \mathscr{O}_{\operatorname{Spec} R} \to \tilde{\phi}_* \mathscr{O}_{\operatorname{Spec} S}$. In light of proposition 1.2.4 it suffices to define $\phi^\#$ on the principal open sets and show that it is compatible with restriction maps. Note first that $\tilde{\phi}_* \mathscr{O}_{\operatorname{Spec} S}(D(f)) = \mathscr{O}_{\operatorname{Spec} S}(D(\phi(f)))$ for any $f \in R$. Recall that the map $\phi \colon R \to S$ induces a map of the localisations $\phi_f \colon R_f \to S_{\phi(f)}$ by $\phi_f(a/f^\alpha) = \phi(a)/\phi(f)^\alpha$, which is a well-defined map of rings. Via this, we have natural maps $\mathscr{O}_{\operatorname{Spec} R}(D(f)) = R_f \to S_{\phi(f)} = \tilde{\phi}_* \mathscr{O}_{\operatorname{Spec} S}(D(\phi(f)))$ induced by ϕ, which we furthermore easily check to be compatible with restrictions. This defines the sheaf map $\phi^\# \colon \mathscr{O}_{\operatorname{Spec} R} \to \tilde{\phi}_* \mathscr{O}_{\operatorname{Spec} S}$.

Finally, we need to check that for any $\mathfrak{p} \in \operatorname{Spec} S$, we have $\phi^\#_{\mathfrak{p}}(\mathfrak{m}_{\tilde{\phi}(\mathfrak{p})}) \subseteq \mathfrak{m}_{\mathfrak{p}}$. But note that for any $\mathfrak{p} \in \operatorname{Spec} R$, the map $\phi^\#_{\mathfrak{p}} \colon (\tilde{\phi}_* \mathscr{O}_{\operatorname{Spec} S})_{\mathfrak{p}} \to O_{\operatorname{Spec} R, \mathfrak{p}} = R_{\mathfrak{p}}$ is precisely the canonical map $S_{\tilde{\phi}(\mathfrak{p})} \to R_{\mathfrak{p}}$, which we know (if we do not, it is a nice exercise to check) to be a homomorphism of local rings. $\square$

Theorem 2.1.11. *There is an equivalence of categories* $\mathbf{Sch}_{\mathrm{Aff}} \leftrightarrow \mathbf{Ring}^{op}$ *given by $X \mapsto \mathscr{O}_X(X)$.*

Proof. The other direction is of course given by $R \mapsto \operatorname{Spec} R$. By the above discussion, it suffices to show that every morphism of affine schemes arises from a ring map. But it is straightforward to show that given a morphism of affine schemes $\phi \colon X \to Y$, this equals the map of schemes induced by $\phi^\#_Y \colon \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)$ from proposition 2.1.10. See [LE06], Lemma 3.3.23 for details. $\square$

2.2 General schemes

We are finally ready to define a general abstract scheme. It is useful to keep in mind the definition of a manifold – and particularly the one given in example 1.2.10 – for intuition.

Definition 2.2.1. A **scheme** is a locally ringed topological space which is locally isomorphic to an affine scheme. More precisely, it consists of a locally ringed space $(X, \mathscr{O}_X)$ along with a covering $\{U_i\}$ and corresponding rings $\{R_i\}$ such that $(U_i, \mathscr{O}_X|_{U_i}) \cong (\operatorname{Spec} R_i, \mathscr{O}_{\operatorname{Spec} R_i})$.

Whenever unambiguous, we will identify X_i with $\operatorname{Spec} R_i$. Again, a morphism of schemes is a morphism of the underlying locally ringed topological spaces. We assemble schemes and their morphisms into a category denoted **Sch**.

Example 2.2.2. Affine schemes are schemes.

Example 2.2.3. Let $X = \operatorname{Spec} \mathbb{Z}[x]$ and $U := D(x) \cup D(p) \subset \mathbb{Z}[x]$ for some prime $p \in \mathbb{Z}$. Intuitively, U looks like $\mathbb{Z}[x]$ with the closed point (x, p) removed. Then $(U, \mathscr{O}_X|_U)$ is evidently a scheme; on the other hand, noting that $\mathscr{O}_X(U) = \mathscr{O}_X(X)$ and appealing to theorem 2.1.11, this cannot be an affine scheme.

We will give more examples once we have examined the properties of schemes more closely.

Definition 2.2.4. Let $U \subset X$ be an open subset of scheme. Then $(U, \mathscr{O}_{X|U})$ – or simply U, if there is no room for confusion – is called an **open subscheme** of X.

Proposition 2.2.5. *An open subscheme is a scheme.*

Proof. Let $Y \subset X$ be an open subscheme, and let $\{X_i\}$ be an affine open covering of X. Then $\{X_i \cap Y\}$ is an open covering of Y, and we can write each $X_i \cap Y$ as a union of principal open sets. By lemma 2.1.9, these are themselves affine schemes, so we have obtained an affine open covering of Y. □

In the same way as we can consider the category of smooth bundles over a manifold, there is a natural notion of schemes over a specified scheme.

Definition 2.2.6. Let S be a scheme. A **scheme over** S, or an S**-scheme**, is a scheme X along with a map $\pi : X \to S$. The map π is called the **structural morphism** of X. Given S-schemes $X \xrightarrow{\pi} S$ and $X' \xrightarrow{\pi'} S$, a **morphism of** S**-schemes** $\phi : X \to X'$ is a map of schemes satisfying $\pi = \pi' \circ \phi$.

In the case where $S = \operatorname{Spec} k$ for some field k, convention dictates that we drop the "Spec" and simply say "k-scheme". The category of S-schemes is denoted by **Sch**$/S$. While defining a new category for every scheme might seem unnecessary, it is actually quite useful. For example, in algebraic geometry we are frequently interested in studying complex curves. Then it seems reasonable to expect that $\operatorname{Spec} \mathbb{C}$ has trivial automorphism group, since it consists of a single point; however, in **Sch** we have that $\operatorname{Aut}(S) \cong \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$, a very large and complicated object. In **Sch**$/\operatorname{Spec} \mathbb{C}$, on the other hand, we have $\operatorname{Aut}(\operatorname{Spec} \mathbb{C}) = *$, as expected.

Example 2.2.7. Any R-algebra A is an R-scheme: by proposition 2.1.10 the canonical map $R \to A$ induces a morphism of schemes $\pi : \operatorname{Spec} A \to \operatorname{Spec} R$.

Example 2.2.8. As a particular case of the previous example, since any ring R admits a natural map $\mathbb{Z} \to R$ determined by $1 \mapsto 1_R$, every affine scheme is a $\operatorname{Spec}\mathbb{Z}$-scheme. More generally, by writing X as a union of affine schemes with corresponding maps into $\operatorname{Spec}\mathbb{Z}$ and gluing via proposition 1.4.5, we see that every scheme is naturally a $\operatorname{Spec}\mathbb{Z}$-scheme.

Example 2.2.9. If $X \xrightarrow{\pi} S$ is an S-scheme and $U \subset X$ an open subscheme, then U is naturally an S-scheme as well, by $\pi|_U : U \to S$.

Definition 2.2.10. A morphism of schemes $f : X \to Y$ is an **open immersion** if the underlying map of topological spaces is an homeomorphism onto an open subset of Y, and f is an isomorphism at the level of stalks.

Example 2.2.11. The inclusion of an open subscheme $Z \to X$ is an open immersion.

Recall that in complex analysis, we define our first non-trivial complex manifold $\mathbb{P}^1_{\mathbb{C}}$ – the Riemann sphere – as $\mathbb{C} \sqcup \{\infty\}$ with the charts z on $\mathbb{C}$ and $1/z$ on $\mathbb{C}^\times \sqcup \{\infty\}$. It should come as no surprise that this has an algebro-geometric analogue,[1] which will be the first example of the following very general construction:

Proposition 2.2.12 (Gluing of S-schemes)**.** *Let X_i be a collection of S-schemes, and for each pair of indices (i,j), an open subscheme X_{ij} and an isomorphism of S-schemes $f_{ij} : X_{ij} \to X_{ji}$ subject to the following conditions:*

(i) $f_{ii} = \mathrm{Id}_{X_i}$;
(ii) $f_{ij} = f_{ji}^{-1}$ *and* $f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$;
(iii) $f_{ik} = f_{jk} \circ f_{ij}$ *on* $X_{ij} \cap X_{ik}$.

Then there exists an S-scheme X, unique up to isomorphism, with a covering of S-subschemes isomorphic to $\{X_i\}$ such that the maps f_{ij} on X_{ij} correspond to the identity map in the image of X_{ij} in X.

It is worth pointing out that conditions (i)–(iii) bear close similarity to the construction of a vector bundle from the transition maps.

Proof. Define $X := \amalg_i X_i / \sim$ where $x_i \sim x_j$ whenever $x_i \in X_i$, $x_j \in X_j$ and $x_j = f_{ij}(x_i)$. Conditions (i)–(iii) show that this determines an equivalence relation. We then have natural inclusions $X_i \xrightarrow{g_i} X$ satisfying $g_i = g_j \circ f_{ij}$ on $X_i \cap X_j$. By proposition 1.2.5, we can glue together the structure sheaves $U_i := g_{i*}\mathscr{O}_{X_i}$ along the morphisms $\{f_{ij}\}$, yielding a ringed topological space $(X, \mathscr{O}_X)$, any stalk $O_{X,x}$ being a local ring because we can find $X_i \ni x$ with $\mathscr{O}_{X_i,x} \cong \mathscr{O}_{X,x}$. Moreover, by considering affine open coverings U_{ij} of each X_i, the images of $\{U_i\}$ in X form an affine open covering, so X is a scheme. From the maps g_i we obtain isomorphisms $X_i \cong U_i$, and if $\pi_i : X_i \to S$ are the structural morphisms, we can glue $\pi_i \circ g_i^{-1}$ to yield a scheme morphism $X \to S$. Finally, X is unique up to isomorphism by construction. □

Of course, by taking $S = \operatorname{Spec}\mathbb{Z}$, we can glue any schemes together without paying attention to structure morphisms.

[1] The charts are, after all, given by rational functions!

Example 2.2.13. Fix a ring R, and define $X = \operatorname{Spec} R[x/y]$ and $Y = \operatorname{Spec} R[y/x]$ considered as subrings of $A := R[x, y, x^{-1}, y^{-1}]$. We then have a natural isomorphism $f\colon X \to Y$ induced by the ring map $y/x \mapsto x/y$, cf. proposition 2.1.10. Let us define the gluing data: set $U = D(x/y) \subset X$ and $V = D(y/x) \subset Y$, and let $f\colon U \to V$ be the isomorphism induced by the equality $R[y/x]_{y/x} \to R[x/y]_{x/y}$ inside A. Conditions (i)–(iii) of proposition 2.2.12 now hold trivially, so we obtain a scheme, denoted by $\mathbb{P}^1_R$.

In the case of $R = \mathbb{C}$, this ought to remind us of projective coordinates on the Riemann sphere: there we have coordinates $[x{:}y]$, and pass to charts by fixing x or y to be zero or non-zero. Of course, we can generalise this construction:

Example 2.2.14. Fix a ring R, and let A be the localisation

$$A := R[x_1, \dots, x_n]_{(x_1,\dots,x_n)} = R[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}].$$

As subrings of A, we take $A_i := R[x_1x_i^{-1}, \dots, x_nx_i^{-1}]$, and set $X_i := \operatorname{Spec} A_i$ and $X_{ij} := D(x_jx_i^{-1}) \subseteq X_i$. Since $\mathcal{O}_{X_i}(X_{ij}) = (A_i)_{(x_jx_i^{-1})} = R[x_1x^{-1}, \dots x_nx_i^{-1}, x_1x_j^{-1}, \dots, x_nx_j^{-1}] = \mathcal{O}_{X_j}(X_{ji})$, we have gluing isomorphisms $X_{ij} \to X_{ji}$ which clearly satisfy the cocycle conditions since they are induced by the identity map.

Definition 2.2.15. The scheme defined in the preceeding example is called **projective n-space over** R, denoted by $\mathbb{P}^n_R$.

2.3 Properties of schemes and the fibre product

There are many conditions one might impose on schemes in order to exclude pathologies. Here we consider a few of them.

Definition 2.3.1. An **affine variety** over a field k is an affine scheme $\operatorname{Spec} A$ where A is a finitely generated k-algebra.

Fix an algebraically closed field k, and let V be an affine variety in the classical sense, that is, the zero-locus of a collection of polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$. One version of Hilbert's Nullstellensatz states that there is a bijective correspondence between affine varieties in k^n and radical ideals in $k[x_1, \dots, x_n]$ (see e.g. Thm. 4.7 in [CLO10]). Explicitly, the datum of an affine variety is equivalent to that of the radical ideal $\mathfrak{a} := \sqrt{(f_1, \dots, f_m)} \leq k[x_1, \dots, x_n]$. But by theorem 2.1.11 this uniquely determines a scheme $X := \operatorname{Spec} R$, where $R := k[x_1, \dots, x_n]/\mathfrak{a}$. Note that R is a finitely generated k-algebra because $k[x_1, \dots, x_n]$ is, so X is indeed an affine variety in the scheme-theoretic sense.

Now recall that for a subfield $L \subseteq k$, an *L-rational point on V* is a tuple $(a_1, \dots, a_n) \in L^n$ such that

$$f_1(a_1, \dots, a_n) = \cdots = f_m(a_1, \dots, a_n) = 0. \tag{2.3}$$

Now there is a bijection between the set of L-rational points on V, $V(L)$, and the set $\operatorname{Hom}_k(R, L)$. In terms of schemes, by theorem 2.1.11 this is in bijection with $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, \operatorname{Spec} R)$. Motivated by this, we define the following:

Definition 2.3.2. Let X and Y be S-schemes. The set $\operatorname{Hom}_S(X, Y)$ is called the set of **Y-points of** X, and we denote it by $X(Y)$.

It is important to note this depends on the choice of structure morphisms $X, Y \to S$. In the "classical case" where Y and S are spectra of rings, we revert to the terminology "L-rational points", and write $X(L)$.

Definition 2.3.3. Let X be a scheme. Given a connected subset $Z \subseteq X$, a point $\eta \in Z$ is called a **generic point** if $\{\eta\}$ is dense in Z.

Recall that a topological space X is *reducible* if $X = X_1 \sqcup X_2$ for closed proper subsets X_1 and X_2, and *irreducible* otherwise.

Proposition 2.3.4. *There is a bijection*

$$\{\textit{generic points of } X\} \leftrightarrow \{\textit{irreducible closed subsets of } X\} \tag{2.4}$$

given by $\eta \mapsto \overline{\{\eta\}}$.

Proof. [LE06], Prop. 2.4.12a). □

It is tempting to define $\pi_0(X)$ as the collection of irreducible closed subsets of X; the proposition above then gives a bijection between the set $\pi_0(X)$ and the set of generic points of X.

If a certain adjective applies to all the local rings of a scheme, we are prone to apply the adjective to the scheme itself. For example:

Definition 2.3.5. A scheme X is **reduced** if all the local rings $\mathcal{O}_{X,x}$ are reduced; that is, contain no nilpotent elements.

Definition 2.3.6. A scheme X is **integral** if for every open subset $U \subseteq X$, $\mathcal{O}_X(U)$ is an integral domain.

Recall that a ring R is *integrally closed* (in its fraction field K) if any $r \in K$ which is the root of some monic polynomial $P \in R[x]$ is also contained in R.

Definition 2.3.7. A scheme X is **normal** if all the stalks $\mathcal{O}_{X,x}$ are integrally closed domains.

Later we will need the following definition:

Definition 2.3.8. A morphism of schemes $f : X \to Y$ is an **affine morphism** if there exists an affine open cover $\{U_i\}$ of Y such that each $f^{-1}(U_i)$ is an affine subscheme of X.

Example 2.3.9. The canonical morphism $X \to \operatorname{Spec} \mathbb{Z}$ is affine if and only if X is affine.

The fibre product

Recall that products in **Ring** often do not preserve nice properties of the constituents, such as being an integral domain – for example, $(0, 1) \cdot (1, 0) = (0, 0) \in \mathbb{Z} \times \mathbb{Z}$ – and the same is true for **Sch**. Trying to define the product of two schemes X and Y in the naïve way as the product of the underlying topological spaces equipped with the product sheaf runs into the problem that $\mathcal{O}_{X\times Y,(x,y)} = \mathcal{O}_{X,x} \times \mathcal{O}_{X,y}$: the product of local rings is not local.[2]

In lieu of this, we can define the following more general notion:

[2] If R has a unique maximal ideal $\mathfrak{m}$, then $R \times R$ has two maximal ideals $R \times \mathfrak{m}$ and $\mathfrak{m} \times R$.

Definition 2.3.10. Let $\mathbf{C}$ be an arbitrary category, let X and Y be objects in Y, and suppose $f\colon X \to Z$ and $g\colon Y \to Z$ are morphisms. Then the **fibre product** of X and Y with respect to f and g, or simply the *fibre product of X and Y* is a solution $(X \times_Z Y, p, q)$ to the following universal mapping problem: for any pair of morphisms $\phi\colon A \to X$ and $\psi\colon A \to Y$ such that $f\phi = g\psi$, there exists a unique $h\colon A \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[dr, dashed, "h"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\psi"'] & & \\ & X \times_Z Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd} \tag{2.5}$$

Example 2.3.11. The fibre product of $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ in **Set** is given by $\{(x, y) \in X \times Y : f(x) = g(y)\}$ and the natural projections onto X and Y.

Theorem 2.3.12. *Fibre products exist in* **Sch**, *and are unique up to unique isomorphism.*

To prove this, it is sensible to start with the case where all the schemes are affine.

Lemma 2.3.13. *Given rings R, S and A, we have* $\operatorname{Spec} R \times_{\operatorname{Spec} A} \operatorname{Spec} S \cong \operatorname{Spec}(R \otimes_A S)$.

Proof. This follows immediately from theorem 2.1.11 by noting that the universal property of fibre products in $\mathbf{Sch}_{\mathrm{Aff}}$ is dual to that of tensor products in **Ring**. This also proves that the fibre product is unique up to unique isomorphism. □

Proof of theorem 2.3.12. Suppose first that S and Y are affine, and that X is an arbitrary scheme covered by affine open schemes $X_i = \operatorname{Spec} R_i$. Then the fibre product $(X_i \times_S Y, p_i, q_i)$ exists for each i, and for every pair i and j we have an isomorphism $p_i^{-1}(X_i \cap X_j) \cong (X_i \cap X_j) \times_S Y$; this gives isomorphisms $f_{ij} : p_i^{-1}(X_i \cap X_j) \to p_j^{-1}(X_i \cap X_j)$. By uniqueness of the isomorphism $(X_i \cap X_j \cap X_k) \times_S Y \cong p^{-1}(X_i \cap X_j \cap X_j)$, we see that $\{f_{ij}\}$ satisfy the cocycle conditions in proposition 2.2.12, so the $X_i \times_S Y$ glue to a unique S-scheme, say, W. By viewing W as a gluing of X-schemes (resp. Y-schemes) through the maps p_i (resp. q_i), we see that these assemble to morphisms $p\colon W \to X$ and $q\colon W \to Y$.

We need to check that (W, p, q) satisfies the universal property of fibre products in **Sch**: let $\phi\colon A \to X$ and $\psi\colon A \to Y$ be morphisms in **Sch** such that $f\psi = g\phi$. Define $\phi_i : f^{-1}(X_i) \to X_i$ and $\psi_i := \psi|_{\phi^{-1}(X_i)}$, and apply the universal property of each $X_i \times_S U$, giving a collection of maps $h_i : A \to X_i \times_S U$ such that

$$\begin{tikzcd} \phi^{-1}(X_i) \arrow[dr, dashed, "h_i"] \arrow[drr, bend left, "\phi_i"] \arrow[ddr, bend right, "\psi_i"'] & & \\ & X_i \times_Z Y \arrow[r, "p_i"] \arrow[d, "q_i"] & X_i \arrow[d, "f_i"] \\ & Y \arrow[r, "g"] & Z. \end{tikzcd} \tag{2.6}$$

Now using proposition 1.4.5 we can glue together h_i to a unique map h, which is readily seen to make the corresponding diagram eq. (2.5) commute.

Next, let us remove that assumption that Y be affine; let Y_i be an affine covering, and construct the fibre products $X \times_S Y_i$ by the above method, noting that the fibre product is symmetric. In a similar way we can glue these, and by a similar argument as above check that $X \times_S Y$ is indeed a fibre product.

Finally, suppose S is no longer affine, and let us cover it with S_i. Letting $f : X \to S$ and $g : Y \to S$ as before be the structure morphisms, we can define $X_i := f^{-1}(S_i)$, $Y_i := g^{-1}(S_i)$ and form the fibre products $(X_i \cap X_j) \times_{S_i} (Y_i \cap Y_j)$. Note that by uniqueness of solutions to universal mapping problems, these are not only fibre products over S_i, but must also equal the fibre products over S. By gluing the $X_i \times_S Y_i$ along the natural inclusions of $(X_i \cap X_j) \times_S (Y_i \cap Y_j)$, we obtain a scheme which one easily checks satisfies the universal property. $\square$

When we take the fibre product of schemes X and Y over an affine scheme Spec R, we are sometimes prone to drop the "Spec" and simply write $X \times_R Y$.

Example 2.3.14. For any ring R and positive integers $n, m \in \mathbb{N}$, we have

$$\mathbb{A}^n_R \times_R \mathbb{A}^m_R \cong \operatorname{Spec} R[x_1, \dots, x_n] \otimes_R R[y_1, \dots, y_m] \cong \operatorname{Spec} R[z_1, \dots, z_{n+m}] = \mathbb{A}^{n+m}_R.$$

The universal property gives many desirable properties of the fibre product for free:

Proposition 2.3.15. *Let X, Y and Z be S-schemes. Then*

(i) $X \times_S S = X$;
(ii) $X \times_S Y = Y \times_S X$;
(iii) $(X \times_S Y) \times_S Z = X \times_S (Y \times_S Z)$.

Proof. Here "=" means that the constituents are isomorphic via a canonical isomorphism. These all follow from showing that the right and left hand sides both satisfy the universal property of fibre products, hence are related by a unique isomorphism by the standard argument. We omit the details. $\square$

Definition 2.3.16. Let $f : X \to Y$ be a morphism of schemes, and construct the fibre product $X \times_Y X$ over the identity map $X \to X$ as in the diagram:

$$\begin{tikzcd} X \arrow[dr, dashed, "\Delta_{X/Y}"] \arrow[drr, bend left, "\mathrm{Id}_X"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\ & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

The morphism $\Delta_{X/Y} : X \to X \times_Y X$ is called the **diagonal morphism of X with respect to ϕ.**

The diagonal morphism is the scheme-theoretic analogue of the diagonal $\Delta : X \to X \times X$ where $\Delta(x) = (x, x)$ of a topological space X.

Proposition 2.3.17. *The diagonal morphism $\Delta_{X/Y}$ is an immersion.*

Proof. Fix affine open $V \subset Y$ and $U \subset f^{-1}(V)$. Then $U \times_V U$ is affine open in $X \times_Y X$. By using proposition 1.4.5, it suffices to show that $U \to U \times_V U$ is a closed immersion for any given U and V. But in this case, when passing to rings the diagonal map corresponds to the multiplication map $R \otimes_{R'} R \to R$, $r \otimes s \mapsto rs$. This is injective, and so $R \cong R \otimes_{R'} R/\mathfrak{a}$ for some ideal $\mathfrak{a} \leq R \otimes_{R'} R$. It is straightforward to show algebraically that $\operatorname{Spec} R \otimes_{R'} R/\mathfrak{a} \cong V(\mathfrak{a})$, which is closed, proving our claim. $\square$

Recall that the topological diagonal $\Delta(X)$ is closed if and only if X is a Hausdorff. A scheme X will usually usually not Hausdorff: if X has a generic point η, then there is no way of separating η from a point x in its closure by open sets. The right analogue turns out to be defined precisely in terms of the diagonal morphism:

Definition 2.3.18. Let $f : X \to Y$ be a morphism of schemes. If $\Delta_{X/Y}$ is a closed map, then we say that f is **separated**.

The fibre product also allows us to settle the problem "how to turn an S-scheme into a S'-scheme?" through the following construction:

Definition 2.3.19. Let X and S' be S-schemes, and let $(X \times_S S', p, q)$ be their fibre product. Regarding $X \times_S S'$ as an S'-scheme with q as its structure morphism is called **base change** by the map $S' \to S$.

Example 2.3.20. For any ring R, one easily checks that $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R \cong \mathbb{P}^n_R$.

However, for base change to be a nice way of passing between categories, we need to be able to transfer maps.

Proposition 2.3.21. *Let $X \to Y$ be a morphism of S-schemes, and suppose $S' \to S$ is also a scheme over S. Then we have a natural map $X \times_S S' \to Y \times_S S'$, and base change is in fact functorial.*

Proof. If all schemes are affine, then we can simply take $X \times_S S' \to Y \times_S S'$ to be the map $f \otimes \mathrm{Id}$. In the general case, we can glue the morphism using proposition 1.4.5. From this and uniqueness of gluing, functoriality is immediate. $\square$

Corollary 2.3.22. *Base change gives a functor* $\mathbf{Sch}/S \to \mathbf{Sch}/S'$.

Definition 2.3.23. We say that a property P is **stable under base change** if the property holding true for an S-scheme $X \to S$ implies it holds true for $X \times_S S'$ formed by base-changing via $S' \to S$.

Definition 2.3.24. Let $\phi\colon X \to Y$ be a morphism of schemes, and fix $y \in Y$. The **fibre of** f **at** y, is by definition $X_p := \mathcal{O}_{X,x} \times_Y \operatorname{Spec} \kappa(y)$, where $\kappa(y) := \mathcal{O}_{Y,y}/\mathfrak{m}_y$ is the residue field at y in Y, and the map $\kappa(y) \to Y$ is given by $* \mapsto y$.

Example 2.3.25. An alternative way to view an S-scheme $X \to S$ is as a *family of schemes* parameterised by the points of S, since we have a correspondence of schemes $s \leftrightarrow X_s$ between points $s \in S$ and fibres X_s of the structure morphism.

Various flavours of finite

As in commutative algebra, finiteness conditions are frequently quite useful.

Definition 2.3.26. A scheme is **Noetherian** if it admits a finite affine open covering $\{X_i\}$ such that each $\mathcal{O}_{X_i}(X_i)$ is a Noetherian ring.

Example 2.3.27. Affine n-space $\mathbb{A}^n_R$ and projective n-space $\mathbb{P}^n_R$ are clearly Noetherian, as is any affine scheme.

We say that a ring homomorphism $R \to S$ is *of finite presentation* if S is isomorphic to a quotient of $R[x_1, \dots, x_n]$ for some $n \in \mathbb{N}$. It is *of finite type* if there exists a surjection of R-algebras $R[x_1, \dots, x_n] \to S$.

Definition 2.3.28. A morphism of schemes $f: X \to Y$ is **locally of finite presentation (resp. finite type)** if for any $x \in X$, there exists an affine open neighbourhood $U \subseteq X$ of x and affine open $V \supset f(U)$ such that the induced map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation (resp. finite type).

Note that being locally of finite presentation implies being of locally finite type.

Example 2.3.29. If $X = \operatorname{Spec} R[x_1, \dots, x_n]/I$ for some $I \leq R[x_1, \dots, x_n]$, then the canonical map $X \to \operatorname{Spec} R$ is of finite presentation. Indeed, we can take X to be the affine open set containing any $x \in X$, and the ring morphism $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(\operatorname{Spec} R) = R \to \mathcal{O}_X(X) = R[x_1, \dots, x_n]/I$ is clearly of finite presentation.

A ring homomorphism $\phi: R \to S$ is *finite* if the action $(r, s) \mapsto \phi(r)s$ makes S into a finite R-module.

Definition 2.3.30. A morphism of schemes $f: X \to Y$ is **finite** if it is affine, and if for any affine open $V \subset Y$ and $U = f^{-1}(V)$, the induced ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is finite.

Proposition 2.3.31. *The properties of being finite, of finite type and of finite presentation are individually stable under composition and base change.*

Proof. See [GW10], appendix C. □

2.4 A menagerie of schemes

Definition 2.4.1. An **arithmetic scheme** is a scheme of finite type over $\operatorname{Spec} \mathbb{Z}$.

We already described $\operatorname{Spec} \mathbb{Z}$ in example 2.1.6, which is certainly an arithmetic scheme; a less trivial example is that of the Gaußian integers $\mathbb{Z}[i]$.

Example 2.4.2. Recall that the prime elements of $\mathbb{Z}[i]$ are given by

(i) primes $p \in \mathbb{Z}$ where $p \equiv 3 \pmod 4$,
(ii) $n + mi$ if $p := n^2 + m^2$ is a prime with $p \equiv 1 \pmod 4$,
(iii) $1 + i$.

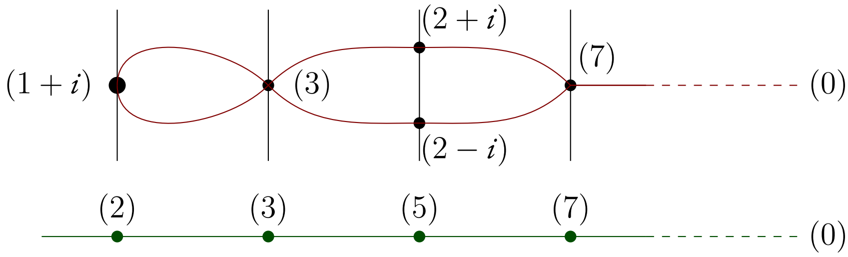
A proof of this can be found in [NS13], Thm. 1.4. To study the geometry of $\operatorname{Spec}\mathbb{Z}[i]$, let us consider the fibres under the canonical map ϕ into $\operatorname{Spec}\mathbb{Z}$. Fix a prime $(p) \in \operatorname{Spec}\mathbb{Z}$. Then

$$(\operatorname{Spec}\mathbb{Z}[i])_{(p)} = \operatorname{Spec}\mathbb{Z}[i] \times_{\mathbb{Z}} \operatorname{Spec}\kappa(p) = \operatorname{Spec}(\mathbb{Z}[i] \otimes \mathbb{F}_p) = \operatorname{Spec}\mathbb{F}_p[i],$$

and consider first the case where $p = 2$. Since $\mathbb{F}_p[i] \cong \mathbb{F}_p[x]/(x^2+1)$, this ring has four elements. But via the automorphism $x \mapsto x+1$, we see that $\mathbb{F}_2[i] \cong \mathbb{F}[x]/x^2$, so the fibre of 2, which consists of only the point $(1+i)$, is a *fat point*, since the fibre is not a field.

Taking $p \cong 3 \pmod 4$, we claim that the fibre of (p) is a field. Indeed, x^2+1 is irreducible in $\mathbb{F}_p[x]$, hence generates a maximal ideal, so $\mathbb{F}_p[x]/(x^2+1) \cong \mathbb{F}_{p^2}$. On the other hand, if $p \equiv 1 \pmod 4$, then x^2+1 is not irreducible over $\mathbb{F}_p$, but decomposes as the product of two linear factors $P_1(x)$ and $P_2(x)$. Then we have a corresponding decomposition of the fibre, as $\mathbb{F}_p[x]/p_1(x) \times \mathbb{F}_p[x]/p_2(x) \cong \mathbb{F}_p \times \mathbb{F}_p$.

We can draw the picture as follows:



We draw the dot at $(1+i)$ slightly thicker to signify that the fibre of 2 is "singular". This gives a geometric interpretation of the statement from algebraic number theory that 2 viewed as an integer ramifies in $\mathbb{Z}[i]$: notice the geometric likeness to covering maps. Note that we have an action of the Galois group $\operatorname{Gal}(\mathbb{Q}[i]/\mathbb{Q})$ acting on each fibre. Of course, $\operatorname{Gal}(\mathbb{Q}[i]/\mathbb{Q})$ is generated by the automorphism $z \mapsto \bar{z}$ which sends $2+i$ to $2-i$, 3 to itself, and so on.

Example 2.4.3. Consider now $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x]$. Recall that a prime ideal $\mathfrak{p} \in \operatorname{Spec}\mathbb{Z}[x]$ takes one of the following forms:

(i) $\mathfrak{p} = (0)$;

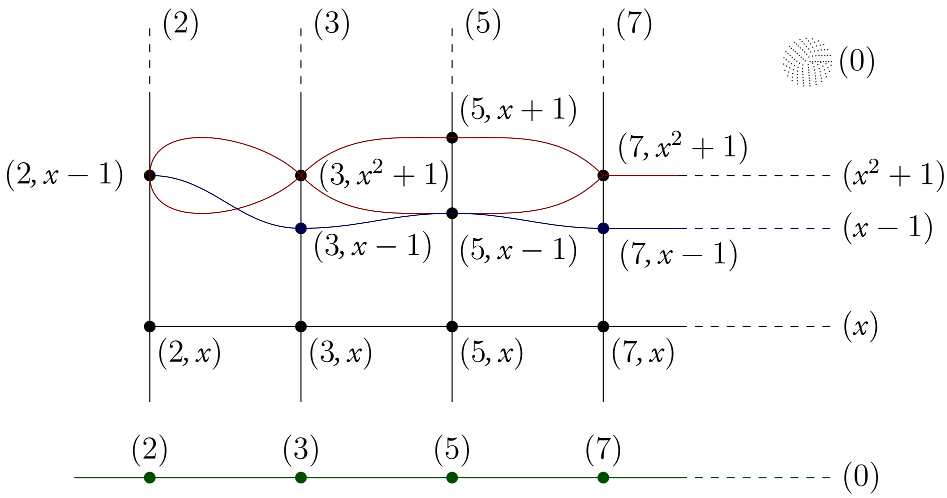
(ii) $\mathfrak{p} = (p)$, where $p \in \mathbb{Z}$ is a prime;

(iii) $\mathfrak{p} = (f)$, where f is a polynomial which is irreducible over $\mathbb{Q}$. This follows from Gauß' lemma.

(iv) $\mathfrak{p} = (p, g)$, where $p \in \mathbb{Z}$ is a prime and g is a polynomial which is irreducible modulo p.

That these are all possibilities is easy to see by considering the fibres under the canonical map

$\operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}$ and splitting into cases.



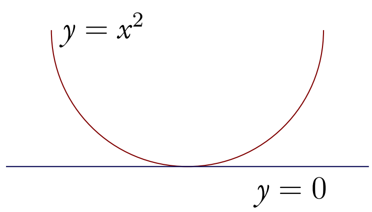
The fibre of a prime $(p) \in \operatorname{Spec}\mathbb{Z}$ corresponds to a vertical line in the drawing: it contains every polynomial g as described in (iv), in addition to the point (p) which is dense in the fibre. On the far right we have the prime ideals of type (iii), polynomials which are irreducible over $\mathbb{Q}$. These all lie in the fibre of $(0) \in \operatorname{Spec}\mathbb{Z}$, along with (0), which is dense (hence the stiple).

This is an example of an *arithmetic surface*, which is the main object of study in Arakelov theory. Although the picture above might not scream "surface", there are two compelling reasons why $\operatorname{Spec}\mathbb{Z}[x]$ should indeed be called so: first of all, at any closed point the structure sheaf has Krull dimension 2, and secondly the maximal chains of proper irreducible subsets have length 2.

Example 2.4.4. Let $f(x,y) = y - x^2$, and $g(x,y) = y$. We form the schemes

$$X := \operatorname{Spec}\frac{\mathbb{C}[x,y]}{(f(x,y))} \quad \text{and} \quad Y := \operatorname{Spec}\frac{\mathbb{C}[x,y]}{(g(x,y))}.$$

Geometrically, we can identify this with the following subset of $\mathbb{C}$:



Note that the classical intersection of X and Y when viewed as curves in $\mathbb{C}$ is simply a point. However, scheme-theoretically, their intersection is $\mathbb{C}[x,y]/(f,g) \cong \mathbb{C}[x]/(x^2)$, which consists of a single point along with extra information arising from the fact that f is tangent to g, namely that of a nilpotent, x. Although we do not yet have the tools to see it, this mirrors the situation with the "singular fibre" in example 2.4.2.

Over an algebraically closed field such as $\mathbb{C}$, there is a well-established area called *intersection theory* describing various forms of intersection and tangency of algebraic varieties.

For arithmetic schemes, the problems is vastly more complicated, and *compactifying* arithmetic surfaces to make them suitable for intersection theory is one of the main ideas behind Arakelov theory.

2.5 Quasi-coherent sheaves on a scheme

Following [Mum13]. Recall from section 1.3 that there is a natural analogue of modules over a sheaf of rings. This turns out to be a very useful tool for studying schemes.

Proposition 2.5.1. *Let $X = \operatorname{Spec} R$ be an affine scheme, and let M be an R-module. Then there exists a sheaf of modules $\widetilde{M}$ of $\mathcal{O}_X$, unique up to isomorphism, such that $\widetilde{M}(D(f)) = M \otimes R_f$.*

Proof. With the data $\widetilde{M}(D(f)) = M \otimes_R R_f$, we obtain as in the construction of the structure sheaf that $M \otimes \mathcal{O}_X$ is a sheaf using proposition 1.2.4. To show that this is indeed a sheaf of modules, it suffices to show that $M \otimes_R R_f$ is an R_f-module for any f, and that multiplication commutes with restriction. But these both follow immediately from definition. □

Definition 2.5.2. Let X be a scheme. A **quasi-coherent sheaf on** X is an $\mathcal{O}_X$-module $\mathscr{F}$ for which there exists an affine open cover $\{U_i = \operatorname{Spec} R_i\}$ of X such that $\mathscr{F}|_{U_i}$ is isomorphic to an $\mathcal{O}_{U_i}$-module of the form $\widetilde{M_i}$ for some R_i-module M_i. If each M_i is finitely generated over R_i, then $\mathscr{F}$ is said to be a **coherent sheaf**.

Example 2.5.3. Take $X = \operatorname{Spec} R$, and let M be an R-module. Then $\widetilde{M}$ is a quasi-coherent sheaf on X, and coherent if and only if M is finitely generated over R.

Example 2.5.4. Fix a ring R, and let X be any scheme. Define a sheaf $\mathscr{F}$ as follows: for any affine open set $U_i = \operatorname{Spec} R_i$ in X, let $\mathscr{F}(U_i) = \operatorname{Hom}_{\mathbf{Ring}}(R, \mathcal{O}_X(U_i)) = \operatorname{Hom}_{\mathbf{Ring}}(R, R_i)$. One easily checks that defines a sheaf of modules on the cover of affine open sets, since

$$\begin{array}{ccc} R_i \times \operatorname{Hom}(R, R_i) & \longrightarrow & \operatorname{Hom}(R, R_i) \\ \downarrow & & \downarrow \\ R_j \times \operatorname{Hom}(R, R_j) & \longrightarrow & \operatorname{Hom}(R, R_i) \end{array} \tag{2.7}$$

commutes: given a restriction $\rho\colon R_i \to R_j$, we have an induced map $\tilde{\rho}\colon \operatorname{Hom}(R, R_i) \to \operatorname{Hom}(R, R_j)$ given by $f \mapsto \rho f$, and we readily verify that $\rho(r_i)\tilde{\rho}(f) = \tilde{\rho}(r_i f)$ for $r_i \in R_i$ and $f \in \operatorname{Hom}(R, R_i)$ since $(\tilde{\rho}(f))(r) = \rho(f(r))$ for any $r \in R$.

Recall that any ring homomorphism $\psi\colon R \to S$ imposes an R-module structure on S given by $r \cdot s = \psi(r)s$. We might expect that the same should hold for schemes, that is, that given a morphism of schemes $\phi\colon X \to Y$, $\phi_*\mathcal{O}_X$ is a sheaf of modules on Y. It is not very difficult to show that this is indeed the case, however it need not be the case that that $\phi_*\mathcal{O}_X$ is a quasi-coherent sheaf on Y, see for example [GW10], exercise 10.14.

For our purposes, the most important example of a quasi-coherent sheaf is the following, which is a purely algebraic analogue of the differentials found in differential geometry.

Let $\phi\colon R \to S$ be a ring homomorphism, and define an S-module $\Omega_{S/R}$ as the free S-module on $\{ds\colon s \in S\}$ modulo the following relations, for $s_1, s_2 \in S$,

(i) $d(s_1 + s_2) = ds_1 + ds_2$,
(ii) $d(s_1 s_2) = s_1 ds_2 + s_2 ds_1$, and
(iii) $d\phi(r) = 0$ for all $r \in R$.

Then $\mathrm{Hom}_{\mathbf{Mod}_S}(\Omega_{S/R}, M)$ is isomorphic to the module of R-derivations $S \to M$ for any S-module M. Explicitly, given a morphism of S-modules $\tau\colon \Omega_{S/R} \to M$, the map $s \mapsto \tau(ds)$ determines an R-derivation, and an R-derivation $D\colon S \to M$ gives a map $\Omega_{S/R} \to M$ generated by $ds \mapsto D(s)$.

Definition 2.5.5. The S-module $\Omega_{S/R}$ is called the **module of Kähler differentials**, or **relative differentials**, of S over R.

Theorem 2.5.6. *Let $m\colon S \otimes_R S \to S$ be the multiplication map, $m(s, s') = ss'$, and define $I := \ker m \leq S \otimes_R S$. Then I/I^2 is canonically isomorphic to $\Omega_{S/R}$.*

Proof. We will construct the isomorphism explicitly: let $\Phi\colon \Omega_{S/R} \to I/I^2$ be determined by $\Phi(ds) = s \otimes 1 - 1 \otimes s$. Note that $s \otimes 1 - 1 \otimes s \in \ker m$, since $m(s \otimes 1 - 1 \otimes s) = s \otimes -s = 0$ since $s \otimes -s = -(s \otimes -s)$. Then Φ is compatible with conditions (i)–(iii).

In the reverse direction, let $A = S \oplus \Omega_{S/R}$ and define $\Psi\colon S \times S \to A$ by $\Psi(s_1, s_2) = (s_1 s_2, s_1 ds_2)$. This is R-bilinear because $\Psi(rs_1, s_2) = (rs_1 s_2, rs_1 ds_2) = r\Psi(s_1, s_2)$ and

$$\Psi(s_1, rs_2) = (rs_1 s_2, s_1 d(rs_2)) = (rs_1 s_2, rs_1 ds_2) = r\Psi(s_1, s_2), \tag{2.8}$$

where we use (iii) in the last equality, so that $d(rs_2) = rds_1 + s_1 dr = rds_1$. Therefore we obtain a map $\widetilde{\Psi}\colon S \otimes_R S \to A$, and for any $i \in I$, the first component of $\widetilde{\Psi}(i)$ is 0 by definition of I. Moreover, since (ii) implies that any square in $\Omega_{S/R}$ is 0, the second component of $\widetilde{\Psi}$ factors through the quotient of I/I^2. It is easily checked that these maps are mutually inverse. □

Example 2.5.7. Let R be some field k, and let $S = k[x_1, \ldots, x_n]$. Then $\Omega_{S/R}$ is the free S-module generated by $dx_1, \ldots, dx_n$ such that for any $f \in S$,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i,$$

which is precisely the analogue of analytic differentials in the special case of polynomials.

Now we want to “globalise” the construction to the case of schemes.

Theorem 2.5.8. *[LE06] Let $f\colon X \to Y$ be a morphism of schemes. Then there exists a unique quasi-coherent sheaf $\Omega^1_{X/Y}$ on X such that for any affine open $V \subset Y$ and $U \subset f^{-1}(V)$ with $x \in U$, we have*

$$\Omega^1_{X/Y}|_U \cong \widetilde{\Omega}_{\mathscr{O}_X(U)/\mathscr{O}_Y(V)} \quad \textit{and} \quad (\Omega^1_{X/Y})_x \cong (\widetilde{\Omega}_{\mathscr{O}_X(U)/\mathscr{O}_Y(V)})_x. \tag{2.9}$$

Our strategy will be the following: first we define the stalks of $\Omega^1_{X/Y}$, and then we patch together in a natural way. We first need the following technical lemma:

Lemma 2.5.9. *Let $\phi\colon R \to R'$ be a ring homomorphism, and fix $\mathfrak{q} \in \operatorname{Spec} R'$ and $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \in \operatorname{Spec} R$. Then we have canonical isomorphisms*

$$\Omega^1_{R'/R} \otimes R'_{\mathfrak{q}} = \Omega^1_{R'_{\mathfrak{q}}/R} = \Omega^1_{R'_{\mathfrak{q}}/R_{\mathfrak{p}}}. \tag{2.10}$$

Proof. See [LE06], Prop. 6.1.8. □

Proof of theorem 2.5.8. For simplicity, write $\Omega_x := \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$ for $x \in X$. Given U, V and x as in the theorem statement, and $\omega \in \Omega^1_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)}$, let ω_x denote the image of ω in $\Omega^1_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)} \otimes \mathcal{O}_{X,x} = \Omega_x$, identifying the two by the previous lemma. Next, for an arbitrary open $U \subset X$, define $\Omega^1_{X/Y}(U)$ to be the set of maps

$$s\colon U \to \coprod_{x \in U} \Omega_x \tag{2.11}$$

such that for any $x \in U$, there exists affine open neighbourhoods $V_y \subset Y$ of $y = f(x)$ and $U_x \subset f^{-1}(V_y)$ and $\omega \in \Omega^1_{\mathcal{O}_X(U_x)/\mathcal{O}_Y(V_y)}$ with $\omega_{x'} = s(x')$ for every $x' \in U_x$. It is straightforward to check (but tedious to write out) that $\Omega^1_{X/Y}$ forms a sheaf of $\mathcal{O}_X$-modules on X with restriction maps given by restriction of domains. By design, the stalks $(\Omega^1_{X/Y})_x$ are isomorphic to Ω_x, and for affine open $V \subset Y$ and $U \subset f^{-1}(V)$ we have a natural map of rings $\Omega^1_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)} \to \Omega^1_{X/Y}(U)$ given by $dr \mapsto (x' \mapsto dr_{x'})$. Noting that this is in fact an isomorphism at the level of stalks, it is accordingly an isomorphism of $\mathcal{O}_X|_U$-modules, proving eq. (2.9). By definition, $\Omega^1_{X/Y}$ is therefore quasi-coherent. □

Definition 2.5.10. Fix a morphism of schemes $f : X \to Y$. The quasi-coherent sheaf $\Omega^1_{X/Y}$ defined above is called the **sheaf of Kähler differentials**, or **relative differentials of degree** 1 of X over Y.

Chapter 3

The Étale Topology

3.1 Insufficiency of the Zariski topology

Compared to the Euclidean topology, the Zariski topology is lacking in many respects. For example, it is never Hausdorff unless the underlying space is finite because every open set contains infinitely many prime ideals. Points need no longer be closed sets, in fact, these are precisely the ideals which are maximal. If we want to transfer ideas from, say, complex differential geometry, we need to work with a finer topology. A more concrete motivation is the following:

The Weil Conjectures

We follow [Har77] and [FK88]. Let X be a scheme of finite type over finite field $k = \mathbb{F}_q$, and define the base-change $\bar{X} = X \times_k \bar{k}$, where $\bar{k}$ is a fixed an algebraic closure of k. Suppose N_r, for $r \in \mathbb{N}$, denotes the number of $\mathbb{F}_{q^r}$-rational points in $\bar{X}$. In classical language, these are precisely the points of $\bar{X}$ with coordinates in $\mathbb{F}_{q^r}$.

Definition 3.1.1. The **zeta function** of X is the formal power series

$$Z(X,t) := \exp\left(\sum_{r\in\mathbb{N}} \frac{N_r t^r}{r}\right) \in \mathbb{Q}[\![t]\!]. \tag{3.1}$$

Here exp denotes the formal power series $\exp(t) = 1 + t/1! + t^2/2! + t^3/3! + \ldots$. Weil conjectured, inspired by direct verification in a few special cases, that the zeta function should obey certain properties analogous to those of the Riemann zeta function. Precisely:

Conjecture (W1)**.** *$Z(X,t)$ is a rational function in t.*

Conjecture (W2)**.** *If E is the self-intersection number of the diagonal in $X \times X$, then Z satisfies*

$$Z\left(X, \frac{1}{q^n t}\right) = \pm q^{nE/2} t^E Z(X,t). \tag{3.2}$$

This is an analogue of the functional equation for the Riemann zeta function,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

The next is inspired by the Riemann hypothesis:

Conjecture (W3). *Let $n := \dim X$. Then Z can be written on the form*

$$Z(X,t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}}, \tag{3.3}$$

where $P_0 = (1-t)$, $P_{2n} = (1-q^n t)$ and $P_i = \prod_j (1-\alpha_{ij}t)$ where a_{ij} are algebraic integers with $|\alpha_{ij}| = q^{i/2}$.

Assuming (W3), we define the *Betti numbers* $\beta_i(X) := \deg P_i$. In the classical setting, if X is a subvariety of $\mathbb{C}$ with the Euclidean subspace topology, we can define the Betti numbers $\beta_i(X)$ as the dimensions of the vector spaces $H^r(X,\mathbb{Q})$. However, over arbitrary fields this fails, and the Zariski topology proves inadequate:

Proposition 3.1.2. *If a topological space X is irreducible, then $H^r(X,\mathscr{F}) = 0$ for all $r > 0$ and every constant sheaf $\mathscr{F}$.*

Since every integral scheme is irreducible, this shows that the Zariski topology is inadequate when we try to apply cohomological methods to $\mathbb{Z}$-schemes. The näive way to fix this problem is to refine the Zariski topology by adding more open sets. This turns out to be a very fruitful approach; the difficulty lies in choosing the correct ones.

Weil's ingenious idea was that if we were to define the "correct" homology theory of varieties over finite fields, then we might have a good chance at proving these result. In his honour, a "good" homology theory, that is, one satisfying a list of axioms including those needed for the resolution of the Weil conjectures, is called a *Weil cohomology theory*. Along came Grothendieck, who with the apparatus of scheme theory was revolutionising algebraic geometry. Equipped with the *étale topology*, he and Michael Artin were able to provide the necessary formalism to prove conjectures (W1) and (W2). However, the Riemann hypothesis remained elusive until it was settled by Pierre Deligne in 1974 [Del74].

3.2 Étale morphisms

Recall from calculus the *inverse function theorem*:

Theorem 3.2.1 (Inverse function theorem). *Fix $x \in \mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}^n$ satisfies*

$$\det \left[\frac{\partial f_i}{\partial x_j}\bigg|_x \right]_{i,j} \neq 0, \tag{3.4}$$

then there exists a neighbourhood U of x such that $f|_U$ is a diffeomorphism onto its image.

Proof. (See [MW97], p.4). □

The idea is simple: to refine the Zariski topology, we add to our topology sets which are the preimages of open sets under maps which satisfy a suitable analogue of the hypothesis of the inverse function theorem. This turns out to be the maps which are *étale*, meaning "still" or "slack", by analogy with the sea. The first ingredient in the definition of étale maps is the idea of an *unramified morphism*:

Unramified maps

Informally, in complex analytic geometry we say that a map of Riemann surfaces is ramified at a point if it "branches out" there. For example, the map $\phi\colon z \mapsto z^n$ in $\mathbb{C}$ is n-to-1 at an arbitrary non-zero point p in $\mathbb{C}$, with the n distinct n-th roots of p mapping to p. However, 0 has only a single preimage, and we say that ϕ is *ramified* at 0, or that 0 is a branch point of ϕ. Topologically, we can think of 0 as having no neighbourhood on which ϕ is injective. If our notion of étale morphisms are meant to mirror local homeomorphisms, the algebro-geometric analogue of ramification certainly needs to be precluded.

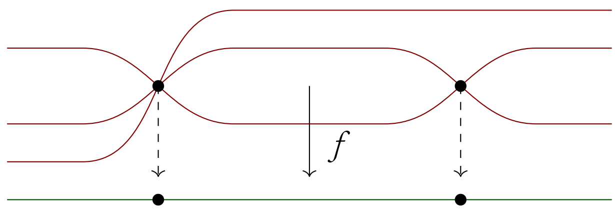


Figure 3.1: A triple cover with two ramified points. f is not a local homeomorphism near the first ramification point since by removing a single point we obtain six connected components in the domain, but only two in the image.

As usual, we start off "locally", in the world of rings: Recall that a morphism of local rings $\phi\colon R \to S$ is *unramified* if $S/\phi(\mathfrak{m}_R)$ is a finite separable field extension of $R/\mathfrak{m}_R$.

Definition 3.2.2. A morphism locally of finite presentation $f\colon X \to Y$ is **unramified at** $x \in X$ if the induced map of local rings $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is unramified. If f is unramified at all $x \in X$, then we say that f is **unramified**.

This definition is not always the easiest to use in practice, but fortunately we have the following:

Proposition 3.2.3. *Let $f\colon X \to Y$ be a morphism locally of finite type. Then the following are equivalent:*

(i) *f is unramified at x,*
(ii) *the stalk $(\Omega^1_{X/Y})_x = 0$,*
(iii) *there exists a neighbourhood U of x restricted to which the diagonal morphism $\Delta_{X/Y}\colon X \to X \times_Y X$ is an open immersion.*

Proof. Unfortunately, we do not have space to develop the machinery required to prove this. See [Mil80], Prop. 1.3.5, or [Sza09], Prop. 5.2.7. □

Example 3.2.4. Proceeding as in example 2.4.2, consider the natural map $f\colon \operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$. In the setting of algebraic number theory one might recall that $(1+i)$ ramifies over $\mathbb{Z}$, so we expect that the same holds for f.

Note first that since $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$ and $0 = d(x^2+1) = 2dx$, we have

$$\Omega_{\mathbb{Z}[i]/\mathbb{Z}} \cong \frac{\mathbb{Z}[i]dx}{\mathbb{Z}[i]2dx} \cong \frac{\mathbb{Z}[i]}{2\mathbb{Z}[i]}.$$

For any $\mathfrak{p} \in \mathbb{Z}[i]$, we find that $(\mathbb{Z}[i]/2\mathbb{Z}[i])_{\mathfrak{p}} = 0$ if $2 \notin \mathfrak{p}$, since it is a field containing a nilpotent, $i+1$. If $2 \in \mathfrak{p}$, then $2 \in (\mathbb{Z}[i]/2\mathbb{Z}[i])_{\mathfrak{p}} \neq 0$, so $(\Omega_{\operatorname{Spec}\mathbb{Z}[i]/\operatorname{Spec}\mathbb{Z}})_{\mathfrak{p}} \neq 0$. Of course, the only prime ideal in $\mathbb{Z}[i]$ containing 2 is $\mathfrak{p} = (1+i)$, which verifies our expectation.

Flat morphisms

The second ingredient in the definition of étale is *flatness*. While many concepts in scheme theory come from geometry, flatness is decidedly algebraic. As memorably put by Mumford, "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers"([Mum13], p. 214). We say that a ring homomorphism $\phi\colon R \to S$ is *flat* if the action defined by $(r, s) \mapsto \phi(r)s$ makes S a flat R-module, that is, a module for which the functor $- \otimes_R S$ is exact. Globally, we take the following:

Definition 3.2.5. A morphism of schemes $f\colon X \to Y$ is **flat** if the corresponding local homomorphisms $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are flat.

We are now set to define étale morphisms:

Definition 3.2.6. A morphism of schemes $f\colon X \to Y$ is **étale** if it is locally of finite presentation, flat and unramified. Similarly, we say that an S-scheme $X \xrightarrow{\pi} S$ is étale whenever π is.[1]

We denote by **Ét**$/S$ the category of étale S-schemes along with arrows given by étale morphisms of S-schemes.

Example 3.2.7. Any open immersion is étale, because it is an isomorphism at the level of stalks.

Definition 3.2.8. An *étale cover* is surjective étale morphism.

Example 3.2.9. Let $S = \operatorname{Spec} k$. Then by unraveling the definitions, we find that an étale cover $X \to S$ is simply a disjoint union of finite separable extensions of k.

Definition 3.2.10. We say that an étale cover $f\colon X \to S$ is **trivial** if X is isomorphic to a disjoint union of copies of S, restricted to each of which f is the identity.

Étale maps obey a few other useful properties:

Proposition 3.2.11. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be étale morphisms.*

(i) *$g \circ f$ is étale;*
(ii) *if X is an S-scheme and $S' \to S$ a morphism, then the induced map $X \times_S S' \to Y$ is also étale.*

Proof. [Mil80], Prop. 1.3.3. □

In light of proposition 3.2.3, it seems reasonable that étale maps should have a straightforward characterisation in terms of the "differential properties" of a scheme.

Theorem 3.2.12. *A morphism of schemes $f\colon X \to Y$ is étale if and only if for each $x \in X$, there exist open affine neighbourhoods $U = \operatorname{Spec} R$ of x and $V = \operatorname{Spec} S$ of $y = f(x)$ such that for some $n \in \mathbb{N}$,*

$$R = S[T_1, \dots, T_n]/(P_1, \dots, P_n) \quad \textit{and} \quad \det \left[\frac{\partial P_i}{\partial T_j}\right]_{i,j} \in R^\times. \tag{3.5}$$

Proof. [Mil80], Cor. 3.16. □

[1] This is slightly more restrictive than more common definitions which require f to be of *finite presentation* instead of locally finite, but this sufficient for our purposes.

3.3 Grothendieck Topologies

Grothendieck observed in his famous *Tohoku* paper ([Gro57]) that all the axioms of a topology were not necessary to define sheaves, and by extension, cohomology theories. In fact, if we adjust the definition of a sheaf somewhat, it suffices to consider appropriately chosen *coverings* of a given scheme. More precisely:

Definition 3.3.1. Let $\mathbf{C}$ be a category. A **Grothendieck topology** J consists of the following data: for each object $U \in \mathbf{C}$, a collection $J(U)$ of sets of maps $\{\phi_i \colon U_i \to U\}$ where each set if called a **covering of** U, satisfying the following conditions:

(i) For any morphism $V \to U$ in $\mathbf{C}$, the fibre products $U_i \times_U V$ exist, and induces a covering $\{U_i \times_U V \to V\}_i$ of V.
(ii) If for each i, $\{V_{ij} \to U_i\}_j$ is a covering of U_i, then $\{V_{ij} \to U\}_{i,j}$ is also a covering of U.
(iii) The class consisting only of the identity map $U \to U$ is a covering of U.

A pair $(\mathbf{C}, J)$ is called a **site**, often abbreviated by $\mathbf{C}$.[2]

Let us check that this indeed generalises the notion of an open cover:

Example 3.3.2. Any "classical" open cover $\{U_i\}$ on a topological space X is a covering.

Proof. Let $\mathbf{U}(X)$ be the poset category which has objects given by open sets of X, and arrows given by inclusions. For any open $V \subseteq X$, the fibre products $U_i \times_X V$ are given by $U_i \cap V$, as seen by the pullback diagram

$$\begin{array}{ccc} U_i \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & X \end{array}$$

Since $U_i \cap V$ are also open subsets of X, they are objects in $\mathbf{U}(X)$, and form an open cover of V. To prove (ii), let $\{V_{ij}\}$ be a covering of U_i for each i. Then V_{ij} is an open cover of U. Finally, U is an open cover of itself, proving (iii). □

Example 3.3.3. Let X be a scheme. The *Zariski site* on X, X_{zar}, is the site associated with the (Zariski) topology on X.

Having defined coverings categorically, it is reasonable to do the same continuous maps.

Definition 3.3.4. Let $\mathbf{C}$, $\mathbf{C}'$ be sites. A **continuous map** $\mathbf{C} \to \mathbf{C}'$ is a functor which preserves fibre products and coverings.

Example 3.3.5. If X and Y are topological spaces, and $f \colon X \to Y$ a continuous map, then f induces a continuous map $\mathbf{U}(Y) \to \mathbf{U}(X)$, defined by pulling back coverings of Y via f.

The key to generalising sheaves to this situation turns out to be the equaliser condition:

[2] The original definition is more general, since it does not require the existence of fibre products. However, this is sufficient for our purposes. We follow Milne.[Mil80]

Definition 3.3.6. A **sheaf** on a site $(\mathbf{C}, J)$ is a contravariant functor $\mathscr{F}\colon \mathbf{C}^{\text{op}} \to \mathbf{C}'$ satisfying the equaliser condition

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j). \tag{3.6}$$

In other words, $\mathscr{F}(U)$ can be identified with the collection of $(s_i) \in \prod_i \mathscr{F}(U_i)$ satisfying $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$. In light of example 3.3.2, we see that this reduces to the original case when considering topological spaces.

The following will not be used later, but is nice to know:

Definition 3.3.7. A **topos** is a category equivalent the category of sheaves on a site.

Definition 3.3.8. The *étale site* on X, $X_{\text{ét}}$, has the underlying category $\mathbf{\acute{E}t}/X$. A covering of X is a surjective family of étale morphisms in $\mathbf{\acute{E}t}/X$, in other words a collection of X-schemes $\{\phi_i\colon U_i \to X\}$ maps such that $\bigcup_i \phi_i(U_i) = X$.

Proposition 3.3.9. *The étale site is indeed a site on X.*

Proof. This is an immediate consequence of proposition 3.2.11. □

This is the first step into developing *ℓ-adic cohomology*, which is the key idea behind the resolution of the Weil conjectures. However, instead of venturing into this vast and complicated area, we will consider another application of étale maps, namely that of the étale fundamental group.

3.4 The Galois theory of finite étale covers

In what follows, we shall see that finite étale covers have many features in common with finite covers of topological space. We start off with a technical lemma:

Lemma 3.4.1. *Let $\psi\colon Y \to X$ and $\phi\colon X \to S$ be morphisms of schemes.*

(i) *If $\phi \circ \psi$ is finite and ϕ is separated, then ψ is finite.*
(ii) *If in addition $\phi \circ \psi$ and ϕ are étale, then so is ψ.*

Proof. (i) By definition of ϕ being separated, the diagonal morphism $\Delta_{X/S}\colon X \to X \times_S X$ is a closed immersion. Now define the *graph of ψ*, Γ_ψ as the fibre product

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\Gamma_\psi"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\mathrm{Id}_Y"'] & & \\
& Y \times_S X \arrow[r, "\mathrm{pr}_2"] \arrow[d] & X \arrow[d, "\phi"] \\
& Y \arrow[r, "\phi\circ\psi"] & S
\end{tikzcd}$$

Then Γ_ψ is finite by proposition 2.3.31. Similarly, pr_2 is a finite map since we can consider it as the base change of $\phi \circ \psi$ by ϕ. Therefore $\mathrm{pr}_2 \circ \Gamma_\psi = \psi$ is finite as well.
(ii): Suppose ϕ is an étale cover. By proposition 3.2.3, $\Delta_{X/S}$ is an isomorphism onto a clopen subset of $X \times_S X$, hence finite étale. Since being finite étale is stable under base change by

proposition 3.2.11 and proposition 2.3.31, so is Γ_ψ. For the same reason, $\mathrm{pr}_2 \circ \Gamma_\psi = \psi$ is finite étale. $\square$

Proposition 3.4.2. *Let $f : X \to S$ be a finite étale cover, and let $s : S \to X$ be a section of f. Then s is an isomorphism onto some clopen subscheme of X.*

Proof. By the previous lemma, s is finite étale. Being a section, it is injective, and so an isomorphism onto its image. The fact that the image is clopen is slightly more subtle, cf. [Sza09], Remark 5.2.2(3), and we omit the proof for the sake of brevity. $\square$

Definition 3.4.3. A **geometric point** $\bar{s}$ of scheme S is a map $\operatorname{Spec} \Omega \to S$, where Ω is some algebraically closed field. If X is an S-scheme, the **geometric fibre of X over** $\bar{s}$ is the fibre product $X \times_S \operatorname{Spec} \Omega$.

Corollary 3.4.4. *If $Z \to S$ is a connected S-scheme, $f_1 : Z \to X$ and $f_2 : Z \to X$ are morphisms satisfying $f_1 \circ \bar{z} = f_2 \circ \bar{z}$ for some geometric point $\bar{z} : \operatorname{Spec} \Omega \to Z$, then $f_1 = f_2$.*

Proof. Since being étale is stable under base change, it suffices to prove this for $S = Z$. But by the proposition above, two sections f_1, f_2 of $X \to S$ are determined by their value at the image of a geometric point, since the connected component of X containing the image of the f_i is uniquely specified by $\bar{z}$. $\square$

Definition 3.4.5. Given a morphism of $f : X \to S$, let $\operatorname{Aut}(X/S)$ be the group of automorphisms $\lambda : X \to X$ satisfying $f(\lambda(x)) = f(x)$ for all $x \in X$.

There is a natural group action of $\operatorname{Aut}(X/S)$ on X by $\lambda \cdot x = \lambda(x)$, and this induces an action on any geometric fibre $X_{\bar{s}} = X \times_S \operatorname{Spec} \Omega$.

Lemma 3.4.6. *Let $f : X \to S$ be a connected finite étale cover, and $\bar{s} : \operatorname{Spec} \Omega \to S$ a geometric point. Then non-trivial elements of $\operatorname{Aut}(X/S)$ act without fixed points on $X_{\bar{s}}$.*

Proof. This is immediate from corollary 3.4.4 by taking $f_1 = f$ and and $f_2 = f \circ \lambda$. $\square$

Since the underlying set of $X_{\bar{s}}$ is finite, we thus have:

Corollary 3.4.7. *If $X \to S$ is a finite étale cover, then $\operatorname{Aut}(X/S)$ is a finite group.*

Just as with sets, there is a natural notion of a quotient of a scheme by a group action.

Theorem 3.4.8. *Let $f : X \to S$ be an affine, surjective map and G a finite subgroup of $\operatorname{Aut}(X/S)$. Then there exists a scheme with underlying set the orbits of X under G, being the unique scheme up to isomorphism satisfying the following universal property: there exists a unique morphism of schemes $\phi : X \to G\backslash X$ such that for any morphism $g : X \to Y$ where Y is affine and surjective, subject to $g(\lambda(x)) = g(x)$ for all $\lambda \in G$, there exists a unique map $\tilde{g} : G\backslash X \to Y$ such that*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\phi"'] & & Y \\ & G\backslash X \arrow[ur, "\tilde{g}"'] & \end{tikzcd}$$

commutes.

Proof. [Sza09], Prop. 5.3.6. $\square$

Lemma 3.4.9. *Let $f\colon X \to S$ be a connected finite étale cover, and $G \subset \mathrm{Aut}(X/S)$. Then the induced morphisms $X \to G\backslash X$ and $G\backslash X \to S$ are finite étale as well.*

Definition 3.4.10. A connected finite étale cover $X \to S$ is called a **Galois cover** if $\mathrm{Aut}(X/S)$ acts transitively on $X_{\bar{s}}$ for any geometric point $\bar{s}\colon \operatorname{Spec}\Omega \to S$.

Theorem 3.4.11. *Let $f\colon X \to S$ be a Galois cover, let $\psi\colon Z \to S$ be a morphism of schemes, and suppose $\phi \circ \pi = \psi$ for some $\pi\colon X \to Z$.*

(i) *π is a finite étale cover, $Z \cong H\backslash X$ for some $H \leq G := \mathrm{Aut}(X/S)$, and there is a bijection*

$$\{\textit{subgroups } H \leq G\} \leftrightarrow \{\textit{intermediate covers } X \to Z \to S\};$$

(ii) *$\psi\colon Z \to S$ is Galois if and only if $H \trianglelefteq G$, in which case $\mathrm{Aut}(Z/S) \cong G/H$.*

Proof. [Sza09], Prop.5.3.8. □

The next lemma shows that in some cases, we need only check a single fibre to determine if $X \to S$ is Galois.

Lemma 3.4.12. *Let $X \to S$ be a connected finite étale map. If $\mathrm{Aut}(X/S)$ acts transitively on $X_{\bar{s}}$ for some $\bar{s}\colon \operatorname{Spec}\Omega \to S$, then X is Galois over S.*

Proof. Consider $\mathrm{Aut}(X/S)\backslash X$. This is connected, and the fibre of $\bar{s}$ consists of a single element. Therefore this is isomorphic to X. By the previous theorem, it follows that $X \to S$ is Galois. □

The following can be seen as an analogue of the result in Galois theory that every field extension has a normal closure.

Theorem 3.4.13. *Let $f\colon X \to S$ be a connected finite étale cover. Then there exists a morphism $P \to X$ such that $P \xrightarrow{\phi\circ\pi} X$ is a Galois cover, and every S-morphism $Q \to X$ where Q is a Galois cover factors through P.*

Proof. Fix geometric points $\bar{s}\colon \operatorname{Spec}\Omega \to S$ and $F = \{\bar{x}_1, \ldots \bar{x}_n\}$ where $\bar{x}_i\colon \operatorname{Spec}\Omega \to X_{\bar{s}}$. By choosing an ordering on F, we get a unique geometric point

$$\bar{x}\colon \operatorname{Spec}\Omega \to X^n := \underbrace{X \times_S \ldots \times_S X}_{n \text{ times}}.$$

Now let $P \subset X^n$ be the connected component containing the image of $\bar{x}$, and let $\pi\colon P \to X$ be the restriction of the projection $X^n \to X$ onto the first component. Note that π is finite étale by virtue of being the base change of f, by proposition 3.2.11 (ii).

We claim that any point of $P_{\bar{s}}$ can be represented as $(\bar{x}_{\sigma(1)}, \ldots, \bar{x}_{\sigma(n)})$ for some permutation $\sigma \in S_n$. Indeed, since each point of $X^n_{\bar{s}}$ arises from an element of F^n, it suffices to show that points in P have distinct coordinates. Note that $\Delta_{X/S}\colon X \to X \times_S X$ is a clopen map because f is separated, hence the preimage of $\Delta(X)$ under the projection $\pi_{ij}\colon X^n \to X \times_S X$ is clopen as well. Now note that $\pi_{ij}^{-1}(\Delta(X)) \cap P = \varnothing$ since otherwise we would have $P \subset \pi_{ij}^{-1}(\Delta(X))$, which is impossible since then $\bar{x}$ would have a repeated coordinate. This proves our claim.

Next we aim to show that $P \to X$ is Galois; note that each permutation σ of the $\bar{x}_i$ induces an automorphism $\lambda_\sigma$of X^n by permuting components, and if $\lambda_\sigma \circ \bar{x} \in P_{\bar{s}}$, then $\lambda_\sigma(P) \cap P \neq \varnothing$. Consequently, $\lambda_\sigma \in \mathrm{Aut}(P/S)$, and so $\mathrm{Aut}(P/S)$ acts transitively on a geometric fibre of P. By lemma 3.4.12, it follows that P is Galois.

Finally, to show the universality, note that for any S-morphism $q\colon Q \to X$ where Q is a Galois cover of S, we can choose a preimage $\bar{y}$ in Q of the geometric point $\bar{x}$. By theorem 3.4.11, q is a surjective morphism which by composing with elements of $\mathrm{Aut}(Q/S)$ gives n morphisms $q_i\colon Q \to X$ satisfying $q_i \circ \bar{y} = \bar{x}_i$. These induce a morphism $Q \to X^n$, and we see that the image lies wholly in P since $\bar{y} \mapsto \bar{x}$. Thus q factors through P, as claimed. □

3.5 The étale fundamental group

In algebraic topology and homotopy theory, we are interested in studying the *homotopy groups* of a given topological space. While we have a natural definition of $\pi_0(X)$ for any scheme X as the set of irreducible components, we quickly run into trouble when trying to define the first fundamental group $\pi_1(X, x)$. This was one of several problems occupying Alexander Grothendieck in the mid-1950's, as he himself writes in a letter to Jean-Pierre Serre: "Obviously, I am looking for an algebraic definition of the fundamental group..." ([GC04], p. 55).

Definition 3.5.1. Let $\bar{s}\colon \operatorname{Spec}\Omega \to S$ be a geometric point, and let the **fibre functor** $\mathrm{Fib}_{\bar{s}}\colon \mathbf{FÉt}/S \to \mathbf{Set}$ be the composition of the base change functor $X \mapsto X \times_s \operatorname{Spec}\Omega$ and the forgetful functor $\mathbf{Sch} \to \mathbf{Set}$ sending a scheme to its underlying set.

Given a functor $F\colon \mathbf{C} \to \mathbf{C}'$, let $\mathrm{Aut}(F)$ be the *automorphism group* of F, namely the group of invertible natural transformations $F \to F$ under composition. Explicitly, each $\phi \in \mathrm{Aut}(F)$ consists of an automorphism ϕ_C of C for each $C \in \mathbf{C}$, and if C is set-valued, we have a natural action of $\mathrm{Aut}(F)$ on each object C by $\phi \cdot c = \phi_C(c)$ for $c \in C$.

Definition 3.5.2. Given a scheme S and a geometric point $\bar{s}\colon \operatorname{Spec}\Omega \to S$, the **étale fundamental group with basepoint** $\bar{s}$, $\pi_1(S, \bar{s})$, is the automorphism group of the fibre functor $\mathrm{Fib}_{\bar{s}}$ on $\mathbf{FÉt}/S$.

Analogously to how the topological fundamental group acts on the covers of a topological space via deck transformations, we have the following:

Theorem 3.5.3 (Grothendieck)**.** *Let S be a connected scheme and $\bar{s}\colon \operatorname{Spec}\Omega \to S$ a geometric point.*

(i) *The group $\pi_1(S, \bar{s})$ is profinite with a continuous action on $\mathrm{Fib}_{\bar{s}}(X)$ for every $X \in \mathbf{FÉt}/S$;*
(ii) *The functor $\mathrm{Fib}_{\bar{s}}$ induces an equivalence of categories between $\mathbf{FÉt}/S$ and the category of finite sets with a continuous left $\pi_1(S, \bar{s})$-action, where connected covers correspond to sets with a transitive action of $\pi_1(S, \bar{s})$, and Galois covers to finite quotients of $\pi_1(S, \bar{s})$.*

Example 3.5.4. From the picture in example 2.1.6, one might anticipate that $\pi_1(\operatorname{Spec}\mathbb{Z}) = 0$. Using Minkowski theory, one can prove the following:

Theorem 3.5.5. *There are no unramified field extensions of $\mathbb{Q}$.*

See for example [NS13], Thm. 2.2.18. for a proof. To apply this, we will state – but unfortunately cannot prove, see [Len08], cor. 6.17 – a key result on the covering theory of normal integral schemes like $\mathbb{Z}$:

Theorem 3.5.6. *Let X be a normal integral scheme with function field k, fix an algebraic closure $\bar{k}$ of k, and let U/k be the maximal unramified separable extension contained in $\bar{k}$. Then $\pi_1(X, \bar{s}) \cong \mathrm{Gal}(U/k)$.*

Now, by Minkowski's theorem the maximal unramified extension is simply $\mathbb{Q}$, which we recall is also the function field of $\mathbb{Z}$. Therefore, $\pi_1(\mathbb{Z}, \bar{s}) = 0$ for any geometric point $\bar{s}$.

Finite étale algebras

To see how this relates to classical Galois theory, let us first consider its reformulation due to Grothendieck. The main reference is [Sza09], chapter 1.

Definition 3.5.7. A finite-dimensional k-algebra A is **étale** over k if it is isomorphic to a finite direct sum of separable extensions of k. If all the separable extensions have finite degree over k, then A is said to be a **finite étale algebra**.

Example 3.5.8. Any separable extension of k can be viewed as a finite étale algebra.

Example 3.5.9. The spectrum of a finite étale algebra A over a field k is a finite étale k-scheme.

Definition 3.5.10. Fix a separable closure k_s of k. The group $\mathrm{Gal}(k) := \mathrm{Gal}(k_s/k)$ consisting of automorphisms of k_s fixing k is called the **absolute Galois group** of k.

In general, this is a very mystical object, but we can prove the following theorem:

Theorem 3.5.11 (Grothendieck's reformulation of the Galois correspondence)**.** *Let k be a field with a separable closure k_s. Then the correspondence*

$$\{\textit{Finite étale k-algebras}\} \leftrightarrow \{\textit{Left } \mathrm{Gal}(k_s/k)\textit{-sets}\}$$
$$A \mapsto \mathrm{Hom}_k(A, k_s)$$

is an anti-equivalence of categories between the category of finite étale k-algebras and the category of $\mathrm{Gal}(k_s/k)$-sets.

To see how this ties in with classical Galois theory, we need the following lemma, which is easily checked:

Lemma 3.5.12. *Let G be a group. Then there is a one-to-one correspondence between subgroups of G and quotients of G as a G-set, given by $H \mapsto G/H$.*

Under this correspondence, we see that Grothendieck's reformulation reduces to:

Corollary 3.5.13 (The classical Galois correspondence)**.** *Let $\mathrm{Gal}(k)$ be the absolute Galois group of k. Then there is a one-to-one correspondence*

$$\{\textit{finite separable extensions of } k\} \leftrightarrow \{\textit{open subgroups of } \mathrm{Gal}(k)\},$$

where a subfield K corresponds to the subgroup of automorphisms which fix K.

Of course, we can also look at a fixed (Galois) subfield and consider its subextensions with corresponding subgroups of the Galois group. For a direct proof, see for example [Nag77], Chapter 7. We are content to show how it follows from Grothendieck's π_1-theorem.

Proof of theorem 3.5.11. In theorem 3.5.3, take $S = \operatorname{Spec} k$, and recall from example 3.2.9 , since the map $X \to S$ is surjective, any finite étale scheme is the spectrum of some finite étale k-algebra. In this case a geometric point $\operatorname{Spec} \Omega \to S$ corresponds to a field extension Ω/k, and the fibre functor sends $\operatorname{Spec} A$ to the underlying set of $\operatorname{Spec} A \otimes_k \Omega$, by definition of the fibre product. In the special case where $A = L$ is some finite separable extension of k, we claim that this set bijectively corresponds to the set $\operatorname{Hom}_k(L, \Omega)$. Recall from Galois theory that a finite separable extension of degree n has exactly n distinct k-algebra morphisms into an algebraic closure Ω (eg. [Sza09], Lemma 1.1.6). But since L breaks into linear factors when tensored with Ω, $\operatorname{Spec} L \otimes \Omega = \amalg_{i=1}^{n} \Omega$, whose spectrum has precisely n points. The image of these morphisms lie in k_s, so we obtain that $\operatorname{Fib}_{\bar{s}}(X) = \operatorname{Hom}_k(L, k_s)$, and so $\pi_1(S, \bar{s}) \cong \operatorname{Gal}(k)$. $\square$

Representable and pro-representable functors

Recall that a functor $F\colon \mathbf{C} \to \mathbf{Set}$ is *representable* if there exists some object $C \in \mathbf{C}$ such that F is naturally isomorphic to $\operatorname{Hom}(C, -)$.

From the proof of theorem 3.5.11 one might suspect that $\operatorname{Fib}_{\bar{s}}$ is representable for $S = \operatorname{Spec} k$ since we can identify $\operatorname{Fib}_{\bar{s}}$ with the functor $X \mapsto \operatorname{Hom}(\operatorname{Spec} k_s, X)$. However, $\operatorname{Spec} k_s$ is *not* finite étale over $\operatorname{Spec} k$. On the other hand, it *is* represented by something that looks like a limit of elements of $\mathbf{FÉt}/S$, which inspires the following definition:

Definition 3.5.14. Let $\mathbf{C}$ be a category, and $F\colon \mathbf{C} \to \mathbf{Set}$ a functor. Then F is **pro-representable** if there exists an inverse system $(C_\lambda, \phi_\lambda)_{\lambda\in\Lambda}$ such that

$$F(X) \cong \varinjlim_{\lambda\in\Lambda} \operatorname{Hom}(C_\lambda, X).$$

Example 3.5.15. Every representable functor is evidently pro-representable.

Example 3.5.16. Consider forgetful functor $F\colon \mathbf{FinGrp} \to \mathbf{Set}$ sending a finite group to its underlying set. This is not representable since for any finite group G we can find some finite H such that $\operatorname{Hom}(G, H) = 0$; taking any H with order coprime to that of G suffices. However, one can show that $G = \operatorname{Hom}(\hat{\mathbb{Z}}, G)$ as sets, where $\hat{\mathbb{Z}}$ is the profinite completion of the integers $\varprojlim \mathbb{Z}/m\mathbb{Z}$ where the limit system is given by the natural projection map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ whenever $n|m$. This is an example of a pro-representable functor which is not representable.

Theorem 3.5.17. *Fix a scheme S and a geometric point $\bar{s}\colon \operatorname{Spec}\Omega \to S$. Then the functor $\operatorname{Fib}_{\bar{s}}\colon \mathbf{FÉt}/S \to \mathbf{Set}$ is pro-representable.*

Proof. Define the inverse system Λ in $\mathbf{FÉt}/S$ by taking objects to be Galois covers $P_\alpha \to S$, with $P_\alpha \prec P_\beta$ if there exists a morphism $P_\beta \to P_\alpha$. Note that this forms a directed set because for any $P_\alpha, P_\beta \in \Lambda$, we can apply theorem 3.4.13 to a connected component Z of $P_\alpha \times_S P_\beta$ to obtain a P_γ and maps $P_\gamma \to Z \to P_\alpha$ and $P_\gamma \to Z \to P_\beta$.

However, this morphism is generally not unique, and we require unique arrows to form a directed system. To remedy this, consider the additional data of an arbitrary $p_\alpha \in \operatorname{Fib}_{\bar{s}}(P_\alpha)$

for every α. Then by corollary 3.4.4 there exists a unique S-automorphism λ of P_β such that $\phi \circ \lambda$ sends p_β to p_α. We can now define $\phi_{\alpha\beta} := \phi \circ \lambda$, and by construction, this is the unique map $P_\beta \to P_\alpha$ such that $\mathrm{Fib}_{\bar{s}}(\phi_{\alpha\beta}(p_\beta)) = p_\alpha$.

As in the classical case, we have for each $X \in \mathbf{FÉt}/S$, $P_\alpha \in \Lambda$, a map $\mathrm{Hom}(P_\alpha, X) \to \mathrm{Fib}_{\bar{s}}(X)$ given by $\phi \mapsto \mathrm{Fib}_{\bar{s}}(\phi)(p_\alpha)$. This respects the inverse system in the sense that a map $\phi_{\alpha\beta}\colon P_\beta \to P_\alpha$ induces a map $\mathrm{Hom}(P_\beta, X) \to \mathrm{Hom}(P_\alpha, X)$ by precomposition. Therefore we obtain a functorial map

$$\varinjlim_{P_\alpha \in \Lambda} \mathrm{Hom}(P_\alpha, X) \to \mathrm{Fib}_{\bar{s}}(X). \tag{3.7}$$

We aim to construct an inverse to this map. Without loss of generality, we can assume that X is connected; recall from theorem 3.4.13 that we can choose a Galois closure $\pi\colon P \to X$. Note that $P = P_\alpha$ for some α, so for any $\bar{x} \in \mathrm{Fib}_{\bar{s}}(X)$ there exists a unique S-automorphism λ such that $\mathrm{Fib}_{\bar{s}}(\pi \circ \lambda)\colon p_\alpha \mapsto \bar{x}$. The map sending $\bar{x}$ to the element corresponding to $\pi \circ \lambda$ in the left hand side of (3.7) gives the required inverse. □

Corollary 3.5.18. *The automorphism groups* $\mathrm{Aut}(P_\alpha)^{op}$ *form an inverse system whose limit is isomorphic to* $\pi_1(S, \bar{s})$.

Proof. We claim that every automorphism of $\mathrm{Fib}_{\bar{s}}$ arises from an automorphism of the inverse system $(P_\alpha)_{\alpha\in\Lambda}$, meaning a collection of automorphisms $\lambda_\alpha \in \mathrm{Aut}(P_\alpha/S)$ compatible with the transition maps. Indeed, any automorphism of $\mathrm{Fib}_{\bar{s}}$ sends the collection (p_α) of distinguished elements to a corresponding system (p'_α), and since P_α are all Galois, each assignment $p_\alpha \to p'_\alpha$ gives rise to an automorphism λ_α of P_α. The λ_α respect the transition maps precisely because (p_α) and (p'_α) are compatible systems.

Now by theorem 3.4.11, for $P_\alpha \prec P_\beta$ we have natural surjections $\mathrm{Aut}(P_\beta/S) \to \mathrm{Aut}(P_\alpha/S)$. Thus we have an inverse system $(\mathrm{Aut}(P_\alpha/S))_{\alpha\in\Lambda}$, and by the above, the automorphisms of $\mathrm{Fib}_{\bar{s}}$ correspond bijectively to automorphisms of the system (P_α), which are precisely the elements of the inverse limit. □

Covering spaces

Theorem 3.5.19 (Generalised Riemann existence theorem)**.** *Let X be a smooth variety over $\mathbb{C}$. Then every finite finite covering space of X has the structure of a smooth variety.*

Proof. Proving this is long and hard, and we point the reader to the corresponding references in [Sza09], Thm. 5.7.4, along with Serre's seminal 'GAGA' paper [Ser55]. □

In light of this and theorem 3.2.12, it is reasonable to suspect that our purely algebraic definition of π_1 might contain as a special case the theory of covering spaces. That is indeed the case.

Example 3.5.20. Recall (a special case of) the Riemann-Hurwitz formula: given a holomorphic map $f\colon X \to Y$ between Riemann surfaces, we have $2 - 2g_Y = \deg f \cdot (2 - 2g_X)$, where g_X is the genus of X.[3] Because of this, we know there are no non-constant holomorphic maps $\mathbb{P}^1_{\mathbb{C}} \to Y$ with $g_Y > 0$, since otherwise we would have $0 < \deg f \cdot (2 - 2g_X) = 0$.

[3] See for example [Sza09] Cor 3.6.12ff. or [Don11] section 7.2.1.

Since any étale covering induces such a map, any such covering of $\mathbb{P}^1_{\mathbb{C}}$ is trivial. Thus all Galois covers of $\mathbb{P}^1_{\mathbb{C}}$ are isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, and so in the limit we obtain $\pi_1(\mathbb{P}^1_{\mathbb{C}}, x) = 0$ for any $x \in \mathbb{P}^1_{\mathbb{C}}$, which agrees with our topological intuition.

We finally prove our main result, Grothendieck's π_1-theorem.

Proof of theorem 3.5.3. (i) Fix an inverse system $(P_\alpha)_{\alpha\in\Lambda}$ which pro-represents $\mathrm{Fib}_{\bar{s}}$. Since the groups $\mathrm{Aut}_S(P_\alpha)$ are finite by corollary 3.4.7, $\pi_1(S, \bar{s})$ is profinite. An action of $\pi_1(S, \bar{s})$ on $\mathrm{Fib}_{\bar{s}}(X)$ is induced by a corresponding automorphism of the inverse system, and this left action is continuous because if $\bar{x} \in \mathrm{Fib}_{\bar{s}}(X)$ comes from $\mathrm{Hom}(P_\alpha, X)$ for some Galois cover P_α, then the automorphism factors through $\mathrm{Aut}(P_\alpha/S)$, which has the discrete topology.

(ii) Recall that proving that a functor is an equivalence of categories is tantamount to showing that it is *essentially surjective*, meaning that any object in the codomain is the image of some element of the domain, and that it is *fully faithful*, that the functor induces a bijection of corresponding Hom-sets.

Let Σ be a finite continuous $\pi_1(S, \bar{s})$-set. By considering each orbit separately, we may assume that the group action is transitive. Fix a point $x \in \Sigma$, and note that by lemma A.3.7, Stab_x is an open subgroup of $\pi_1(S, \bar{s})$. Let $\pi_\alpha\colon \pi_1(S, \bar{s}) \to \mathrm{Aut}(P_\alpha/S)^{\mathrm{op}}$ be the natural projection maps, and note that $(N_\alpha \coloneqq \ker \pi_\alpha)_{\alpha\in\Lambda}$ form a basis of open neighbourhoods of 1 in $\pi_1(S, \bar{s})$. Then Stab_x contains some N_α, and we can consider the image H of Stab_x in $\mathrm{Aut}(P_\alpha/S)^{\mathrm{op}}$. By lemma 3.4.9, we obtain an action of H^{op} on P_α, and define X to be the quotient set. Then $\Sigma \cong \mathrm{Fib}_{\bar{s}}(X)$, so $\mathrm{Fib}_{\bar{s}}$ is essentially surjective.

Finally, we prove fully faithfulness. Fix finite étale S-schemes X and Y, and $\Phi\colon \mathrm{Fib}_{\bar{s}}(X) \to \mathrm{Fib}_{\bar{s}}(Y)$ be a $\pi_1(S, \bar{s})$-equivariant map. Up to considering orbits separately, this is determined by the action at some $x \in \mathrm{Fib}_{\bar{s}}(X)$. Since Φ is $\pi_1(S, \bar{s})$-equivariant, we have a natural inclusion $\mathrm{Stab}_{\Phi(x)} \subset \mathrm{Stab}_x \subset \pi_1(X, \bar{s})$, and by theorem 3.4.11 these determine a unique map $X \to Y \to S$. Thus we have a bijection $\mathrm{Hom}_{\mathbf{F\acute{E}t}}(X, Y) \to \mathrm{Hom}(\mathrm{Fib}_{\bar{s}} X, \mathrm{Fib}_{\bar{s}} Y)$, as required. □

Proposition 3.5.21. *Let S be a connected scheme. Given geometric points $\bar{s}\colon \operatorname{Spec}\Omega \to S$ and $\bar{s}'\colon \operatorname{Spec}\Omega' \to S$, there exists a natural isomorphism of functors $\mathrm{Fib}_{\bar{s}} \xrightarrow{\sim} \mathrm{Fib}_{\bar{s}'}$.*

Proof. From the proof above, it is clear that the inverse systems have the same objects, but the morphisms might differ. So, let $(P_\alpha, \phi_{\alpha\beta})$ and $(P_\alpha, \psi_{\alpha\beta})$ be inverse systems defining $\mathrm{Fib}_{\bar{s}}$ and $\mathrm{Fib}_{\bar{s}'}$, respectively. Associated with $(P_\alpha, \phi_{\alpha\beta})$ we have distinguished points $p_\alpha \in \mathrm{Fib}_{\bar{s}}(P_\alpha)$. Fix $\lambda_\beta \in \mathrm{Aut}(P_\beta/S)$. Now define λ_α to be the unique isomorphism sending p_α to $p'_\alpha \coloneqq \mathrm{Fib}_{\bar{s}}(\phi_{\alpha\beta})(p_\beta)$. By corollary 3.4.4 applied to $\bar{z} = p_\alpha, f_1 = \psi_{\alpha\beta} \circ \lambda_\beta$ and $f_2 = \lambda_\alpha \circ \phi_{\alpha\beta}$ we have a commutative diagram

$$\begin{array}{ccc} P_\beta & \xrightarrow{\lambda_\beta} & P_\beta \\ \downarrow{\scriptstyle \phi_{\alpha\beta}} & & \downarrow{\scriptstyle \psi_{\alpha\beta}} \\ P_\alpha & \xrightarrow{\lambda_\alpha} & P_\alpha \end{array}$$

and defining maps $\rho_{\alpha\beta}\colon \mathrm{Aut}(P_\beta/S) \to \mathrm{Aut}(P_\alpha/S)$ by $\lambda_\beta \mapsto \lambda_\alpha$ we obtain an inverse system $(\mathrm{Aut}(P_\alpha/S), \rho_{\alpha\beta})$ of non-empty finite sets. It is straightforward to verify that the inverse limit of such a system has a non-empty limit, so in particular there exists an element λ in the limit which defines an isomorphism $(P_\alpha, \phi_{\alpha\beta}) \to (P_\alpha, \psi_{\alpha\beta})$. □

By considering the isomorphism $\phi \mapsto \lambda^{-1} \circ \phi \circ \lambda$, we obtain the following:

Corollary 3.5.22. *If S is connected and $\bar{s}\colon \Omega \to S$ and $\bar{s}' : \Omega' \to S$ are geometric points, then $\pi_1(S, \bar{s}) \cong \pi_1(S, \bar{s}')$.*

Appendix A

Commutative Algebra

A.1 The Zariski topology

Definition A.1.1. Let R be a ring. The (prime) **spectrum** of R is the set of prime ideals in R, denoted by $\operatorname{Spec} R$.

Example A.1.2. Let k be a field. Then $\operatorname{Spec} k$ consists of a single point, corresponding to the unique prime ideal (0).

Example A.1.3. For $R = \mathbb{Z}$, we have that $\operatorname{Spec} \mathbb{Z} = \{(p) : p \text{ is prime}\} \cup 0$.

Example A.1.4. Let k be a field, and $R = k[x]$. Then every point of $\operatorname{Spec} R$ is uniquely identified with a polynomial irreducible over k, as $k[x]$ is a PID.

Example A.1.5. As a special case of the previous example, consider $\mathbb{A}^1_{\mathbb{C}} \coloneqq \operatorname{Spec} \mathbb{C}[x]$. Since the irreducible polynomials over $\mathbb{C}$ are all monomials, we see that there is a correspondence $\operatorname{Spec} \mathbb{C}[x] \leftrightarrow \mathbb{C} \cup *$ determined by $(x - a) \mapsto a$. However, 0 is also prime, so we treat this as corresponding with $*$ above. Note that 0 unlike the other elements $\mathfrak{p} \in \operatorname{Spec} \mathbb{C}[x]$ does not correspond to a maximal ideal.

Definition A.1.6. Let R be a ring. The set $V(f) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} R : f \in \mathfrak{p}\}$ is called a **principal closed set** in $\operatorname{Spec} R$, and its complement, denoted by $D(f) \coloneqq V(f)^c$, is called a principal open set.

We will promptly justify the name:

Proposition A.1.7. *The collection of principal closed sets $V(f)$ generate a topology on* $\operatorname{Spec} R$.

Proof. In order for $\{D(f) : f \in R\}$ to be a basis of open sets on $\operatorname{Spec} R$, we need to check that intersection of two principal sets contains a principal open set, and that the collection covers $\operatorname{Spec} R$; the latter is evident since $\operatorname{Spec} R = D(1)$. By definition, $D(f) \cap D(g)$ is the set of prime ideals containing neither f nor g. But by definition of being prime, this implies that $fg \notin D(f) \cap D(g)$, so $D(fg) \subset D(f) \cap D(g)$. $\square$

Definition A.1.8. The topology so defined is called the **Zariski topology** on $\operatorname{Spec} R$.

Proposition A.1.9. *Every ring homomorphism $\phi \colon R \to S$ induces a map $\tilde{\phi} \colon \operatorname{Spec} S \to \operatorname{Spec} R$ which is continuous with respect to the Zariski topology. Morover, this is* functorial *in the sense that if $R \xrightarrow{\phi} S \xrightarrow{\psi} A$ are ring morphisms, then $\widetilde{\psi \circ \phi} = \tilde{\phi} \circ \tilde{\psi}$.*

Proof. Let us first check that $\tilde{\phi}\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is well-defined: if $\mathfrak{p} \in \operatorname{Spec} S$, define $\tilde{\phi}(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. Suppose $ab \in \phi^{-1}(\mathfrak{p})$. Then $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, so $\phi(a)$ or $\phi(b) \in \mathfrak{p}$, and so either a or b is in $\phi^{-1}(\mathfrak{p})$, so $\phi^{-1}(\mathfrak{p})$ is indeed a prime ideal.

To show continuity, it suffices to show that for any $f \in R$, there exists $g \in S$ such that $\tilde{\phi}^{-1}(D(f)) \subset D(g)$. But taking $g = \phi(f)$, we see that $\tilde{\phi}^{-1}(\mathfrak{p})$ contains f if and only if $\mathfrak{p}$ contains $\phi(f) = g$. Finally, the functoriality condition is easily verified. □

Corollary A.1.10. *We can regard* Spec *as a functor* Spec : **Ring** → **Top**.

In algebraic geometry, there is an unfortunate convention of meaning "compact Hausdorff" when one says "compact"; the replacement for "compact" is the following:

Definition A.1.11. A topological space is **quasi-compact** if every open covering has a finite subcovering.

Proposition A.1.12. *The topological space* Spec R *is quasi-compact for any ring* R.

Proof. Let $\{U_i\}$ be an open covering of $\operatorname{Spec} R$ where R is Noetherian, and write $U_i = \bigcup_j D(f_{ij})$. Then $D(1) = \operatorname{Spec} R = \bigcup_{i,j} D(f_{ij})$, so $1 \in (f_{i,j})_{i,j}$, and by definition there exists a finite subcollection, say $f_1, \ldots, f_n$, so that $1 = f_1 r_1 + \ldots + f_n r_n$, hence $\{D(f_i)\}_{i=1}^n$ is a finite subcover. □

A.2 Localisation

Definition A.2.1. Let R be a commutative ring, and let $S \subseteq R$ be a multiplicative set, that is, a set containing 1 and is closed under multiplication. The **localisation of** R **at** S, written $(j, S^{-1}R)$, is the solution to the following universal mapping problem: for any ring T, if $f\colon R \to T$ maps every element of S to a unit of T, then there exists a unique ring homomorphism $g\colon S^{-1}R \to T$ such that the following diagram commutes.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "j"'] & T \\ S^{-1}R \arrow[ur, dashed, "\exists! g"'] & \end{tikzcd}$$

Remark. While it's hardly obvious from the definition, an informal description of the localisation of R at S is a ring consisting of elements of R where those in S are treated as units. For example, if R is an integral domain, then we want the localisation of R at $S = R \setminus 0$ to be the field of fractions of R. The unique map g is then defined by $g(r/s) = f(r)f(s)^{-1}$.

If we drop the assumption of R being an integral domain, the condition $\frac{a}{b} = \frac{c}{d}$ iff $ac - bd = 0$ cannot possibly hold. For example, if $R = \mathbb{Z}/6\mathbb{Z}$ and $S = \{1, 2, 4\}$, then we would obtain $\frac{0}{1} = \frac{0}{2} = \frac{3}{1}$ which would seem to imply that $0 = 3$. This corresponds to the fact that the map j is not injective in general.

We ought to prove the existence of such an object:

Proposition A.2.2. *The localisation of* R *at* S *exists.*

Proof. Define an equivalence relation on $R \times S$ by $(r_1, s_1) \sim (r_2, s_2)$ if there exists a $t \in S$ such that $t(r_1 s_2 - r_2 s_1) = 0$, and let $S^{-1}R \coloneqq R \times S/\sim$. Let $j \colon R \to R \times S$ be defined by $(r, 1)$, and let $g \colon (r, s) \mapsto f(r)f(s)^{-1}$. Then $g \circ j \colon r \mapsto f(r)$, so $(j, R \times S/\sim)$ satisfies the universal property. □

Lemma A.2.3. *The localisation $S^{-1}R$ is trivial if and only if $0 \in S$.*

Proof. If $0 \in S$, then $(r, s) \sim (0, 1)$ for any $(r, s) \in R \times S$. If $0 \notin S$, then $(0, 1) \in R^{-1}S$. □

Example A.2.4. Let $\mathfrak{p}$ be a prime ideal in R, and take $S = R \setminus \mathfrak{p}$. Then $S^{-1}R$ is called the *localisation of R at $\mathfrak{p}$*.

Recall that a *local ring* is a ring with a unique maximal ideal.

Proposition A.2.5. *The localisation $R_\mathfrak{p}$ is a local ring.*

Proof. If $I \leq R$ is not contained in $\mathfrak{p}$, then $j(i)$ is a unit for any $i \in I$, and so $j(I) = R_\mathfrak{p}$. Thus any ideal in $R_\mathfrak{p}$ is contained in $j(\mathfrak{p})$, so this is the maximal ideal. □

Example A.2.6. Fix $f \in R$, and let $S = \{1, f, f^2, \ldots\}$. Then S is a multiplicative set, and $R_f \coloneqq S^{-1}R$ is called the *localisation of R at f*. By Lemma A.2.3, R_f is trivial if and only if f is nilpotent.

Example A.2.7. Let S consist of the elements of R that are not zero-divisors. Then $Q(R) \coloneqq S^{-1}R$ is called the *total ring of fractions of R*.

Lemma A.2.8. *Let R be a ring, $\mathfrak{m}$ a maximal ideal in R, and let $j \colon R \to R_\mathfrak{m}$ be the localisation map. Then $R_\mathfrak{m}/j(\mathfrak{m}) = R/\mathfrak{m}$.*

Proof. Let $R \to R_\mathfrak{m}$ be the natural map, and by composing with the quotient map we have a ring morphism $\phi \colon R \to R_\mathfrak{m}/j(\mathfrak{m})$. Since $\operatorname{Im}\phi$ is an ideal in $R_\mathfrak{m}/j(\mathfrak{m})$ and ϕ is not the zero-map, ϕ is surjective. Now we see that $\mathfrak{m} \leq \ker\phi$, which by maximality implies that $\mathfrak{m} = \ker\phi$. We thus have an isomorphism $R_\mathfrak{m}/j(\mathfrak{m}) = R/\mathfrak{m}$, as required. □

Proposition A.2.9. *Let $D(f) \subset D(g)$ for some $f, g \in R$. Then there exists $r \in R$ and $n \in \mathbb{N}$ such that $f^n = gr$.*

Proof. [LE06], Lemma 2.1.6b) □

A.3 Topological groups

Definition A.3.1. A **topological group** is a group element in the category of topological spaces. In other words, it is a topological space G along with continuous maps $m \colon G \times G \to G$ and $i \colon G \to G$, and a distinguished element $e \in G$, such that $m(g, e) = m(e, g) = g$, $m(g, i(g)) = e$, and $m(m(g, g'), g'') = m(g, m(g', g''))$, for all $g, g', g'' \in G$.

Of course, m and i are just multiplication and inversion, respectively, and we tend to write $gg' \coloneqq m(g, g')$ and $g^{-1} \coloneqq i(g)$. Alternatively, we can define a topological group as a group where the underlying set has a topology, with respect to which the group operations are continuous.

Example A.3.2. Let G be any finite group equipped with the discrete topology. Then m and i are automatically continuous, and so G can be regarded as a topological group.

Example A.3.3. Let $G = \mathbb{C}$ with the standard topology. Then it is easy to verify that $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous, so $\mathbb{C}$ is also a topological group. Similarly, we can show that $\mathbb{C}^\times$ with operations $(x, y) \mapsto xy$ and $x \mapsto 1/x$ is a topological group.

When considering topological groups, we can often reduce local problems to considering neighbourhoods of the identity. More precisely, if N is a neighbourhood of g, then $g^{-1}N$ is a neighbourhood of e, open if and only if N is open.

Definition A.3.4. A **profinite group** is a topological group for which the underlying space is compact Hausdorff and totally disconnected.

Proposition A.3.5. *Let G be a Hausdorff group. Then the following are equivalent:*

(i) *G is profinite,*
(ii) *G is compact and G admits a basis of neighbourhoods of the identity consisting of clopen normal subgroups,*
(iii) *G is a topological inverse limit of finite discrete groups.*

Proof. [Neu08], Prop. 1.1.3. $\square$

Definition A.3.6. Let G be a group. Then a G-**set** S is a set S along with a group action $G \circlearrowright S$. A G-**equivariant map** is a set map $f : S \to S'$ for which $g \cdot f(s) = f(g \cdot s)$ for all $g \in G, s \in S$.

The category of G-sets with morphisms given by G-equivariant maps, denoted by $\mathbf{Set}_G$, is a full subcategory of **Set**. If a group G acts on a topological space X, we say that the *action is continuous* if the associated map $G \times X \to X$ is.

Lemma A.3.7. *Let X be a topological space equipped with the discrete topology, and suppose a topological group G acts continuously on X. Then* $\mathrm{Stab}_x := \{g \in G : gx = x \text{ for all } x \in X\}$ *is an open subgroup of G.*

Proof. Fix $x \in X$, let $m: G \times X \to X$ denote the multiplication map, and let $i_x: G \to G \times X$ be the inclusion $g \mapsto (g, x)$. Then $m \circ i_x$ is continuous, and since $\{x\}$ is open, $\mathrm{Stab}_x = (m \circ i_x)^{-1}(x)$ is open. $\square$

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