Proper base change I Étale cohomology study group

Wojtek Wawrów

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Review: edge maps in spectral sequences

Theorem (Grothendieck spectral sequence)

Suppose $A \xrightarrow{G} B \xrightarrow{F} C$ is a chain of functors between categories. Under appropriate conditions, for any $a \in A$ we have a spectral sequence

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These are the *edge maps*. In our case of interest these are the maps

$$R^n f_* \circ g'_* \to R^n(f_*g'_*) = R^n(g_*f'_*) \to g_* \circ R^n f'_*.$$



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Suppose $f : X \to Y$ is a proper morphism and \mathcal{F} is a torsion étale sheaf. Then the base change morphism is an isomorphism.

There are other conditions which imply this isomorphism, e.g. g being smooth (assuming extra technical conditions.)

Suppose we have a Cartesian diagram of topological spaces:



We have to check the isomorphism on stalks: for all $y \in Y'$ we want an isomorphism

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Observe that the fibers $X_{g(y)}$ and X'_{y} are isomorphic, and the isomorphism identifies $\mathcal{F}|_{X_{g(y)}}$ with $(g'^*\mathcal{F})|_{X'_{y}}$. It would be enough to have an isomorphism $(R^nf_*\mathcal{F})_{g(y)} \cong H^n(X_{g(y)}, \mathcal{F}|_{X_{g(y)}})$ —base change for inclusion of a point.

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You can prove this for higher cohomology too. See Milne's *Lectures on Étale Cohomology*, Section 17.

Let (A, \mathfrak{m}) be a henselian local ring, $X \to \operatorname{Spec}(A)$ proper, Z the special fiber. For any (Zariski) sheaf \mathcal{F} on X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

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Sections on geometric fibers

Proposition

In the above situation, let $i_Z : Z \to X$ be the inclusion. For an étale sheaf \mathcal{F} , we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i_Z^{-1}\mathcal{F})$.

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Corollary

 $f : X \to Y$ proper, \overline{y} a geometric point of Y. For any étale sheaf \mathcal{F} we have an isomorphism $(f_*\mathcal{F})_{\overline{y}} \to \Gamma(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$.

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Corollary (Base change in degree 0)

Base change map $g^{-1}f_*\mathcal{F} \to f'_*g'^*\mathcal{F}$ is an isomorphism.

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Definition

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Base change holds for f iff for all injective \mathbb{Z}/ℓ -sheaves \mathcal{I} on X_{et} and $g: Y' \to Y$, $g'^{-1}\mathcal{I}$ is f'_* -acyclic: $R^n f'_*(g'^{-1}\mathcal{I}) = 0$.

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One direction clear. For the converse, use colimits and induction to go from \mathbb{Z}/ℓ to arbitrary torsion. Then use acyclic resolutions $g'^{-1}\mathcal{I}^{\bullet}$ to compute cohomology and use n = 0 case.

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Proposition

If f_1 is surjective and base change works for $f_1, f_2 \circ f_1$, then it works for f_2 .

Let $f : X \to Y$ be finite. Then $R^n \mathcal{F} = 0$ for all n > 0 and all étale sheaves \mathcal{F} on X.

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Corollary

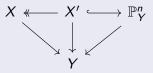
Base change works for finite morphisms.

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Can assume Y affine. Chow's lemma:

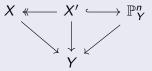


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Hence enough to show for $\mathbb{P}_Y^n \to Y$. We have a finite surjective map $(\mathbb{P}_Y^1)^n \to \mathbb{P}_Y^n$. Finally we can reduce to \mathbb{P}^1 by induction.