Infinite-Dimensional Lie Algebras

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Abstract

Notes from a course at the University of Edinburgh given during the spring of 2020 by Dr. Andrea Appel et al., supplemented by some stuff from various other textbooks. Topics marked in the index by an asterisk were not taught.

Contents

1 Lie algebra basics

1.1 Basic concepts

Recall that an *algebra* over a field k is a k-vector space with an additional product structure. Equivalently, it is a k-vector space which is also a ring. Equivalently, it is a ring into which there is a ring homomorphism from k.

Definition 1.1. A **Lie algebra** is a vector space $\mathfrak{g}$ over a field k equipped with a *bracket* $[-,-]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear and skew-symmetric, and obeys the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \tag{1.1}$$

In fact, by skew-symmetry, linearity is sufficient to ensure bilinearity. The *dimension* of a Lie algebra is the dimension of the underlying vector space. If X and Y are subsets of $\mathfrak{g}$, $[X, Y]$ denotes the subspace spanned by the elements $\{[x, y] : x \in X, y \in Y\}$.

Example 1.2. Any vector space is a Lie algebra with the bracket defined by $[x, y] = 0$. It is then called an *abelian Lie algebra*.

Example 1.3. Let $V = \mathbb{R}^3$, and $[v, w] \coloneqq v \times w$, the cross product. A straightforward albeit tedious computation shows that this forms a Lie algebra.

Example 1.4. Any k-algebra is a Lie algebra under the bracket $[x, y] = xy - yx$, which is called the *commutator bracket*. In particular, $\mathrm{Mat}_n(k)$ is a Lie algebra for any field k and natural number n, and we denote it by $\mathfrak{gl}_n(k)$.[1]

Definition 1.5. A **morphism of Lie algebras** is a k-linear map $\phi: \mathfrak{g} \to \mathfrak{h}$ which satisfies $\phi([x, y]_\mathfrak{g}) = [\phi(x), \phi(y)]_\mathfrak{h}$ for any $x, y \in \mathfrak{g}$. As usual, an isomorphism is an invertible morphism.

Example 1.6. For any k-vector space V, $\mathrm{End}(V)$ has a Lie algebra structure given by the bracket $[f, g] = f \circ g - g \circ f$. It is a nice exercise to show that this is isomorphic to $\mathfrak{gl}_n(k)$ if $n = \dim V < \infty$.

Let us denote the category of Lie algebras over a field k by $\mathbf{Lie}_k$. Because $\mathbf{Lie}_k$ is a full subcategory of $\mathbf{Vec}_k$, we get a lot of structure for free. For example, we have kernels, cokernels and direct sums. We also have interesting subobjects:

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ a vector subspace of $\mathfrak{g}$. Then $\mathfrak{h}$ is a **Lie subalgebra** of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, and an **ideal** of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

Example 1.8. The Lie algebra $\mathfrak{gl}_n(k)$ has a natural Lie subalgebra $\mathfrak{sl}_n(k)$ defined by $\{x \in \mathfrak{gl}_n(k) : \mathrm{tr}(x) = 0\}$. As the notation suggests, this is isomorphic to the tangent space at the identity of $\mathrm{Sl}_n(k)$.

[1] This might seem confusing, but arises from the fact that $\mathfrak{gl}_n$ is isomorphic to the tangent plane of the Lie group Gl_n. The convention is to use gothic letters for Lie algebras and regular ones for Lie groups, although historically it was the other way around.

For an ideal $\mathfrak{i} \subseteq \mathfrak{g}$, we can define a quotient object by taking the vector space $\mathfrak{g}/\mathfrak{i}$ along with the bracket $[\bar{x}, \bar{y}] \coloneqq \overline{[x, y]}$.

Definition 1.9. The ideal $\mathfrak{z} \coloneqq \{x \in \mathfrak{g} : [x, \mathfrak{g}] = 0\}$ (gothic letter z) is called the **centre** of $\mathfrak{g}$.

Naïvely, one might expect that quotienting a Lie algebra by its centre gives a new Lie algebra without a centre; however, this is not generally the case, as the next example shows. This is the case for the commutator bracket, though.

Example 1.10. Consider the vector space spanned by the basis $u_1, \dots, u_n, v_1, \dots, v_n, z$,

$$\mathcal{H}eis_{2n+1} \coloneqq \bigoplus_{i=1}^{n} \mathbb{C}u_i \oplus \bigoplus_{i=1}^{n} \mathbb{C}v_i \oplus \mathbb{C}z, \tag{1.2}$$

along with the bracket defined by $[u_i, v_j] = \delta_{i,j} z$ and $[\mathcal{H}eis_n, z] = 0$, where $\delta_{i,j}$ is the Kronecker delta. This is called the *Heisenberg algebra* of dimension n. Note that the centre of $\mathfrak{h}$ is $\mathbb{C}z$, but $\mathfrak{h}/\mathbb{C}z$ is abelian, so $\mathcal{H}eis_n$ does not have trivial centre. We will later study the Heisenberg algebra with $n = \infty$.

In particular, the 3-dimensional Heisenberg algebra is isomorphic to the subalgebra of $\mathfrak{sl}_3(k)$ consisting of elements of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with basis} \quad \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \tag{1.3}$$

and one easily checks that these satisfy the relations above.

Example 1.11. It is a nice exercise to show that the centre of $\mathfrak{gl}_n(k)$ consists only of $\mathbb{C}\mathbb{I}_n$, where $\mathbb{I}_n$ is the $n \times n$-identity matrix. One can also show that $\mathfrak{sl}_n$ has no centre, and we have the decomposition $\mathfrak{gl}_n = \mathfrak{sl}_n \times \mathbb{C}\mathbb{I}_n$.

1.2 Representation theory of Lie algebras

Definition 1.12. Let $\mathfrak{g}$ be a Lie algebra, V a vector space and $\pi \colon \mathfrak{g} \longrightarrow \mathrm{End}(V)$ a Lie algebra morphism. The pair (π, V) is called a **representation** of $\mathfrak{g}$, and V the **representation space** of (π, V).

We often think of the vector space V as having the structure of a $\mathfrak{g}$-module, and are prone to write $x \cdot v$ or even xv to mean $\pi(x)v$.

Example 1.13. For any vector space V, we have the trivial representation $\pi(x) \colon v \mapsto 0$.

Example 1.14. Taking $V = \mathfrak{g}$, we define the *adjoint representation* given by $\pi \colon x \mapsto [x, -]$. Note that the Jacobi identity is equivalent to the statement that π is indeed a Lie algebra morphism.

Definition 1.15. Let V and V' be representations of a Lie algebra $\mathfrak{g}$. An **intertwining**, or Lie algebra morphism ϕ is a linear map ϕ such that $\phi(xv) = x\phi(v)$ for all $v \in V$.

Definition 1.16. Let $\pi\colon \mathfrak{g} \to \mathrm{End}_k(V)$ be a representation. A **subrepresentation of** V is a subspace V' fixed by π, that is, one which satisfies $xv' \in V'$ for all $v' \in V'$. A representation is **irreducible** if it contains no subrepresentations other than itself and 0.

Comparing definitions, we see that the category of representations of $\mathfrak{g}$ is naturally isomorphic to the category of $\mathfrak{g}$-modules. In practice, we often use both languages, as in the proof of the following:

Proposition 1.17 (Schur's lemma)**.** *Let V and V' be representations of a Lie algebra $\mathfrak{g}$ over a field k, and $\phi\colon V \to W$ a non-trivial intertwining.*

(i) *If V is irreducible, then ϕ is injective.*
(ii) *If W is irreducible, then ϕ is surjective.*
(iii) *If k is algebraically closed and V a finite-dimensional irreducible representation, then every intertwining $\phi\colon V \to V$ acts by scalar multiplication.*

Proof. For (i) and (ii), one checks that $\ker\phi$ and $\mathrm{Im}\,\phi$ are submodules of V and W, respectively, and that $\ker\phi = 0$ (resp. $\mathrm{Im}\,\phi = W$) if and only if ϕ is injective (resp. surjective).

To prove (iii), note first that by (i) and (ii), ϕ is an isomorphism. Therefore, it has non-zero eigenvalues λ_i, and by the Cayley-Hamilton theorem we have $\det(\phi - \lambda_i \mathbb{I}) = 0$. But $\phi - \lambda_i\mathbb{I}$ is also an intertwining, and by the above, it must be identically 0. Thus $\phi = \lambda\mathbb{I}$, as claimed. □

1.3 The universal enveloping algebra

Basic UEA

An issue with $\mathfrak{g}$ is that it does not carry a useful structure of an algebra. Often, it is useful to replicate the behaviour of Lie algebras which are actually algebras. For example, it would be convenient to be able to write $[x, y] \cdot v = (xy - yx) \cdot v$ in the notation established above. With some care, we can do this by introducing a monoidal structure on $\mathfrak{g}$. This idea naturally leads to the following:

Definition 1.18. Let $\mathfrak{g}$ be a Lie algebra. The **universal enveloping algebra** $\mathfrak{U}(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$, $T(\mathfrak{g}) \coloneqq \bigoplus_{n\geq 1} \mathfrak{g}^{\otimes n}$, quotiented by the ideal generated by $x \otimes y - y \otimes x - [x, y]$, for $x, y \in \mathfrak{g}$.

It is a good exercise to convince oneself that this is a pretty reasonable way to solve the problem above. In particular, every representation (π, V) of $\mathfrak{g}$ induces a representation $(\widetilde{\pi}, V)$ of $\mathfrak{U}(\mathfrak{g})$ by $\widetilde{\pi}(x_1 \dots x_n) \coloneqq \pi(x_1) \dots \pi(x_n)$, and any representation of $\mathfrak{U}(\mathfrak{g})$ defines a representation of $\mathfrak{g}$, so that studying representations of $\mathfrak{g}$ is equivalent to studying those of $\mathfrak{U}(\mathfrak{g})$. More generally, any Lie algebra morphism $\mathfrak{g} \to \mathfrak{h}$ induces a map $\mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{h})$ in this fashion.

It is also worth noting that if $\mathfrak{g}$ is already an algebra, then $\mathfrak{g} = \mathfrak{U}(\mathfrak{g})$. This is because the pedantic way to define $\mathfrak{U}(\mathfrak{g})$ is via a universal property, and the quotient of the tensor algebra above happens to satisfy this. But so does the algebra, so $\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{g}$ by the standard argument.

Lemma 1.19. *In the universal enveloping algebra of* $\mathfrak{g}$*, we have*

$$[a, bc] = [a, b]c + b[a, c] \quad \text{and} \quad [ab, c] = [a, c]b + a[b, c] \tag{1.4}$$

for all $a, b, c \in \mathfrak{U}(\mathfrak{g})$.

Proof. Observe that

$$[a, bc] = abc - bca = abc - bac + bac - bca = [a, b]c + b[a, c].$$

The second is proven similarily. □

Intermediate UEA

Universal property

Advanced UEA

UEA gives an equivalence of categories $\mathbf{Rep}_{\mathfrak{g}} \simeq \mathbf{Rep}_{\mathfrak{U}(\mathfrak{g})}$; the functor $\mathbf{Alg}_k \to \mathbf{Lie}_k$ sending A to A with the commutator bracket has a left adjoint.

Since $\mathfrak{U}(\mathfrak{g})$ has elements of the form $\sum c_i x_{i,1} \ldots x_{i,n_i}$, one might expect it to have a natural grading. But the Lie bracket allows us to possibly "reduce the degree" of an element by replacing a product of elements of $\mathfrak{g}$ with a single bracket, so this is somewhat optimistic. It is, however, impossible to raise the degree beyond a certain point, so we obtain a substitute by talking about maximal degree. This gives a decomposition

$$\mathfrak{U}(\mathfrak{g})_0 \subset \mathfrak{U}(\mathfrak{g})_1 \subset \ldots \subset \mathfrak{U}(\mathfrak{g})_i \subset \mathfrak{U}(\mathfrak{g})_{i+1} \subset \ldots, \qquad \mathfrak{U}(\mathfrak{g}) = \bigcup_{i \geq 0} \mathfrak{U}(\mathfrak{g})_i, \tag{1.5}$$

called a *filtration*, which in particular allows us to perform induction on $\mathfrak{U}(\mathfrak{g})$. This allows us to prove the following:

Theorem 1.20 (Poincaré-Birkhoff-Witt)**.** *Let* $\mathfrak{g}$ *be a Lie algebra with a (possibly infinite) basis* $x_1, x_2, \ldots$.

(i) $\mathfrak{U}(\mathfrak{g})$ *has a basis of the form*

$$\{x_1^{m_1} x_2^{m_2} \ldots : m_i \in \mathbb{Z}_{\geq 0}, m_i = 0 \text{ for all but finitely many } i\}; \tag{1.6}$$

(ii) *If* $\mathfrak{g}$ *has a vector space decomposition* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$*, then*

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}_1) \ldots \mathfrak{U}(\mathfrak{g}_n) := \langle \{a_1 \ldots a_n : a_i \in \mathfrak{U}(\mathfrak{g}_i)\} \rangle,$$

inducing an isomorphism of vector spaces $\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{U}(\mathfrak{g}_1) \otimes \ldots \otimes \mathfrak{U}(\mathfrak{g}_n)$*;*

(iii) *Given Lie subalgebras* $\mathfrak{h}, \mathfrak{m}, \mathfrak{n}$ *satisfying* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} \oplus \mathfrak{n}$*, we have* $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{h})(\mathfrak{m}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n})$.

Definition 1.21. A Lie algebra is **simple** if it has no proper ideals. It is **semisimple** if it can be decomposed as the direct sum of simple algebras.

Example 1.22. The Lie algebra $\mathfrak{sl}_n(k)$ is simple, while $\mathfrak{gl}_n(k)$ is not since it has $\mathfrak{sl}_n$ as a proper ideal. However, one can show that $\mathfrak{gl}_n(k) = \mathfrak{sl}_n(k) \oplus \mathbb{C}\mathbb{I}_n$.

Proof. □

Fact 1.23 (Weyl's complete irreducibility theorem)**.** *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lia algebra over an algebraically closed field k, and let $V \in \mathbf{Rep}_{\mathfrak{g}}$. If W is a subrepresentation of V, then there exists a subspace W' of V such that $V = W \oplus W'$.*

Using *Dynkin diagrams*, one can prove the following:

Fact 1.24 (Classification of complex finite-dimensional simple Lie algebras)**.** *Any finite-dimensional simple Lie algebra over $\mathbb{C}$ is one of the following:*

(i) *A_n for $n \geq 2$, where $A_n := \mathfrak{sl}_{n+1}(\mathbb{C})$;*
(ii) *B_n for $n \geq 2$, where $B_n := \mathfrak{so}_{2n+1}(\mathbb{C}) := \{x \in \mathfrak{sl}_{2n+1}(\mathbb{C}) : x^t = -x\}$;*
(iii) *C_n for $n \geq 3$, where $C_n := \mathfrak{sp}_{2n}(\mathbb{C}) := \{x \in \mathfrak{gl}_{2n} : J_n x^t = xJ_n\}$ for $J_n := \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$;*
(iv) *D_n for $n \geq 4$, where $D_n := \mathfrak{so}_n$;*
(v) *The exceptional Lie algebras E_n for $n \in \{6, 7, 8\}$, F_4 and G_2.*

For a proof, see Humphreys *Introduction to Lie Algebras and Representation Theory*, pp. 55 ff.

Example 1.25. Let us consider $\mathfrak{sl}_2(\mathbb{C})$ in detail: let $E_{ij} := [\delta_{ij}]$, and note that $\mathfrak{sl}_2$ is generated by $e = E_{12}, f = E_{21}$ and $h = E_{11} - E_{22}$, subject to the relations

$$[h, e] = 2e \qquad [h, f] = -2f \quad \text{and} \quad [e, f] = h. \tag{1.7}$$

Example 1.26. More generally, for $\mathfrak{sl}_n(k)$ we have generators $e_i := E_{i,i+1}, f_i := E_{i+1,i}$ and $h_i := E_{ii} - E_{i+1,i+1}$, and it is easy to compute their brackets explicitly: $[h_i, e_j] = a_{ji}e_j$, $[h_i, f_j] = -a_{ij}f_j$ and $[e_i, f_j] = \delta_{ij}h_i$, where

$$a_{ij} = \begin{cases} 2 & \text{if} \quad i = j \\ -1 & \text{if} \quad |i - j| = 1 \\ 0 & \text{if} \quad |i - j| > 1. \end{cases} \tag{1.8}$$

Definition 1.27. The matrix $[a_{ij}]$ constructed from the a_{ij} as defined in eq. (1.8) is called the **Cartan matrix** of $\mathfrak{sl}_n$.

Explicitly,

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}. \tag{1.9}$$

Theorem 1.28 (Serre)**.** *If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$, then $\mathfrak{g}$ admits similar object, called a Chevalley presentation.*

(Serre-Chevalley relations?) In fact, it is possible to reconstruct a finite-dimenisonal simple Lie algebra from its Cartan matrix.

1.4 Representations of $\mathfrak{sl}_2(\mathbb{C})$

2 Kac-Moody algebras

2.1 Reconstructing a Lie algebra

We now study a method of reconstructing a simple Lie algebra like $\mathfrak{sl}_2$ from its Cartan matrix. This method will turn out to work for any matrix, so we actually get a method for constructing an entire class of Lie algebras.

Step 1:

Through the construction, we fix $A \in \mathrm{Mat}_n(\mathbb{C})$, where $n \in \mathbb{N}$ is some positive integer.

Definition 2.1. Let $A = [a_{ij}] \in \mathrm{Mat}_n(\mathbb{C})$. A **realisation of** A is a triple $\Sigma = (\mathfrak{h}, \Pi, \Pi^\vee)$, where

- $\mathfrak{h}$ is a vector space,
- $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ is a set of linearly independent elements,
- $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ is a set of linearly independent elements,

and $\alpha_i(\alpha_j^\vee) = a_{ji}$. The set Π is called the set of **simple roots** of A, and $\Pi^\vee$ the set of **simple coroots**.

Usually, we do not explicitly pick α_i and $\alpha_i^\vee$, but rather *define* $\mathfrak{h} := \oplus_{i=1}^n \mathbb{C}\alpha_i^\vee$.

Lemma 2.2. *Let* $\Sigma = (\mathfrak{h}, \Pi^\vee, \Pi)$ *be a realisation of* A*. Then* $\dim \mathfrak{h} \geq 2n - \mathrm{Rank}(A)$*, and we can always find a realisation with* $\dim \mathfrak{h} = 2n - \mathrm{Rank}(A)$*.*

Such a realisation is a called a *minimal realisation*.

Proof. □

We call a *morphism of realisations* $(\mathfrak{h}, \Pi, \Pi^\vee) \to (\widetilde{\mathfrak{h}}, \widetilde{\Pi}, \widetilde{\Pi}^\vee)$ a linear map $\phi\colon \mathfrak{h} \to \widetilde{\mathfrak{h}}$ such that $\phi(\alpha_i^\vee) = \widetilde{\alpha}_i^\vee$ and $\phi^*(\widetilde{\alpha}_i) = \alpha_i$, where $\phi^* : \widetilde{\mathfrak{h}}^* \to \mathfrak{h}^*$ is the adjoint map of ϕ. An isomorphism is, as usual, a morphism with an inverse which is also a morphism.

Proposition 2.3. *For any* $A \in \mathrm{Mat}_n(\mathbb{C})$*, there exists a unique minimal realisation up to isomorphism.*

Proof. Since the underlying vector spaces are isomorphic, we have an obvious map sending root to root and coroot to coroot, and an obvious inverse. □

Out of negligence, we call this *the* realisation of A.

Step 2:

Definition 2.4. Let $\Sigma = (\mathfrak{h}, \Pi, \Pi^\vee)$ be a realisation of A. Then $\widetilde{\mathfrak{g}}(A) \equiv \widetilde{\mathfrak{g}}(\Sigma)$ is the free Lie algebra generated by e_i, f_i and $\mathfrak{h}$ for $i = 1, \dots, n$, modulo the ideal generated by the relations

- $[e_i, f_j] = \delta_{i,j}\alpha_i^\vee$,
- $[h, e_i] = \alpha_i(h)e_i$,
- $[h, f_i] = -\alpha_i(h)f_i$.

Definition 2.5. Fix a realisation $\Sigma = (\mathfrak{h}, \Pi, \Pi^\vee)$. Then

- The set $Q \coloneqq \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ is called the **root lattice** of $\widetilde{\mathfrak{g}}(A)$, and $Q_+ \coloneqq \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$ the **positive root lattice** of $\widetilde{\mathfrak{g}}(A)$.
- Dually, we define $Q^\vee \coloneqq \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ as the **coroot lattice** of $\widetilde{\mathfrak{g}}(A)$, and $Q_+^\vee \coloneqq \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i^\vee$ the **positive coroot lattice** of $\widetilde{\mathfrak{g}}(A)$.
- Regarded as a subalgebra of $\widetilde{\mathfrak{g}}(A)$, $\mathfrak{h}$ is called the **Cartan subalgebra** of $\widetilde{\mathfrak{g}}(A)$.
- For $\alpha \in Q$, define the **weight space** $\widetilde{\mathfrak{g}}_\alpha \coloneqq \{x \in \widetilde{\mathfrak{g}}(A) : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$.
- The set $R \coloneqq \{\alpha \in Q \setminus \{0\} : \widetilde{\mathfrak{g}}_\alpha \neq 0\}$ is called the **set of roots** of $\widetilde{\mathfrak{g}}(A)$.

Example 2.6.

Theorem 2.7. *Fix $A \in \mathrm{Mat}_n(\mathbb{C})$ and a realisation $\Sigma = (\mathfrak{h}, \Pi, \Pi^\vee)$.*

(i) *As a vector space, we have a decomposition $\widetilde{\mathfrak{g}}(A) = \widetilde{n}_+ \otimes \mathfrak{h} \otimes \widetilde{n}_-$, where $\widetilde{n}_+$ is the free Lie algebra generated by $\{e_i\}$ and $\widetilde{n}_-$ that of $\{f_i\}$.*
(ii) *We have $\widetilde{n}_+ = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \widetilde{\mathfrak{g}}_\alpha$ and $\widetilde{n}_- = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \widetilde{\mathfrak{g}}_{-\alpha}$.*
(iii) *There is an inclusion $[\widetilde{\mathfrak{g}}_\alpha, \widetilde{\mathfrak{g}}_\beta] \subset \widetilde{\mathfrak{g}}_{\alpha+\beta}$.*

Note that the defining relations of $\widetilde{\mathfrak{g}}(A)$ mirror the presentation of $\mathfrak{sl}_2$. However, in the case of $\mathfrak{sl}_n$ for $n > 2$, we have further relations in the presentation, and in general it is not necessarily true that $\widetilde{\mathfrak{g}}(A)$ is simple. The final step is to remedy this:

Step 3:

Let $\mathfrak{r}_{\max} \coloneqq \sum \mathfrak{r} \subset \widetilde{\mathfrak{g}}(A)$ where the sum is taken over all ideals $\mathfrak{r} \subset \mathfrak{g}$ such that $\mathfrak{r} \cap \mathfrak{h} = 0$. Then $\mathfrak{r}_{\max}$ is the maximal ideal of $\widetilde{\mathfrak{g}}(A)$ which intersects $\mathfrak{h}$ trivially.

Definition 2.8. Let $A \in \mathrm{Mat}_n(\mathbb{C})$. The **Kac-Moody algebra** associated with A is defined to be $\mathfrak{g}(A) \coloneqq \widetilde{\mathfrak{g}}(A)/\mathfrak{r}_{\max}$.

Here $\mathfrak{h}$ plays the rôle of the Cartan subalgebra in $\mathfrak{sl}_n$. Let us consider some examples:

Example 2.9. Let $A = [2]$, and fix $\Pi = \{\alpha\}$, $\Pi^\vee = \{\alpha^\vee\}$ and $\mathfrak{h} \coloneqq \mathbb{C}\alpha$. Then $\widetilde{g}(A)$ is the free Lie algebra generated by e, f and $h = \alpha$, subject to the relations $[e, f] = 2\alpha^\vee$, $[\alpha, e] = 2e_i$ and $[\alpha, f] = -2f$. We then have a natural surjection $\mathfrak{sl}_2 \to \widetilde{g}(A)$ by $e \mapsto e, f \mapsto f$ and $h \mapsto \alpha^\vee$, and this is in fact an isomorphism since $\mathfrak{sl}_2$ is simple. It is straightforward to show that $\widetilde{\mathfrak{g}}(A)$ has no ideals meeting $\mathfrak{h}$ trivially, so $\widetilde{\mathfrak{g}}(A) = \mathfrak{g}(A)$.

Example 2.10. In a similar fashion, we can take $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ for which $\mathfrak{g}(A) \cong \mathfrak{sl}_3$.

Example 2.11. If we take $A := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, it turns out that $\mathfrak{g}(A)$ is infinite-dimensional.

Although one might think that this is a pathological example, it is in fact a very important object to which we will return later.

2.2 The Serre relations

Throughout this section we will assume that $A = [a_{ij}] \in \mathrm{Mat}_n(\mathbb{Z})$ is symmetrisable, $a_{ii} = 2$ for $i = 1, \ldots, n$, and $a_{ij} \leq 0$ for $i \neq j$.

The terminology given in definition 2.5 transfers in a natural way from $\widetilde{\mathfrak{g}}(A)$ to $\mathfrak{g}(A)$. For example, we say that $\alpha \in Q$ is a *root of* $\mathfrak{g}(A)$ if $\mathfrak{g}_\alpha := \{x \in \mathfrak{g}(A) : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\} \neq 0$. We also define the *multiplicity of* α to be $\mathrm{mult}(\alpha) := \dim \mathfrak{g}_\alpha$. Note that this is finite-dimensional, since for $\alpha = \sum_{i=1}^n k_i \alpha_i \in Q$, we have $\mathrm{mult}(\alpha) \leq n^{|\mathrm{ht}(\alpha)|}$, where $\mathrm{ht}(\alpha) := \sum_{i=1}^n k_i$.

Fact 2.12. *The partially defined map* $(-,-)$ *extends to a non-degenerate symmetric invariant bilinear form* $(-,-) : \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$ *on all of* $\mathfrak{g}(A)$.

Recall that $(-,-)$ is *non-degenerate* if for fixed x, $(x, y) = 0$ for all $y \in \mathfrak{g}(A)$ implies $x = 0$, *symmetric* if $(x, y) = (y, x)$, and *invariant* if $([x, y], z) = (x, [y, z])$.

Corollary 2.12.1. *For* $\alpha, \beta \in Q$, *we have* $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) \neq 0$ *if and only if* $\alpha + \beta = 0$. *In particular,* $\mathfrak{g}_\alpha \cong \mathfrak{g}_{-\alpha}^*$.

Let $\nu : \mathfrak{h} \to \mathfrak{h}^*$ be defined by $\nu(h) = (h, -)$. For any $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$, we have that $[x, y] = (x, y)\nu^{-1}(\alpha)$.

Theorem 2.13 (The Serre relations)**.** *In* $\mathfrak{g}(A)$, *we have*

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \quad \textit{and} \quad \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \tag{2.1}$$

for all $i, j = 1, \ldots, n$.

Example 2.14. In $\mathfrak{sl}_3$, we have $[e_1, [e_1, e_2]] = 0$.

To prove this, we need the following lemma:

Lemma 2.15. *For* $x \in n_+$, $x = 0$ *if and only if* $[f_i, x] = 0$ *for all* i. *Similarly, if* $x \in n_-$, *then* $x = 0$ *if and only if* $[e_i, x] = 0$ *for every* i.

Proof. □

Proof of theorem 2.13. We prove the second equality; the first is completely analogous. As proven in the workshops, the inclusion of the 1×1-matrix $[2]$ into the (i, i)-th entry of A induces an injection $\mathfrak{sl}_2^{(i)} \to \mathfrak{g}(A)$, which makes $\mathfrak{g}(A)$ an $\mathfrak{sl}_2$-module, hence a representation. Here $\mathfrak{sl}_2^{(i)}$ denotes the specific copy of $\mathfrak{sl}_2$ generated by $\langle e_i, f_i, \alpha_i^\vee \rangle$.

For any $\mathfrak{sl}_2$-module V where $h \cdot v = \lambda v$ and $e \cdot v = 0$, we find that

$$-2fv = [h, f]v = hfv - fhv = hfv - f\lambda v,$$

so $hfv = (\lambda - 2)fv$, and similarly

$$\lambda v = hv = [e, f]v = efv - fev = efv,$$

whence $efv = \lambda v$.

Define $v := f_j$ and $w_m := f_i^m v$. Using lemma 2.15, we are done if we can show that $[e_k, w_{1-a_{ij}}] = 0$ for all k. By the above relations, since $h \cdot v = \alpha_i^\vee \cdot v = -a_{ji}f_j$ and $e_i v = 0$, we have

$$\alpha_i^\vee \cdot w_m = (\lambda - 2m)w_m \quad \text{and} \quad e_i \cdot w_m = m(\lambda - m + 1)w_{m-1}.$$

for $i \neq j$. In particular, for $m = 1 - a_{ij}$, we have $e_i w_m = 0$. If $k \neq i, j$, then $[e_k, f_j] = [e_k, f_j] = 0$, so $e_k w_m = 0$. If $a_{ij} \neq 0$ then $i \neq j$ and $1 - a_{ij} \geq 2$, so $[f_i, e_j] = 0$, and

$$e_i \cdot w_{1-a_{ij}} = \mathrm{ad}^{-a_{ij}}([e_j, f_j]) = \mathrm{ad}^{-a_{ij}}(\alpha_j^\vee) = [f_i, [f_i, \ldots, [f_i, \alpha_j^\vee]] \ldots] = 0.$$

Finally, if $a_{ij} = 0$, then since $w_1 = [f_i, f_j]$,

$$e_j w_{1-a_{ij}} = [e_j, [f_i, f_j]] = [f_i, \alpha_j^\vee] = a_{ji} f_i = 0$$

since $a_{ji} = 0$ if and only if $a_{ij} = 0$. □

Fact 2.16 (Gabber-Kac). *If A is symmetrisable and satisfies the conditions above, then the Serre relations, along with the defining ones, uniquely characterise $\widetilde{\mathfrak{g}}(A)$.*

2.3 Integrable representations and the Weyl group

Now, consider the adjoint action of $\mathfrak{g}(A)$ on itself. By the Serre relations, we know that the operators e_i and f_i are *locally nilpotent*, i.e. that there exists an $N \gg 0$ such that $e_i^N \cdot x = f_i^N \cdot x = 0$. Thus we can define the following:

Definition 2.17. Let x be a locally nilpotent element of a Lie algebra $\mathfrak{g}$. The **exponential of** x is defined to be $\exp(x) := \sum_{n \geq 0} \frac{x^n}{n!}$.

The sum is finite precisely because x^n is eventually 0. In particular, we get another action of $\mathfrak{g}(A)$ on itself, by $(x, y) \mapsto \exp(x) \cdot y = [\exp(x), y]$.

Example 2.18. If $x \in \mathfrak{g}_\alpha$, then $\exp(h) \cdot x = e^{\alpha(h)}x$.

Definition 2.19. Let V be a representation of $\mathfrak{g}(A)$. We say that V is **integrable** if (i) we have a decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \quad \text{into } \textbf{weight spaces} \quad V_\lambda := \{v \in V : h \cdot v = \lambda v \text{ for all } h \in \mathfrak{h}\},$$

and (ii), e_i and f_i act locally nilpotently on V for all i.

Example 2.20. In particular, we see that the adjoint action of $\mathfrak{g}(A)$ is integrable, and it is a good exercise to check that every finite-dimensional representation is integrable as well.

Integrable representations are well-behaved in the sense that they decompose nicely under the action of $\mathfrak{sl}_2^{(i)} \subset \mathfrak{g}(A)$:

Proposition 2.21. *Let V be an integrable representation of $\mathfrak{g}(A)$. Then, as an $\mathfrak{sl}_2^{(i)}$-module, V decomposes as*

$$V = \bigoplus_{d \geq 0} W_d^{\oplus m_d}, \tag{2.2}$$

for $m_d \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$, where each W_d is an irreducible $d{+}1$-dimensional $\mathfrak{sl}_2^{(i)}$-representation.

Proof. For any $v \in V$,

$$U_v := \operatorname{span}\{f_i^k e_i^m v \colon k, m \geq 0\}, \tag{2.3}$$

and note if $v = \lambda w$ for $\lambda \in \mathbb{C}^\times$, then $U_v = U_w$. Each U_v is finite-dimensional precisely because f_i and e_i act locally nilpotently, and is closed under the action of $\mathfrak{sl}_2^{(i)}$. The result then follows from the Weyl reducibility theorem, fact 1.23. □

This gives a concrete way of studying integrable representations.

Definition 2.22. The **weights** of V is by definition the set

$$\operatorname{wt}(V) := \{\mu \in \mathfrak{h}^* \colon V_\mu \neq 0\}. \tag{2.4}$$

Proposition 2.23. *Fix i, and set*

$$M := \{t \in \mathbb{Z} \colon \lambda + t\alpha_i \in \operatorname{wt}(V)\} \quad \text{and} \quad m_\lambda(t) := \dim_{\mathbb{C}} V_{\lambda + t\alpha_i}. \tag{2.5}$$

Then the following hold:

(i) *For some $p, q \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$, $M = [-p, q] \cap \mathbb{Z}$;*
(ii) *If $m_\lambda(t) < \infty$, then $p, q < \infty$;*
(iii) *If $p, q < \infty$, then $p - q = \lambda(\alpha_i^\vee)$;*
(iv) *$e_i \colon V_{\lambda+t\alpha_i} \to V_{\lambda+(t+1)\alpha_i}$ is an injection if $t \in [-p, -\frac{1}{2}\lambda(\alpha_i^\vee)]$;*
(v) *$t \mapsto m_\lambda(t)$ is symmetric around $-\frac{1}{2}\lambda(\alpha_i^\vee)$;*
(vi) *If $\lambda, \lambda + \alpha_i \in \operatorname{wt}(V)$, then $e_i(V_\lambda) \neq 0$.*

Example 2.24.

Definition 2.25. Let $\mathfrak{g}(A)$ be a Kac-Moody algebra. The **fundamental reflections** of $\mathfrak{g}(A)$ are the maps $r_i \in \operatorname{End}(\mathfrak{h}^*)$ defined by

$$r_i \colon \lambda \mapsto \lambda - \lambda(\alpha_i^\vee)\alpha_i. \tag{2.6}$$

The name arises from the fact that $r_i(\alpha_i) = \alpha_i - 2\alpha_i = -\alpha_i$. Moreover, r_i fixes the elements $\lambda \in \mathfrak{h}^*$ which satisfy $\lambda(\alpha_i^\vee) = 0$.

Definition 2.26. The **Weil group** W associated to $\mathfrak{g}(A)$ is the subgroup of $\operatorname{End}(\mathfrak{h}^*)$ generated by the fundamental reflections $\{r_i\}$.

Example 2.27. Taking $A = [2]$, we recall that $\mathfrak{g}(A) = \mathfrak{sl}_2$, and we see that since $\mathfrak{h}^* = \mathbb{C}\alpha$, we have

$$r = r_1 : c\alpha \mapsto c\alpha - c\alpha(\alpha^\vee)\alpha = -c\alpha, \tag{2.7}$$

which has order 2. Therefore $W \simeq C_2$.

Proposition 2.28. *If V is an integrable representation of $\mathfrak{g}(A)$, $\lambda \in \mathfrak{h}^*$ and $w \in W$, then the following hold:*

(i) $m_\lambda(t) = m_{w(\lambda)}(t)$;
(ii) *there is a natural action of W on R, the set of roots of $\mathfrak{g}(A)$;*
(iii) *for any $\alpha \in R$, we have* $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w(\alpha)}$.

2.4 A loop algebra realisation

As promised in example 2.11, we now return to study the infinite-dimensional Kac-Moody algebra more closely. Let

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

note that $\mathrm{rk}(A) = 1$, and form $\widehat{\mathfrak{g}} := \mathfrak{g}(A)$ in the usual manner. In other words, fix indeterminates $\Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee\}$, $\Pi = \{\alpha_1, \alpha\}$, d and Λ. Taking

$$\mathfrak{h} = \mathbb{C}\alpha_0^\vee \oplus \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}d \quad \text{and} \quad \mathfrak{h}^* = \mathbb{C}\alpha_0 \oplus \mathbb{C}\alpha_1 \oplus \mathbb{C}\Lambda, \tag{2.8}$$

we obtain a realisation of A by requiring that $\alpha_i(\alpha_j^\vee) = (-1)^{i+j}2$,

$$\alpha_0(d) = \Lambda(\alpha_0^\vee) = 1 \quad \text{and} \quad \alpha_1(d) = \Lambda(\alpha_1^\vee) = \Lambda(d) = 0. \tag{2.9}$$

Since A is symmetrisable, we obtain a non-degenerate bilinear form with associated matrix

$$\begin{pmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.10}$$

By convention, we define $c := \alpha_0^\vee + \alpha_1^\vee$ and $\delta := \alpha_0 + \alpha_1$. One easily verifies that this gives orthogonal decompositions

$$\mathfrak{h} = \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}c \oplus \mathbb{C}d \quad \text{and} \quad \mathfrak{h}^* = \mathbb{C}\alpha_1 \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda, \tag{2.11}$$

by direct computiation.

However, this description of $\mathfrak{g}(A)$ leaves a lot to be desired; for example, it is difficult to explicitly see when the bracket of two elements is 0. Therefore, it is desirable to come up with an alternative description.

Let $\mathcal{L} := \mathbb{C}[t, t^{-1}]$ be the collection of Laurent polynomials in t, and for $P = \sum c_i t^i$, define $\mathrm{Res}\, P := c_{-1}$. Of course, this is simply the residue at 0 in the complex-analytic sense. Note that $\mathrm{Res}\colon \mathcal{L} \to \mathbb{C}$ is a linear functional which obeys

$$\mathrm{Res}\, t^{-1} = 1 \quad \text{and} \quad \mathrm{Res}\, \frac{\mathrm{d}P}{\mathrm{d}t} = 0 \tag{2.12}$$

for any $P \in \mathcal{L}$. Now define

$$\varphi \colon \mathcal{L} \times \mathcal{L} \to \mathbb{C} \quad \text{by} \quad (P, Q) \mapsto \varphi(P, Q) = \operatorname{Res}\left(\frac{\mathrm{d}P}{\mathrm{d}t} Q\right). \tag{2.13}$$

Lemma 2.29. *The map φ satisfies*

$$\varphi(P, Q) = -\varphi(Q, P) \quad \textit{and} \quad \varphi(PQ, R) + \varphi(QR, P) + \varphi(RP, Q) = 0.$$

Proof. By the product rule,

$$\frac{\mathrm{d}PQ}{\mathrm{d}t} = \frac{\mathrm{d}P}{\mathrm{d}t} Q + \frac{\mathrm{d}Q}{\mathrm{d}t} P$$

so

$$0 = \operatorname{Res} \frac{\mathrm{d}PQ}{\mathrm{d}t} = \operatorname{Res}\left(\frac{\mathrm{d}P}{\mathrm{d}t} Q\right) + \operatorname{Res}\left(\frac{\mathrm{d}Q}{\mathrm{d}t} P\right) = \varphi(PQ) + \varphi(QP),$$

giving the first equality. Proceeding analogously with the second, using the product rule twice on $\frac{\mathrm{d}PQR}{\mathrm{d}t}$ and taking residues gives the result. □

Now, for the sake of convenience, write $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ and set $\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{L}$. Recall that a pure tensor in $\mathcal{L}\mathfrak{g}$ then looks like $x \otimes P$ for $x \in \mathfrak{g}$, $P \in \mathcal{L}$, and we define a bracket on $\mathcal{L}\mathfrak{g}$ by

$$[x \otimes P, y \otimes Q]_0 := [x, y]_{\mathfrak{g}} \otimes PQ. \tag{2.14}$$

Finally, define the map

$$\psi \colon \mathcal{L}\mathfrak{g} \otimes \mathcal{L}\mathfrak{g} \to \mathbb{C} \quad \text{by} \quad (x \otimes P, y \otimes Q) \mapsto (x, y)\varphi(P, Q). \tag{2.15}$$

Using lemma 2.29 it is easy to check the following:

Lemma 2.30. *We have*

$$\psi(a, b) = -\psi(a, b) \quad \textit{and} \quad \psi([a, b], c) + \psi([b, c], a) + \psi([c, a], b) = 0$$

for all $a, b, c \in \mathcal{L}\mathfrak{g}$.

Definition 2.31. Let $\mathfrak{g}$ be a Lie algebra. A **central extension** of $\mathfrak{g}$ consists of another Lie algebra $\mathfrak{g}'$ and a surjection $\pi \colon \mathfrak{g}' \to \mathfrak{g}$ such that $\dim \ker \pi = 1$, and $[x, k] = 0$ for all $x \in \mathfrak{g}$ and $k \in \ker \pi$.

Like any surjection, π fits into the following exact sequence.

$$0 \to \ker \pi \to \mathfrak{g}' \xrightarrow{\pi} \mathfrak{g} \to 0. \tag{2.16}$$

Example 2.32. Any Lie algebra $\mathfrak{g}$ admits a trivial central extension given by the natural projection $\pi \colon \mathfrak{g}' := \mathfrak{g} \oplus \mathbb{C}K \to \mathfrak{g}$, where $\mathfrak{g}'$ has the bracket $[x, y]_{\mathfrak{g}'} = [x, y]_{\mathfrak{g}}$ and $[x, K]_{\mathfrak{g}'} = 0$ for any $x, y \in \mathfrak{g}$.

We will use this to enlarge $\mathcal{L}\mathfrak{g}$: Define the central extension $\widetilde{\mathcal{L}\mathfrak{g}} := \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$ fitting into the exact sequence

$$0 \to \mathbb{C}c \to \widetilde{\mathcal{L}\mathfrak{g}} \to \mathcal{L}\mathfrak{g} \to 0, \tag{2.17}$$

with bracket extending $[-,-]_0$ in eq. (2.14) given by $[a,c] = 0$ and $[a,b] = [a,b]_0 + \psi(a,b) \cdot c$ for all $a, b \in \mathcal{L}\mathfrak{g}$. Finally, extend this further by another element d,

$$\widehat{\mathfrak{g}} := \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d, \tag{2.18}$$

with bracket extending the previous defined by $[d,c] = 0$ and $[d, x \oplus P] = x \oplus t\frac{\mathrm{d}P}{\mathrm{d}t}$. In other words, d acts as a sort of differential operator.

Fact 2.33. *The vector space* $\widehat{\mathfrak{g}}$ *is a Lie algebra, and futhermore is isomorphic to* $\mathfrak{g}(A)$*, where*

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The first part is a straightforward verification of the axioms. As for the second, the isomorphism is given by

$$\begin{aligned}
(e_1, \alpha_1^\vee, f_1) &\mapsto (e_1, \alpha_1^\vee, f_1) \\
(c, d) &\mapsto (c, d) \\
e_0 &\mapsto f_1 \otimes t \\
f_0 &\mapsto e_1 \otimes t^{-1},
\end{aligned}$$

but we omit the details.

Definition 2.34. The Kac-Moody algebra $\mathfrak{g}(A)$ is called the **affine** $\mathfrak{sl}_2$ Lie algebra, and $\widehat{\mathfrak{g}}$ the **loop algebra realisation of** $\mathfrak{g}(A)$.

2.5 Roots of Lie algebras

Recall that if we have a Lie algebra $\mathfrak{g} = n_+ \oplus \mathfrak{h} \oplus n_-$, then there is a natural action of $\mathfrak{h}$ on $n_\pm$, and a corresponding eigenspace decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$, where $\alpha \in \mathfrak{h}^*$ and

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h,x] = \alpha(h)x\}.$$

Here we consider some examples:

Example 2.35 (Roots of $\mathfrak{sl}_2$)**.** Recall that $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$, and that $\mathfrak{h} = \mathbb{C}h$ is the Cartan subalgebra of $\mathbb{C}$, where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and $[e,h] = 2e$, $[h,h] = 0$ and $[f,h] = -2f$. Therefore, if α is the linear map $\lambda h \mapsto 2\lambda$, then $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{g}_0$. The roots of $\mathfrak{sl}_2$ are $\{\alpha, -\alpha\}$.

Example 2.36 (Roots of $\mathfrak{sl}_3$). Recall that a basis for $\mathfrak{sl}_3$ is given by

$$h_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad E_{ij} \text{ for } i \neq j.$$

By direct computation, we find that $[h_1, E_{23}] = -E_{23}$ and $[h_2, E_{23}] = 2E_{23}$, and by defining α_2 by $\alpha_2(h_1) = -1$ and $\alpha_2(h_2) = 2$, we see that $E_{23} \in \mathfrak{g}_{\alpha_2}$. Likewise, we can define α_1 by $\alpha_1(h_1) = 2$ and $\alpha_1(h_2) = -1$, and check that $E_{12} \in \mathfrak{g}_{\alpha_1}$. Next, $E_{13} \in \mathfrak{g}_{\alpha_1+\alpha_2}$ because $[h_1, E_{13}] = [h_2, E_{13}] = 1$ and $(\alpha_1 + \alpha_2)(h_1)E_{13} = (\alpha_1 + \alpha_2)(h_2)E_{13} = E_{13}$. It is easy to check that we have corresponding negative roots if the indices are decreasing instead of increasing. Finally, $h_1, h_2 \in \mathfrak{g}_0$, and so the roots are given by $\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$.

Example 2.37 (Affine $\mathfrak{sl}_2$). Let us consider $\widehat{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$, the loop algebra realisation of $\mathfrak{g}\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. For convenience, let us fix the roots and coroots of $\mathfrak{g}(A)$. Define

$$\alpha_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x, \qquad \alpha_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = b \quad \text{and} \quad \Lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z,$$

and

$$\alpha_0^\vee = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \qquad \alpha_1^\vee = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and compare with eq. (2.10). Note that under the isomorphism in Fact 2.33, we compute

$\alpha_0(h) = -2$	$\alpha_0(c) = 0$	$\alpha_0(d) = 1$
$\alpha_1(h) = 2$	$\alpha_1(c) = 0$	$\alpha_1(d) = 0$
$\Lambda(h) = 1$	$\Lambda(c) = 1$	$\Lambda(d) = 0$

Let us first find the weight space of e_1:

$$\begin{aligned} [h, e_1] &= 2e_1 = \alpha_1(h)e_1, \\ [d, e_1] &= 0 = \alpha_1(d)e_1, \\ [c - h, e_1] &= -2e_1 = \alpha_1(c - h)e_1, \end{aligned}$$

and so we conclude that $e_1 \in \mathfrak{g}_{\alpha_1}$. Similarly, we compute

$$\begin{aligned} [h, f_1] &= -2f_1 = -\alpha_1(h)f_1, \\ [d, f_1] &= 0 = -\alpha_1(d)f_1, \\ [c - h, f_1] &= -2f_1 = -\alpha_1(c - h)f_1, \end{aligned}$$

so $f_1 \in \mathfrak{g}_{-\alpha_1}$.

Next, let us find the weight space of e_0, which is sent to $f_1 t$ in the loop algebra. We compute

$$[h, f_1 t] = -2 f_1 t = \alpha_0(h) f_1 t,$$
$$[c, f_1 t] = 0 = \alpha_0(c) f_1 t,$$
$$[d, f_1 t] = f_1 t = \alpha_0(d) f_1 t,$$

and so $f_1 t \in \mathfrak{g}_{\alpha_0}$. More generally, consider $f_1 t^k$, and for convenience we drop the subscript. Then we see that

$$[h, ft^k] = -2ft^k,$$
$$[c, ft^k] = 0,$$
$$[d, ft^k] = kft^k,$$

and suppose λ is the corresponding root. Since $\lambda(c) = 0, \lambda = A\alpha_0 + B\alpha_1$ for some $A, B \in \mathbb{C}$, and from the remaining two equations we get $\lambda(d) = A = k$ and $\lambda(h) = -2A + 2B = -2$ hence $B = k - 1$. Therefore $ft^k \in \mathfrak{g}_{k(\alpha_0+\alpha_1)-\alpha_1}$. By a similar procedure we find that $et^k \in \mathfrak{g}_{k(\alpha_0+\alpha_1)+\alpha_1}$ and $t^k \in \mathfrak{g}_{k(\alpha_0+\alpha_1)}$. Since these generate $\widehat{\mathfrak{g}}$, we conclude that the roots of affine $\mathfrak{sl}_2$ are

$$\{k(\alpha_0 + \alpha_1), k(\alpha_0 + \alpha_1) \pm \alpha_1\}. \tag{2.19}$$

3 More Lie algebras

3.1 The Witt and Heisenberg algebras

Let $A \equiv \mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials over $\mathbb{C}$ defined in section 2.4, and consider $\mathrm{Der}(A) \subset \mathrm{End}(A)$, the vector space of derivations of A. Explicitly, this is the collection of linear endomorphisms ϕ satisfying the *Leibniz rule* $\phi(fg) = f\phi(g) + g\phi(f)$ for any $f, g \in \mathbb{C}[t, t^{-1}]$.

Lemma 3.1. *The space* $\mathrm{Der}\, A$ *is a subalgebra of* $\mathrm{End}(A)$, *and has a basis given by* $L_n := -t^{n+1}\frac{\mathrm{d}}{\mathrm{d}t}$, *where* $[L_m, L_n] = (m-n)L_{m+n}$ *for each* $n \in \mathbb{Z}$.

Proof. The operators L_n are evidently linearly independent; to see that they span $\mathrm{Der}\, A$, note that if $\phi(z) = -\sum a_n z^{n+1}$ for $a_n \in \mathbb{C}$, then inductively one sees that $\phi(z^k) = kz^{k-1}\phi(z) = \frac{\mathrm{d}z^k}{\mathrm{d}z}\phi(z)$ for any $k \in \mathbb{Z}$ using the Leibniz rule. By linearity we therefore have that

$$\phi(f) = \frac{\mathrm{d}f}{\mathrm{d}z}\phi(z) = -\sum a_n z^{n+1}\frac{\mathrm{d}f}{\mathrm{d}z} = \sum a_n L_n(f), \tag{3.1}$$

so $\phi = \sum a_n L_n$. □

Definition 3.2. The Lie algebra realised in this way, that is, by the basis $\{L_n\}_{n\in\mathbb{Z}}$ with bracket $[L_m, L_n] = (m-n)L_{m+n}$, is called the Witt algebra, denoted $\mathcal{Witt}$.

Example 3.3. Suppose $z = e^{i\vartheta}$. A quick computation shows that

$$L_n = -e^{i(n+1)\vartheta}\left(\frac{\mathrm{d}z}{\mathrm{d}\vartheta}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}\vartheta} = ie^{in\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}. \tag{3.2}$$

If $\frac{\mathrm{d}}{\mathrm{d}\vartheta}$ is regarded as a tangent field on the unit circle in $\mathbb{C}$, then $\mathcal{W}itt$ consists precisely of elements of the form $\sum a_n ie^{in\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}$, that is, complex vector fields with a finite Fourier expansion.

3.2 The Virasoro algebra

Recall that we can enlarge a Lie algebra by adding a central element. In this section we do this to $\mathcal{W}itt$ in order to obtain the Virasoro algebra.

Proposition 3.4. *There exists a non-trivial central extension*

$$0 \to \mathbb{C}c \to \mathcal{V}ir \xrightarrow{\pi_{\mathcal{V}ir}} \mathcal{W}itt \to 0 \tag{3.3}$$

which is unique in the sense that for any other non-trivial central extension $(\mathfrak{g}, \pi)$ *of* $\mathcal{W}itt$*, there exists an isomorphism of Lie algebras* $\phi\colon \mathfrak{g} \to \mathcal{V}ir$ *such that* $\pi = \pi_{\mathcal{V}ir} \circ \phi$*.*

Explicitly, $\mathcal{V}ir$ *has a basis* $\{c\} \cup \{L_n\}_{n\in\mathbb{Z}}$*, and bracket defined by*

$$[c, \mathcal{V}ir] = 0 \quad \textit{and} \quad [L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}\cdot c. \tag{3.4}$$

Definition 3.5. The central extension so defined is called the **Virasoro algebra**.

Proof. The proof is slightly tedious and omitted for the time being. □

There is another scenic route to the Virasoro algebra, namely via the Heisenberg algebra:

Definition 3.6. The **Heisenberg algebra** is the Lie algebra $\mathcal{H}eis$ generated by $\{\hbar\} \cup \{a_n\}_{n\in\mathbb{Z}}$ with bracket given by

$$[\hbar, \mathcal{H}eis] = 0 \quad \text{and} \quad [a_m, a_n] = m\delta_{m,-n}\hbar, \tag{3.5}$$

for $m, n \in \mathbb{Z}$.

One might note that $\hbar$ in $\mathcal{H}eis$ looks an awful lot like a central extension element. Indeed, we can form $\mathcal{H}eis$ as the central extension of the abelian Lie algebra with basis $\{a_n\}_{n\in\mathbb{Z}}$.

Another way to arrive at the Heisenberg algebra is to consider the abelian Lie algebra $\mathfrak{gl}_1 := \mathbb{C}a$ with bilinear form $(a, a) = 1$. Define the loop algebra of $\mathfrak{gl}_1$ as in section 2.4, namely $\mathcal{L}\mathfrak{gl}_1 = \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}a$, and take a central extension, $\widetilde{\mathcal{L}\mathfrak{gl}_1} = \mathcal{L}\mathfrak{gl}_1 \oplus \mathbb{C}K$. This we equip with the bracket given by $[xt^m, yt^n] = \delta_{m,-n}m \cdot (x, y)K$.

Proposition 3.7. *The map*

$$\begin{aligned} \mathcal{H}eis &\to \widetilde{\mathcal{L}\mathfrak{gl}_1} \\ a_n &\mapsto at^n \\ \hbar &\mapsto K \end{aligned}$$

is an isomorphism of Lie algebras.

Proof. Indeed, we see that $[at^n, at^m] = \delta_{m,-n} mK$ and $[K, at^n] = 0$. The map is evidently a bijection, so our result is immediate. □

Now, we let us consider the following representation of $\mathcal{H}eis$: Fix $\mu, h \in \mathbb{C}$, and let $B(\mu, h) := \mathbb{C}[x_1, x_2, \ldots]$ be the space of polynomials in countably infinitely many variables, along with the action of $\mathcal{H}eis$ defined by

$$\hbar \cdot f = hf \quad \text{and} \quad a_n \cdot f = \begin{cases} \frac{\partial f}{\partial x_n} & \text{if } n > 0, \\ \mu f & \text{if } n = 0, \\ -hnx_{-n} f & \text{if } n < 0. \end{cases} \tag{3.6}$$

Lemma 3.8. *The space $B(\mu, h)$ is indeed a $\mathcal{H}eis$-representation, and for any $v \in B(\mu, h)$, $a_n \cdot v = 0$ for $n \gg 0$.*

Proof. The only situation where differentation and multiplication fails to commute is when the variables in question are the same. Therefore,

$$\begin{aligned} \left[\frac{\partial}{\partial x_n}, hnx_n\right](f) &= \frac{\partial}{\partial x_n}\left(hnx_n f\right) - hnx_n \frac{\partial f}{\partial x_n} \\ &= hn\left(\frac{\partial x_n}{\partial x_n}\right) f + hnx_n \frac{\partial f}{\partial x_n} - hnx_n \frac{\partial f}{\partial x_n} \\ &= hn \cdot f, \end{aligned}$$

which follows from the product rule of $\frac{\partial}{\partial x_n}$, is the only verification required. The second claim follows from noting that only finitely many variables appear in each vector $f(x_1, x_2, \ldots)$, so choosing n greater than all of these kills f. □

Definition 3.9. The **normally ordered product** of two elements $a_i, a_j \in \operatorname{End} B(\mu, h)$ is

$$:a_i a_j: \;:= \begin{cases} a_i a_j & \text{if} \quad i \leq j, \\ a_j a_i & \text{if} \quad i > j. \end{cases} \tag{3.7}$$

This enigmatic definition will be put to use immediately:

Theorem 3.10. *Let V be any $\mathcal{H}eis$-representation with the property that for any $v \in V$, there exists an $n \gg 0$ such that $a_n \cdot v = 0$, and suppose $\hbar$ acts as the identity on V. Then the operators*

$$L_n := \frac{1}{2} \sum_{k \in \mathbb{Z}} :a_{-k} a_{n+k}: \tag{3.8}$$

satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}. \tag{3.9}$$

In other words, $B(\mu, 1)$ is a $\mathcal{V}ir$-representation with c acting as the identity.

Definition 3.11. A $\mathcal{V}ir$-representation V is said to have **central charge** $c \in \mathbb{C}$ if $c_{\mathcal{V}ir}$ acts by multiplication by c.

In other words, $B(\mu, 1)$ is a $\mathcal{V}ir$-representation of central charge 1.

Example 3.12. Every $\mathcal{W}itt$-representation induces one of $\mathcal{V}ir$ with central charge 0.

To prove theorem 3.10, we need the following lemma:

Lemma 3.13. *For $k, n \in \mathbb{Z}$, we have $[a_k, L_n] = ka_{k+n}$.*

Proof. To deal with the infinite sums, we need some sort of cutoff-function. Define $\psi\colon \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| < 0, \end{cases} \tag{3.10}$$

and let $\varepsilon > 0$. Setting

$$L_n(\varepsilon) \coloneqq \frac{1}{2}\sum_{j\in\mathbb{Z}} {:}a_{-j}a_{j+n}{:}\ \psi(\varepsilon j), \tag{3.11}$$

we see that $L_n(\varepsilon)$ is a finite sum, and that $L_n(\varepsilon)v = L_n v$ for ε sufficiently small. First, observe that

$$\begin{aligned} L_n(\varepsilon) - \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}\psi(\varepsilon j) &= \frac{1}{2}\sum_{j\in\mathbb{Z}}({:}a_{-j}a_{j+n}{:} - a_{-j}a_{j+n})\psi(\varepsilon j) \\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}}(a_{j+n}a_{-j} - a_{-j}a_{j+n})\mathbb{1}_{\{-j>j+n\}}\psi(\varepsilon j) \\ &= \frac{1}{2}\sum_{-j>j+n}[a_{j+n}, a_{-j}]\psi(\varepsilon j) \\ &= \frac{1}{2}\sum_{0>2j+n}\delta_{n,0}j\psi(\varepsilon j), \end{aligned}$$

and since this is a scalar, we conclude that for any $x \in \mathcal{H}eis$,

$$0 = \left[x, L_n(\varepsilon) - \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}\psi(\varepsilon j)\right] = [x, L_n] - [x, \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}\psi(\varepsilon j)]$$

so that by lemma 1.19,

$$\begin{aligned}
[a_k, L_n] &= [a_k, \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}\psi(\varepsilon j)] \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}}[a_k, a_{-j}a_{j+n}]\psi(\varepsilon j) \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}}\left([a_k, a_{-j}]a_{j+n} + a_{-j}[a_k, a_{j+n}]\right)\psi(\varepsilon j) \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}}\left(\delta_{k,j}ka_{j+n} + a_{-j}\delta_{k,-(j+n)}k\right)\psi(\varepsilon j) \\
&= \frac{1}{2}ka_{k+n}\psi(\varepsilon k) + \frac{1}{2}ka_{k+n}\psi(-\varepsilon(n+k)),
\end{aligned}$$

and sending ε to 0 proves the claim. □

Proof of theorem 3.10. Using the lemma above along with lemma 1.19, we find that

$$\begin{aligned}
[L_m(\varepsilon), L_n] &= \frac{1}{2}\sum_{j\in\mathbb{Z}}[a_{-j}a_{j+m}, L_n]\psi(\varepsilon j) \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}}\left([a_{-j}, L_n]a_{j+m} + a_j[a_{j+m}L_n]\right)\psi(\varepsilon j) \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}}\left(-ja_{-j+n}a_{j+m} + a_j(j+m)a_{j+m+n}\right)\psi(\varepsilon j).
\end{aligned}$$

Now, note that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}(j+m):a_{-j}a_{j+m+n}:\psi(\varepsilon j) &= \sum_{-2j\le m+n}(j+m)a_{-j}a_{j+m+n}\psi(\varepsilon j) + \sum_{-2j>n+m}(j+m)a_{j+m+n}a_{-j}\psi(\varepsilon j) \\
&= \sum_{j\in\mathbb{Z}}(j+m)a_{-j}a_{j+m+n}\psi(\varepsilon j) + \sum_{-2j>n+m}(j+m)(a_{j+m+n}a_{-j} - a_{-j}a_{j+m+n})\psi(\varepsilon j) \\
&= \sum_{j\in\mathbb{Z}}(j+m)a_{-j}a_{j+m+n}\psi(\varepsilon j) + \sum_{-2j>n+m}(j+m)[a_{j+m+n}, a_{-j}]\psi(\varepsilon j) \\
&= \sum_{j\in\mathbb{Z}}(j+m)a_{-j}a_{j+m+n}\psi(\varepsilon j) + \delta_{m,-n}\sum_{j<0}(j+m)j\psi(\varepsilon j).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} -j:a_{-j+n}a_{j+m}:\psi(\varepsilon j) &= \sum_{j\in\mathbb{Z}}(-j)a_{-j+n}a_{j+m}\psi(\varepsilon j) + \sum_{-j+n>j+m}(-j)[a_{-j+n}, a_{j+m}]\psi(\varepsilon j) \\
&= \sum_{j\in\mathbb{Z}}(-j)a_{-j+n}a_{j+m}\psi(\varepsilon j) + \sum_{-j+n>j+m}\delta_{j-n,j+m}(j+m)(-j)\psi(\varepsilon j) \\
&= \sum_{j\in\mathbb{Z}}(-j)a_{-j+n}a_{j+m}\psi(\varepsilon j) + \delta_{m,-n}\sum_{j<-m}(j+m)(-j)\psi(\varepsilon j).
\end{aligned}$$

Therefore,

$$\begin{aligned}
[L_m(\varepsilon), L_n] =& \frac{1}{2}\sum_{j\in\mathbb{Z}} \big(-ja_{-j+n}a_{j+m} + a_j(j+m)a_{j+m+n}\big)\,\psi(\varepsilon j) \\
=& \frac{1}{2}\sum_{j\in\mathbb{Z}}(j+m) : a_{-j}a_{j+m+n} \colon \phi(\varepsilon j) - \frac{1}{2}\sum_{j\in\mathbb{Z}} j : a_{-j+n}a_{j+m} \colon \phi(\varepsilon j) \\
& - \frac{\delta_{m,-n}}{2}\sum_{j<0}(j+m)j\psi(\varepsilon j) - \frac{\delta_{m,-n}}{2}\sum_{j<-m}(j+m)(-j)\psi(\varepsilon j) \\
=& \frac{1}{2}\sum_{j\in\mathbb{Z}}(j+m) : a_{-j}a_{j+m+n} \colon \phi(\varepsilon j) - \frac{1}{2}\sum_{j\in\mathbb{Z}}(j+n) : a_{-j}a_{j+n+m} \colon \phi(\varepsilon(j+n)) \\
& - \frac{\delta_{m,-n}}{2}\Big(\sum_{j<0}(j+m)j - \sum_{j<-m}(j+m)j\Big)\,\psi(\varepsilon j).
\end{aligned}$$

Now, by sending ε to 0 and summing the series using the Faulhaber formulae, we see that this equals

$$\begin{aligned}
&\frac{1}{2}\sum_{j\in\mathbb{Z}}(m-n) : a_{-j}a_{j+m+n} \colon - \frac{\delta_{m,-n}}{2}\sum_{-m<j<0}(j+m)j \\
&= (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12},
\end{aligned}$$

and this is what we wanted to show. □

Remark. There is a significantly more general version of this theorem where $\mathcal{V}ir$ acts on a general semisimple Lie algebra, not just $\mathcal{H}eis$, called the *Sugawara construction*. More details can be found in Chap.1 of Kac, Raina - *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, 1987.

Definition 3.14. The $\mathcal{V}ir$-representations $B(\mu, 1)$ coming from the Heisenberg algebra are called **oscillator representations** of $\mathcal{V}ir$.

3.3 Highest weight representations

We shall now continue our study of representations of the Virasoro algebra $\mathcal{V}ir$.

Definition 3.15. Let V be a $\mathcal{V}ir$-representation. A vector $v \in V$ is **singular of weight** $(h, c) \in \mathbb{C}\times\mathbb{C}$ if $L_0 v = hv$, $c_{\mathcal{V}ir} = cv$, and $L_n v = 0$ for all $n > 0$. A singular vector v of weight (h, c) is a **highest weight vector** if

$$V = \operatorname{span}\{L_{-n_1}L_{-n_2}\cdots L_{-n_k}v \colon k \geq 0, n_1, \ldots, n_k > 0\}, \tag{3.12}$$

and V is a **highest weight module** of weight (h, c) if such a vector exists.

Highest weight modules have a decomposition theory similar to semi-simple Lie algebras and Kac-Moody algebras:

Theorem 3.16. *Let V be a highest weight module for $\mathcal{V}ir$ with highest weight (h, c). Then the following hold:*

(i) *V has central charge c;*
(ii) *there is a decomposition*
$$V = \bigoplus_{k \geq 0} V_{h+k}, \tag{3.13}$$
where each V_{h+k} is finite-dimensional;
(iii) $\dim_{\mathbb{C}} V_h = 1$.

Proof. Note that $[c_{\mathcal{V}ir}, L_n] = 0$ since $c_{\mathcal{V}ir}$ is a central element. Therefore, $c_{\mathcal{V}ir} L_n \cdot v = L_n c_{\mathcal{V}ir} \cdot v$. Now, let v_0 denote the highest weight vector of V. Then

$$\begin{aligned} c_{\mathcal{V}ir} \cdot v &= c_{\mathcal{V}ir}(L_{-n_1} L_{-n_2} \dots L_{-n_k} v_0) \\ &= L_{-n_1} L_{-n_2} \dots L_{-n_k} c_{\mathcal{V}ir} v_0 \\ &= L_{-n_1} L_{-n_2} \dots L_{-n_k} c v_0 \\ &= c(L_{-n_1} L_{-n_2} \dots L_{-n_k} v_0) = cv \end{aligned}$$

since both $c_{\mathcal{V}ir}$ and the scalar c commutes with the L_{-n_i}. This proves (i).

Let V_λ denote the eigenspace of L_0 with eigenvector λ. Note that

$$[L_0, L_n] \cdot v = (n - 0) L_{n+0} \cdot v \quad \text{and} \quad [L_0, L_n] \cdot v = L_0 L_n \cdot v - L_n hv,$$

so $L_0 L_n \cdot v = (h - n) L_n v$. Therefore, by writing $v \in V$ as $L_{-n_1} \dots L_{-n_k} v_0$, we have inductively that

$$L_0 v = (h + n_1 + \dots + n_k) v,$$

and so any $v \in V$ is an eigenvector of L_0. Since a given non-zero eigenvector can only have a single eigenvalue, this proves (ii). Moreover, the only vector with eigenvalue h is v_0, so we immediately obtain (iii). □

The above result gives a lot of information about the structure of $\mathcal{V}ir$-representations. For example:

Proposition 3.17. *Let V be a highest weight Virasoro module. Then V has a unique maximal submodule $V'' \subset V$, and $V' \coloneqq V/V''$ is an irreducible highest weight representation of same weight as V.*

Proof. Let V'' be the sum of all proper submodules of V. Then V'' is evidently maximal, and we claim that $V'' \subsetneq V$. Indeed, note that no proper submodule of V contains V_h, since this generates all of V. Therefore any proper submodule is contained in $\bigoplus_{k>0} V_{h+k}$, and therefore their sum is as well.

Now, note that if $W \subset V'$ were a proper submodule, then its preimage under the quotient map would be a proper submodule containing V'', contradicting maximality of V''. □

Given that highest weight representations are so useful, it is natural to ask for which pairs $(h, c) \in \mathbb{C} \times \mathbb{C}$ they exist.

Proposition 3.18. *Let $(h, c) \in \mathbb{C} \times \mathbb{C}$. Then there exists a highest weight Virasoro module $M = M(h, c)$ with highest weight (h, c), and this is moreover universal in the sense that for any highest weight module V of same weight, there exists a unique morphism of $\mathcal{V}ir$-modules $M \to V$ sending the highest weight vector of M to that of V.*

Corollary 3.18.1. *For all pairs $(h, c) \in \mathbb{C}\times\mathbb{C}$ there exists a unique irreducible $\mathcal{V}ir$-module $V(h, c)$ of highest weight (h, c).*

Definition 3.19. The highest weight $\mathcal{V}ir$-module $M(h, c)$ in the above proposition is called the **Verma module** of highest weight (h, c).

Proof of proposition 3.18. □

The Verma modules leave something to be desired in practical applications. For more explicit examples of highest weight modules, we return to the oscillator representation of $\mathcal{V}ir$. Recall that given $\mathcal{H}eis$-representation $B(\mu, h)$ with underlying vector space $\mathbb{C}[x_1, x_2, \ldots]$, then this is naturally a $\mathcal{V}ir$-representation. Now, fix $h = 1$ and define

$$B'(\mu, 1) \coloneqq \operatorname{span}\{L_{-n_1} \ldots L_{-n_k} \cdot 1 \colon 0 < n_1 \leq \ldots \leq n_k\}. \tag{3.14}$$

Then we have the following:

Proposition 3.20. *The $\mathcal{V}ir$-module $B'(\mu, 1)$ is an irreducible highest weight module of weight $(\mu^2/2, 1)$.*

3.4 The affine Lie algebra $\widehat{\mathfrak{gl}}_\infty$

3.5 Bi-infinite matrix algebras

Definition 3.21. The Lie algebra $\mathfrak{gl}_\infty$ is defined to be the space of $\mathbb{Z} \times \mathbb{Z}$-matrices over $\mathbb{C}$ with all but finitely many entries 0.

Example 3.22. Consider the matrices

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 691 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then the first is not in $\mathfrak{gl}_\infty$ since it has infinitely many non-zero entries, but the second is.

Note that matrix multiplication is well-defined, as any infinite sum in the entries of the product has only finitely many non-zero entries.

Proposition 3.23. *The set $\mathfrak{gl}_\infty$ is a Lie algebra, with a basis given by E_{ij}, where $E_{ij} = [\delta_{i,k}\delta_{j,l}]_{k,l}$ is the matrix with a single* 1 *in the* (i,j)*-th entry, and* 0 *elsewhere.*

Proof. Clearly any matrix in $\mathfrak{gl}_\infty$ can be written as a finite linear combination of the E_{ij}. As in the case of finite-dimensional matrices, we find by evaluating the commutator bracket $[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{il}E_{kj}$ that $\mathfrak{gl}_\infty$ is closed under this bracket, so it is indeed a Lie algebra. □

Definition 3.24. The **principal gradation** on $\mathfrak{gl}_\infty$ is the grading defined by extending linearly $\deg E_{ij} = i - j$. More precisely, let $\mathfrak{g}_k := \operatorname{span}\{E_{ij} : i - j = k\}$. Then $\mathfrak{gl}_\infty = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$.

In other words, the degree k elements are those whose entries all lie in the (signed) k-th diagonal above the *principal diagonal* $i = j$.

One easily checks that this is indeed a grading on $\mathfrak{gl}_\infty$, since $\deg[E_{ij}, E_{lk}] = \deg(\delta_{jk}E_{il} - \delta_{il}E_{kj}) = \delta_{jk}(i - l) + \delta_{il}(k - j)$. There are now three cases: $j = k$, in which case $\deg[E_{ij}, E_{lk}] = i - l$; $i = l$, giving $\deg[E_{ij}, E_{lk}] = j - k$, and finally $[E_{ij}, E_{lk}] = 0$ otherwise. In all cases, we see that $[E_{ij}, E_{lk}] \in \mathfrak{g}_m$ where $m = i - j + l - k$. Note that by convention $0 \in \mathfrak{g}_m$ for all m.

In a similar vein, we can define the Lie group corresponding to $\mathfrak{gl}_\infty$:

Definition 3.25. We define Gl_∞ to be the set of invertible $\mathbb{Z}\times\mathbb{Z}$-matrices which differ from the identity matrix in only finitely many entries.

Thus in the example above, the first but not the second matrix is in Gl_∞. It is straightforward to check that this is a group under multiplication, and it is in fact an infinite-dimensional Lie group.

For our purposes, however, both $\mathfrak{gl}_\infty$ and Gl_∞ are too small to fit copies of $\mathcal{H}eis$ and $\mathcal{V}ir$. We enlarge by defining the following:

Definition 3.26. Let $\mathfrak{gl}_\infty^\Delta$ denote the set of $\mathbb{Z}\times\mathbb{Z}$-matrices with only finitely many non-zero diagonals.

Example 3.27. The set of *bi-infinite Toeplitz matrices* is by definition the collection of matrices

$$A = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & a_0 & a_1 & a_2 & \cdots \\ \cdots & a_{-1} & a_0 & a_1 & \cdots \\ \cdots & a_{-2} & a_{-1} & a_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

that is, matrices which are constant along each diagonal. These are generally not in $\mathfrak{gl}_\infty^\Delta$, unless $a_n = 0$ for all $|n| \gg 0$.

Note that $\mathfrak{gl}_\infty^\Delta$ contains $\mathfrak{gl}_\infty$ as a proper subalgebra. Moreover, it is also spanned by the E_{ij}, although we ignore the slightly dodgy problem of dealing with infinite sums; for example, the identity matrix $\mathbb{I} = [\delta_{ij}]_{ij}$ is written as $\sum_{i\in\mathbb{Z}} E_{ii}$.

Proposition 3.28. *The vector space $\mathfrak{gl}_\infty^\Delta$ is a Lie algebra.*

Proof. The space is clearly closed under subtraction, so by showing that it is closed under multiplication, we obtain that it is a Lie algebra with the commutator bracket. For $x, y \in \mathfrak{gl}_\infty^\Delta$, choose positive integers n_x and n_y such that $x_{ij} = 0$ for $|i-j| > n_x$ and $y_{ij} = 0$ for $|i-j| > n_y$. Let $|i-j| > n_x + n_y$, note that

$$[xy]_{i,j} = \sum_{n\in\mathbb{Z}} x_{i,n} y_{n,j},$$

and suppose for the sake of contradiction that both $x_{i,n}$ and $y_{i,n}$ are non-zero for any given $n \in \mathbb{Z}$. Then $|i-n| \leq n_x$ and $|j-n| \leq n_y$, and so by the triangle inequality we have that $|i-j| \leq |i-n| + |j-n| \leq n_x + n_y$, contrary to assumption. Thus $[xy]_{i,j} = 0$ for all $|i-j| \gg 0$, so $xy \in \mathfrak{gl}_\infty^\Delta$, as required. □

However, this is still too small. We enlarge just a bit by considering the central extension

$$0 \to \mathbb{C}c \to \widehat{\mathfrak{gl}}_\infty^\Delta \to \mathfrak{gl}_\infty^\Delta \to 0,$$

where we define the bracket as $[c, x] = 0$, $[x, y] = xy - yx + \gamma(x, y)c$, where γ is defined by

$$\gamma(E_{ij}, E_{ji}) = \begin{cases} 1 & \text{if} \quad i \leq 0, j \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$\gamma(E_{ij}, E_{mn}) = 0$ for $(i, j) \neq (n, m)$, and extending linearly. Note that in light of Workshop 3, this does indeed define a bracket.

3.6 Realisations of $\mathcal{H}eis$ and $\mathcal{V}ir$ in $\widehat{\mathfrak{gl}}_\infty^\Delta$.

In this section, we will let $V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}v_j$, where $\{v_j\}$ is a set of indeterminates. Since V is naturally isomorphic to the space of $\mathbb{Z} \times 1$-matrices with finitely many non-zero entries, there is a natural action of $\mathfrak{gl}_\infty^\Delta$ on V, explicitly by

$$A \cdot v = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & A_{-1,-1} & A_{-1,0} & A_{-1,1} & \cdots \\ \cdots & A_{0,-1} & A_{0,0} & A_{0,1} & \cdots \\ \cdots & A_{1,-1} & A_{1,0} & A_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ v_{-1} \\ v_0 \\ v_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \sum_{m\in\mathbb{Z}} A_{-1,m} v_m \\ \sum_{m\in\mathbb{Z}} A_{0,m} v_m \\ \sum_{m\in\mathbb{Z}} A_{0,m} v_m \\ \sum_{m\in\mathbb{Z}} A_{1,m} v_m \\ \vdots \end{pmatrix}.$$

Define the *shift operators* $\Lambda_k : v_j \mapsto v_{j-k}$. Since $E_{mn} \cdot v_k = \delta_{nk} v_m$, we can realise these in terms of an action of $\mathfrak{gl}_\infty$ by $\Lambda_k = \sum_{i\in\mathbb{Z}} E_{i,i+k}$. One can check that $[\Lambda_j, \Lambda_k] = 0$. So, let $\mathfrak{a}$ denote the abelian Lie subalgebra generated by these Λ_k, and note that we obtain a corresponding subalgebra $\widehat{\mathfrak{a}}$ inside $\widehat{\mathfrak{gl}}_\infty^\Delta$ by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \widehat{\mathfrak{gl}}_\infty^\Delta & \longrightarrow & \mathfrak{gl}_\infty^\Delta & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \widehat{\mathfrak{a}} & \longrightarrow & \mathfrak{a} & \longrightarrow & 0
\end{array}$$

Proposition 3.29. *The following holds:*

(i) $\gamma(\Lambda_n, \Lambda_k) = n\delta_{n,-k}$, *and*
(ii) $\mathfrak{a} \cong \mathcal{H}eis$.

Proof. (i) follows from a straightforward computation:

$$\gamma\left(\sum_{m\in\mathbb{Z}} E_{m,m+n}, \sum_{m'\in\mathbb{Z}} E_{m',m'+k}\right) = \sum_{m,m'\in\mathbb{Z}} \gamma\big(E_{m,m+n}, E_{m',m'+k}\big) \tag{3.15}$$

Note that $\gamma(E_{m,m+n}, E_{m',m'+k})$ is non-zero only if $m = m' + k$ and $m' = m + n$, or equivalently, $-k = n$. Moreover, we require that $m \leq 0$ and $m + n \geq 1$, which holds true only for $m = 0, -1, \ldots, -n + 1$, and so we obtain $\gamma(\Lambda_n, \Lambda_k) = n\delta_{n,-k}$, as required.

Comparing generators and relations, we see that $\widehat{\mathfrak{a}} \cong \mathcal{H}eis$ with $\hbar = 1$. □

Since we can fit $\mathcal{H}eis$ inside $\widehat{\mathfrak{gl}}_\infty^\Delta$, it is only reasonable to assume that it can accommodate $\mathcal{W}itt$ as well.

Theorem 3.30. *We have:*

(i) *There exists a family of embeddings*

$$i_{\alpha,\beta}\colon \mathcal{W}itt \to \mathfrak{gl}_\infty^\Delta \quad \text{given by} \quad L_n \mapsto \sum_{k\in\mathbb{Z}} \big(k - \alpha - \beta(n+1)\big) E_{k-n,k}.$$

(ii) *Let $\widehat{\mathcal{W}itt}$ be the central extension of the image of $\mathcal{W}itt$ in $\widehat{\mathfrak{gl}}_\infty^\Delta$. Then $\gamma(L_i, L_j) = \delta_{i,-j}\frac{i^3-i}{12}c_\beta + h_0$, where*

$$c_\beta = 12\beta^2 - 12\beta - 2 \quad \text{and} \quad h_0 = \frac{1}{2}\alpha(\alpha + 2\beta - 1) \tag{3.16}$$

(iii) *Defining $\widehat{L}_n := L_n + \delta_{n,0}h_0 c$, we have*

$$[\widehat{L}_n, \widehat{L}_m] = (n-m)\widehat{L}_{n+m} + \frac{m^3 - m}{12}\delta_{m,-n}c_\beta c, \tag{3.17}$$

so $\widehat{\mathcal{W}itt} \cong \mathcal{V}ir$.

4 The boson-fermion correspondence

In this section we will realise the action of $\mathcal{H}eis$ on $B(\mu, 1)$ in terms of the *fermionic Fock space*, obtained as a certain limit of spaces of infinite vectors.

4.1 The fermionic Fock space

A combinatorial description

Let $\lambda = (\lambda_0, \dots, \lambda_{n-1})$ be a *decreasing partition* of a positive integer k, namely, a tuple of integers $\lambda_0, \dots, \lambda_{n-1}$ such that $\lambda_i \geq \lambda_j$ for $i \leq j$. For example, for $k = 12$ we can take $\lambda = (5, 3, 3, 1)$. To each such partition, we can assign a *Young tableau*, namely a diagram such as

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} 1 & 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} 1 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & \cdots\cdots \quad \lambda_0 \\ \hline 1 & 2 & 3 & \cdots\cdots & \lambda_1 \\ \hline \vdots & \vdots & \vdots & & \\ \hline 1 & 2 & \cdots & \lambda_{n-1} & \\ \hline \end{array}$$

Young tableaux occur more naturally in the study of representations of the symmetric groups S_k, and even in algebraic geometry, more specifically the Schubert calculus.

To each decreasing partition λ, we can associate

4.2 Representations of $F^{(m)}$

4.3 The boson-fermion correspondence

Theorem 4.1 (The boson-fermion correspondence). *We have an isomorphism of $\mathcal{H}$eis-modules $\sigma_m \colon F^{(m)} \xrightarrow{\sim} B^{(m)}$.*

4.4 Application: Schur polynomials

5 *Further topics

5.1 *The Weyl algebra

Definition 5.1. The **Weyl algebra** A_n is the subalgebra of $\operatorname{End}(k[x_1, \dots, x_n])$ generated by the operators $\frac{\partial}{\partial x_i}$ and $\widehat{x_i}$ which act by

$$\frac{\partial}{\partial x_i} x_j^k := \delta_{i,j} k x_j^{k-1} \quad \text{and} \quad \widehat{x_i} x_j := x_i x_j.$$

For convenience we tend to drop the hat on $\widehat{x_i}$, and write ∂_i for $\frac{\partial}{\partial x_i}$. Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we define the operators $x^\alpha = x^{\alpha_1} \dots x^{\alpha_n}$ and $\partial_\alpha = \partial_1^{\alpha_n} \dots \partial_n^{\alpha_n}$. Under the commutator bracket, A_n obtains the structure of a Lie algebra, which is infinite-dimensional:

Proposition 5.2. *A basis for A_n is given by $\mathscr{B} := \{x^\alpha \partial_\beta \colon \alpha, \beta \in \mathbb{N}^n\}$.*

Proof. The elements of $\mathscr{B}$ are evidently linearly independent, and we claim that they also span A_n. Note first that if $|\beta| \geq |\alpha|$, then

$$\partial_\beta(x^\alpha) = \begin{cases} \beta! := \prod_{i=1}^n \beta_i! & \text{if} \quad \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

and for any $i \in \mathbb{N}$ and $f \in k[x_1, \ldots, x_n]$ we have by the product rule $\partial_i x^\alpha(f) - x^\alpha \partial_i(f) = f\partial_i(x^\alpha)$. This allows us to move [...] $\square$

The Weyl algebra has a natural grading defined as follows:

Definition 5.3. Let $D = \sum_{(\alpha,\beta)\in\mathscr{A}} x^\alpha \partial_\beta$ be an element of A_n. The **degree of** D is the number $\deg D := \max_{(\alpha,\beta)\in\mathscr{A}}(|\alpha| + |\beta|)$.

Proposition 5.4. *Let $D, D' \in A_n$. Then*

(i) $\deg(D + D') \leq \max(\deg D, \deg D')$,
(ii) $\deg(DD') = \deg D + \deg D'$,
(iii) $\deg[D, D'] \leq \deg D + \deg D' - 2$.