Singular minimisers in the Calculus of Variations

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1 Introduction

1.1 Motivation

Minimisation problems are ubiquitous in mathematics as well as in physics. Extremal properties have been investigated throughout the history of mathematics, from Dido's problem in Virgil's *Aeneid* of finding the greatest area enclosed by a line of a given perimeter, to the Lagrangian formulation of classical mechanics and up until the present day. The area of mathematics studying such problems is called the Calculus of Variations, coined by Euler. Indeed, many of the great mathematicians through history, including Euler himself, Lagrange, Weierstrass, Hilbert and Carathéodory made valuable contributions to the area, the two last in particular in what is known as *field theories*. Employers of these so-called classical methods often neglected the possibility that the minimum of a variational problem might not be attained by any function, which famously resulted in Riemann's flawed proof of the Riemann mapping theorem. Weierstrass observed that Riemann's assumption that the Dirichlet problem always had a solution, was false. In the early years of the 20th century, the Calculus of Variations was put on rigourous footing by Tonelli, who gave sufficient conditions for the existence of minimisers of variational problems. His approach, which was purely functional-analytic in nature, is an example of a *direct method*.

Our variational problems take the following form: minimise

$$\mathscr{L}(u) := \int_a^b L(x, u(x), u'(x))dx$$

among all functions $u \in \mathscr{A}$ where $\mathscr{A}$ is a subset of some function space consisting of the functions satisfying specified boundary conditions $u(a) = A$ and $u(b) = B$. The map $L\colon [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called the *Lagrangian* of the problem, and $\mathscr{L}(u)$ is the *energy* of u.

Tonelli also proved that if the Lagrangian of a minimisation problem is sufficiently well-behaved, then the minimiser of the problem is smooth everywhere apart from an exceptional set of Lebesgue measure 0, called the *singular set*. A minimiser with a non-empty singular set is said to be a *singular minimiser*. In a variational problem, it is possible that the minimum given by the singular minimiser is strictly lower than that of minimisers in spaces of more regular functions. When this occurs, we say that the variational problem exhibits the *Lavrentiev phenomenon*. The first example of such an occurence was given by Lavrentiev [Lav27] after a challenge posed by Tonelli in Moscow in the 1920's, whence the name stems.

In the event of the Lavrentiev phenomenon occuring, it can also happen that the energy of any sequence of regular functions converging to the minimiser blows up, which severly impacts the efficacy of numerical approximation schemes. This is called the *repulsion property*, and is evident in many examples of the Lavrentiev phenomenon. These two phenomena have been linked to the theory of nonlinear elasticity and specifically to fractures in materials, which is of great importance in Material Science. Thus finding necessary and sufficient conditions for the non-occurence of the Lavrentiev phenomenon is of no small importance also outside the Calculus of Variations.

1.2 Acknowledgements

Most of the material in Section 2.2 and Proposition 9 are from Richard Gratwick's handwritten notes from 2016, although inaccuracies and errors are, of course, solely mine. The ideas of Section 3.2 are also Richard's but with some details filled out by myself. Apart from this and unless referenced, the remaining contents of this document are my own. The project was enabled by funding from the University of Edinburgh School of Mathematics, and College of Science and Engineering vacation scholarships.

1.3 Regarding notation and assumptions

Since this project is a continuation of a previous summer project under the same supervisor, we leave out some definitions from real and functional analysis which are readily found in for example [Eva98]. For convenience, however, it is useful to define the function spaces over which we shall work: We are concerned mainly with the Sobolev spaces $W^{n,p}(U)$, $U \subseteq \mathbb{R}$ which will be the (equivalence classes of) functions $f \colon \mathbb{R} \to \mathbb{R}$ for which the *Sobolev norm*

$$\|f\|_{W^{n,p}(U)} := \left(\sum_{k=1}^{n} \|f^{(k)}\|^{p}_{L^p(U)} \right)^{1/p}$$

is finite when $1 \leq p < \infty$, and $\|f\|_{W^{n,\infty}(U)} := \sum_{k=1}^{n} \|f^{(k)}\|_{L^\infty(U)}$. Since the L^p-norms are defined up to a set of measure 0, the derivatives should be interpreted in a weak sense, namely, g is the *weak derivative* of f on $U \subseteq \mathbb{R}$ if

$$\int_U g(x)\varphi(x)dx = -\int_U f(x)\varphi'(x)dx$$

for all $\varphi \in C_0^\infty(U)$. The space $\mathrm{AC}(a,b)$ of absolutely continuous functions on $[a,b]$ can be identified with the space $W^{1,1}(a,b)$ by choosing the continuous representative of each equivalence class (cf. [BGH98]), and we have a similar identification of $\mathrm{Lip}\,(a,b)$, the space of Lipschitz functions on $[a,b]$, with $W^{1,\infty}(a,b)$.

We assume the *Lagrangian* $L \colon \mathbb{R}^3 \to [0,\infty)$ to be continuous; $\xi \mapsto L(x,y,\xi)$ to be convex and have superlinear growth, $L(x,y,\xi) \geq \vartheta(\xi) \geq 0$ where $\vartheta(\xi)/|\xi| \to \infty$ as $\xi \to \infty$.

We consider the variational problem

$$\mathscr{L}(u) := \int_a^b L(x,u(x),u'(x))dx, \tag{1}$$

where $u \colon [a,b] \to \mathbb{R}$, $u \in \mathrm{AC}(a,b)$, the space of absolutely continuous functions on $[a,b]$, $a < b$, unless otherwise specified. We will also adopt the notation $\mathscr{L}(u;a,b)$ when we want to emphasise the domain of integration in (1).

When minimising $\mathscr{L}$ over all functions u satisfying, say, $u \in \mathrm{AC}(a,b)$ such that $u(a) = A$, $u(b) = B$, we are wont to say that u is *admissible* if it satisfies the implied boundary conditions.

We conclude the section by putting the assumptions above to use:

Theorem 1 (Tonelli's existence theorem)**.** *Let L be as above. Then the variational problem* (1) *has a minimiser in the class of functions $\{u \in \mathrm{AC}(a,b) : u(a) = A, u(b) = B\}$, for any fixed $A, B \in \mathbb{R}$.*

Proof. See [BGH98], Thm. 3.7. ■

Theorem 2 (Tonelli's partial regularity theorem)**.** *Suppose L satisfies the conditions above. Then the derivative of any minimiser of* (1) *exists everywhere, taking possibly the values $\pm\infty$, and is continuous almost everywhere.*

Proof. See [BGH98], Thm 4.6, or the paper [Fer12]. ■

2 Towards a necessary and sufficient condition for non-occurence of the Lavrentiev phenomenon

It was first observed by Lavrentiev in [Lav27] that the minimum of $\mathscr{L}(u)$ over all $u \in \mathrm{AC}(a,b)$ satisfying $u(a) = A$ and $u(b) = B$ for fixed $(a, A, b, B) \in \mathbb{R}^4$ can be strictly smaller than the corresponding minimum over admissible Lipschitz functions. As such, the occurence of this is called the *Lavrentiev phenomenon*. The difference between the respective inifima is called the *Lavrentiev gap*. A straightforward example with a polynomial Lagrangian was given by Manià [Man34],

$$\mathscr{L}(u) = \int_0^1 (u(x)^3 - x)^2 (u'(x))^6 dx \quad \text{with} \quad u(0) = 0,\, u(1) = 1,$$

which is minimised by $x^{1/3}$ over $\mathrm{AC}(0,1)$, but bounded away from the minimum when we pass to the class of admissible Lipschitz functions. Manià's example also exhibits the *repulsion property*, namely that for any sequence of Lipschitz functions (u_n) tending uniformly to the minimiser $u(x) = x^{1/3}$, $\mathscr{L}(u_n) \to \infty$ as $n \to \infty$. Note, however that this does not satisfy the superlinearity condition. This was mended by Ball & Mizel in [BM87] by adding a correction term in the third variable. They also came up with the general class of variational problems

$$\mathscr{L}(u) = \int_{-1}^1 (x^4 - u^6)^2 |u'|^s + \epsilon (u')^2 dx \quad \text{subject to} \quad u(-1) = k_1,\, u(1) = k_2,$$

parameterised by $s > 3$, $\epsilon > 0$. For $s \geq 27$, $-1 \leq k_1 < 0 < k_2 \leq 1$ and ϵ sufficiently small, the problem exhibits the Lavrentiev phenomenon not only for admissible Lipschitz functions, but across admissible $\mathrm{AC}(0,1)$ and $W^{1,q}(-1,1)$ for any $q \geq 3$, in addition to being superlinear. It is particularly interesting because the *singular set* of the minimiser, that is, $E(u) \coloneqq \{x \in [a,b] \mid |u'(x)| = \infty\}$, is in the interior of the interval. More examples are given in Section 4. It is also interesting to note that while the examples given in the last few years have either simple polynomial Lagrangians or very complicated inductive constructions, Lavrentiev's had neither.

2.1 Sufficient conditions for the non-occurence of the Lavrentiev phenomenon

Much work has been put into finding conditions preventing the Lavrentiev phenomenon from occuring. Lavrentiev himself gave some conditions in his original paper on the phenomenon [Lav27]. Angell finds a criterion denoted "condition $(\mathcal{D})$" [Ang79]; Clarke and Vinter show that if L is autonomous, that is, does not depend on the first variable, then the phenomenon does not occur [CV85]. Extending this, Ferriero shows that we have non-occurence whenever L depends only on two adjecent derivatives [Fer05].

2.2 The Lavrentiev set

Definition 3. The *value function* $S\colon \mathbb{R}^4 \to \mathbb{R} \cup \{\pm\infty\}$ is the map

$$S(a, A, b, B) \coloneqq \inf \{\mathscr{L}(u) \mid u \in \mathrm{AC}(a,b),\ u(a) = A,\ u(b) = B\}.$$

The idea of using a value functional was introduced already by [BM92], a paper investigating the Lavrentiev phenomenon from an abstract point of view, yielding an explicit expression for the Lavrentiev gap under certain condition. However, they considered instead the infimum over suitable Lipschitz functions with one fixed endpoint.

Definition 4. Given $\epsilon > 0$, the *ϵ-Lavrentiev set of L* is the set

$$\begin{aligned}\mathrm{Lav}_\epsilon(L) \coloneqq \{(a, A) \in \mathbb{R}^2 : \exists\, (b, B) \in \mathbb{R}^2 \text{ s.t. } \mathscr{L}(u; a, b) \geq S(a, A, b, B) + \epsilon,\\ \forall u \in \mathrm{AC}(a, b) \text{ s.t. } u(a) = A,\ u(b) = B,\ |u'(a)| < \infty\},\end{aligned}$$

and the *Lavrentiev set of L* is defined to be $\mathrm{Lav}(L) \coloneqq \bigcup_{\epsilon>0} \mathrm{Lav}_\epsilon$.[1]

Proposition 5. *Suppose $(a_0, A_0) \in \mathrm{Lav}(L)$, that is, $(a_0, A_0) \in \mathrm{Lav}_\epsilon(L)$ for some $\epsilon > 0$. Then S has a jump discontinuity at (a_0, A_0, b_0, B_0).*

Proof. Fix $M > 0$, let $0 < \delta < \epsilon/2K$, where

$$K \coloneqq \sup\{L(x, y, z) : (x, y, z) \in [a_0, b_0] \times [-\|u_0\|_\infty - A_0 - 1, \|u_0\|_\infty + A_0 + 1] \times [-M, M]\}.$$

for u_0 attaining $S(a_0, A_0, b_0, B_0)$. Fix any a satisfying $a_0 < a < a_0 + \delta$ and A with $|A - A_0| \leq M|a - a_0|$. Let $u \in AC(a, b_0)$ attain $S(a, A, b_0, B_0)$, and define $v \in AC(a_0, b_0)$ by

$$v(x) = \begin{cases} \frac{A_0 - A}{a_0 - a}(x - a) + A & \text{if } \quad x \in [a_0, a] \\ u(x) & \text{if } \quad x \in (a, b_0]. \end{cases}$$

Intuitively, v is the affine extension of u to a_0. Then $|v'(x)| \leq M$ for $x \in [a_0, a]$, so $\mathscr{L}(v; a_0, b_0) \geq S(a_0, A_0, b_0, B_0) + \epsilon$. But

$$\begin{aligned}\mathscr{L}(v; a_0, b_0) &= \int_{a_0}^{a} L(x, v(x), v'(x))dx + \mathscr{L}(v; a, b_0)\\ &\leq |a - a_0| \cdot K + \mathscr{L}(v; a, b_0)\\ &\leq \epsilon/2 + S(a, A, b_0, B_0),\end{aligned}$$

by our choice of δ. It follows that

$$\begin{aligned}S(a, A, b_0, B_0) &\geq \mathscr{L}(v; a_0, b_0) - \epsilon/2\\ &\geq S(a_0, A_0, b_0, B_0) + \epsilon - \epsilon/2\\ &= S(a_0, A_0, b_0, B_0) + \epsilon/2,\end{aligned}$$

which is what we wanted. ∎

The following theorem is similar to Sychev's result in [Syc93], Lemma 1.6(2), that S is continuous in the absence of a Lavrentiev phenomenon; in short, he shows that S is upper semicontinuous by choosing an admissible smooth function arbitrarily close to $S(a_0, A_0, b_0, B_0)$ in energy. Since S is unconditionally lower semicontinuous as shown in 1.6(1), this gives his result.

Proposition 6. *Suppose $(a_0, A_0, b_0, B_0) \in \mathbb{R}^4$, $a_0 < b_0$ and that there exists $\epsilon > 0$ such that $S(a, A, b_0, B_0) \geq S(a_0, A_0, b_0, B_0) + \epsilon$ for (a, A) sufficiently close to (a_0, A_0). Then $(a, A) \in \mathrm{Lav}(L)$.*

Proof. Suppose $u \in \mathrm{AC}(a_0, b_0)$ satisfies $u(a_0) = A_0$, $u(b_0) = B_0$ and $|u'(a_0)| \leq M < \infty$ for some constant M. Choose $x_0 \in (a_0, b_0)$ satisfying the following conditions:

(i) $\mathscr{L}(u; a_0, x_0) = \int_{a_0}^{x_0} L(x, u(x), u'(x))dx < \epsilon/4$,

[1] If we did not have Theorem 2, the derivative would have had to be interpreted in a suitable weak sense, for example as $\limsup_{x\to a} \frac{u(x)-u(a)}{x-a}$.

(ii) $|a_0 - x_0| < \epsilon/4K$, where

$$K := \sup\{L(x,y,z) : (x,y,z) \in [a_0,b_0] \times [\|u\|_\infty - A_0 - 1, \|u\|_\infty + A_0 + 1] \times [-M-2, M+2]\},$$

(iii) $\left|\frac{u(a)-u(a_0)}{a-a_0} - u'(a_0)\right| < 1$ for all $a \in (a_0, x_0]$, so that in particular $\left|\frac{u(a)-u(a_0)}{a-a_0}\right| \leq M+1$.

Now choose $a \in (a_0, x_0)$ and $A < A_0$ sufficiently close to (a_0, A_0) such that $|a - a_0| < |a_0 - x_0|$ and

$$\left|\frac{A-u(x_0)}{a-x_0} - \frac{A_0 - u(x_0)}{a_0 - x_0}\right| \leq 1. \tag{2}$$

Next, define an admissible $v \in AC(a, b_0)$ by

$$v(x) := \begin{cases} \frac{A-u(x_0)}{a-x_0}(x-x_0) + u(x_0) & \text{for} \quad x \in [a, x_0), \\ u(x) & \text{for} \quad x \in [x_0, b_0], \end{cases}$$

that is, the extension of u along a straight line to a_0. Note that on $[a, x_0]$,

$$|v'| = \left|\frac{A-u(x_0)}{a-x_0}\right| \leq \left|\frac{A_0-u(x_0)}{a_0-x_0}\right| + 1 \leq M+2,$$

by the choice of a and condition (iii). Estimating the integral, we get that

$$|\mathscr{L}(v; a, x_0)| \leq |a - x_0| K < \epsilon/4.$$

Condition (i) now yields, since $u \equiv v$ on $[x_0, b_0]$, that

$$\begin{aligned} |\mathscr{L}(u; a_0, b_0) - \mathscr{L}(v; a, b_0)| &\leq |\mathscr{L}(u; a_0, x_0)| + |\mathscr{L}(v; a, x_0)| \\ &< \epsilon/4 + \epsilon/4 \\ &= \epsilon/2. \end{aligned}$$

It follows that

$$\begin{aligned} \mathscr{L}(u; a_0, b_0) &\geq \mathscr{L}(v; a, b_0) - \epsilon/2 \\ &\geq S(a, A, b_0, B_0) - \epsilon/2 \\ &\geq S(a_0, A_0, b_0, B_0) - \epsilon/2 + \epsilon \\ &= S(a_0, A_0, b_0, B_0) + \epsilon/2, \end{aligned}$$

so $(a_0, A_0) \in \mathrm{Lav}_{\epsilon/2}(L)$, hence in $\mathrm{Lav}(L)$. ■

Corollary 7. *Let $\epsilon > 0$. Then $(a, A) \in \mathrm{Lav}_\epsilon$ if and only if $\{(a, A) : a < a_0, A \leq A_0\} \ni (a, A) \mapsto S(a, A, b_0, B_0)$ is discontinuous at (a_0, A_0).*

We can sharpen this slightly, at the cost of conciseness:

Theorem 8. *The following are equivalent:*

(i) $(a_0, A_0) \in \mathrm{Lav}(L)$.

(ii) *There exists $\epsilon > 0$ such that for any $M > 0$, $a_0 < a < a_0 + \delta$ for some $\delta > 0$ and $|A - A_0| \leq M|a - a_0|$ implies that $S(a, A, b_0, B_0) \geq S(a_0, A_0, b_0, B_0) + \epsilon$.*

(iii) *There exist $\epsilon > 0$ such that for all $a > a_0$ and $A > A_0$ sufficiently close, $S(a, A, b_0, B_0) \geq S(a_0, A_0, b_0, B_0) + \epsilon$.*

(iv) *There exists an $\epsilon > 0$ such that $\limsup_{a > a_0, (a,A) \to (a_0, A_0)} S(a, A, b_0, B_0) \geq S(a_0, A_0, b_0, B_0) + \epsilon$.*

(v) *$\{(a, A) : a > a_0, A \geq A_0\} \ni (a, A) \mapsto S(a, A, b_0, B_0)$ is discontinuous at (a_0, A_0).*

(vi) $(a, A) \mapsto S(a, A, b_0, B_0)$ *is discontinuous at* (a_0, A_0).

Proof. (i) $\Rightarrow$ (ii) is a slightly more verbose version of Proposition 5. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are immediate from definitions, as is (v) $\Rightarrow$ (vi). (v) $\Rightarrow$ (i) follows from lower semicontinuity of the value functional and Proposition 6: in particular, if (a_n, A_n) tends to (a_0, A_0) from above, then by the discontinuity of S, $\lim_{n\to\infty} S(a_n, A_n, b_0, B_0) \neq S(a_0, A_0, b_0, B_0)$. As S is lower semicontinuous, $\liminf_{(a,A)\to(a_0,A_0)} S(a, A, b_0, B_0) \geq S(a_0, A_0, b_0, B_0)$. By discontinuity, the liminf is strictly smaller than the limsup, so for $\epsilon > 0$ sufficiently small $\limsup_{(a,A)\to(a_0,A_0)} S(a, A, b_0, B_0) > S(a_0, A_0, b_0, B_0)+\epsilon$, which gives the conditions for Proposition 6. ∎

2.3 Structural properties of $\mathrm{Lav}(L)$

In general, it is difficult to describe the structure of $\mathrm{Lav}_\epsilon(L)$ as a set, but the following is quite effortless to prove:

Proposition 9. *Let* $\epsilon > 0$, *and let* L *be a Lagrangian satisfying the above conditions. Then the set* $\mathrm{Lav}_\epsilon(L)$ *is closed.*

Proof. Suppose $(a_n, A_n)_{n\in\mathbb{N}} \subseteq \mathrm{lav}_\epsilon$ with $(a_n, A_n) \to (a_0, A_0)$. Choose $0 < \eta < \epsilon$, and note that $\mathrm{lav}_\eta \subseteq \mathrm{lav}_\epsilon$. Then for any n there exist (a'_n, A'_n) such that $S(a'_n, A'_n, b_0, B_0) \geq S(a_n, A_n, b_0, B_0) + \eta$ and $(a'_n, A'_n) \to (a_0, A_0)$. It follows that

$$\begin{aligned}\mathscr{L}(u; a_0, b_0) &\geq \limsup_{(a,A)\to(a_0,A_0)} S(a, A, b_0, B_0)\\ &\geq \limsup_{n\to\infty} S(a'_n, A'_n, b_0, B_0)\\ &\geq \limsup_{n\to\infty} S(a_n, A_n, b_0, B_0) + \eta\\ &\geq \liminf_{n\to\infty} S(a'_n, A'_n, b_0, B_0) + \eta\\ &\geq S(a_0, A_0, b_0, B_0) + \eta,\end{aligned}$$

the last step following from the lower semicontinuity of S. Since this holds for every $\eta < \epsilon$, we conclude that $(a_0, A_0) \in \mathrm{Lav}_\epsilon(L)$. ∎

By observing that $\mathrm{Lav}(L) = \bigcup_{n\in\mathbb{N}} \mathrm{Lav}_{1/n}$, we obtain the following:

Corollary 10. *The set* $\mathrm{Lav}(L)$ *is* F_σ, *a countable union of closed sets.*

Note that $\mathrm{Lav}(L)$ need not, a priori, be closed, since we could conceivably pick a sequence of points $x_n \in \mathrm{Lav}_{1/n}(L) \setminus \mathrm{Lav}_{1/(n-1)}(L)$ which would not converge to a point in $\mathrm{Lav}(L)$.

Ball & Nadirashvili introduce in [BN93] the *universal singular set* of a Lagrangian L, which is defined to be the points in the plane for which there exists a singular minimiser of the associated variational problem with respect to its own boundary conditions. That is,

$$\begin{aligned}\mathrm{uss}(L) \coloneqq \{(x, u(x)) \in \mathbb{R}^2 : |u'(x)| = \infty,\ &\mathscr{L}(u; a, A, b, B) = S(a, A, b, B),\\ &a \leq x \leq b,\ A \leq u(x) \leq B\}.\end{aligned}$$

It follows from definitions that $\mathrm{Lav}(L) \subseteq \mathrm{uss}(L)$. Thus we have the following:

Theorem 11. *The set* $\mathrm{Lav}(L)$ *is purely unrectifiable, is of first category and has Lebesgue measure* 0.

Proof. These properties are proven in [CKO+08], [BN93] and [Syc94], respectively, for $\mathrm{uss}(L)$, and are all hereditary. ∎

2.4 The p-Lavrentiev set

We can generalise slightly, inspired by the footnote following Definition 4:

Definition 12. Given $\epsilon > 0$ and $p > 1$, the (p, ϵ)-Lavrentiev set of L is the set

$$\begin{aligned}\mathrm{Lav}^p_\epsilon(L) := \{(a, A) \in \mathbb{R}^2 : \exists\, (b, B) \in \mathbb{R}^2 \text{ s.t. } \mathscr{L}(u; a, b) \geq S(a, A, b, B) + \epsilon,\\ \forall u \in \mathrm{AC}(a, b) \text{ s.t. } u(a) = A,\ u(b) = B,\ \limsup_{\delta\to 0} \left\| u' \mathbb{1}_{[a,a+\delta]} \right\|_p < \infty\},\end{aligned}$$

and the *p-Lavrentiev set of* L is defined to be $\mathrm{Lav}^p(L) := \bigcup_{\epsilon>0} \mathrm{Lav}^p_\epsilon(L)$.

Note that $\mathrm{Lav}^\infty_\epsilon$ is our original ϵ-Lavrentiev set.

Proposition 5 holds true for Lav^p_ϵ as well, the proof in this case being virtually the same except that the bound on the gradient of the affine extention is replaced by the following:

$$\begin{aligned}\left(\int_a^{a+\delta} \left|\frac{A_0 - A}{a_0 - a}(x - a) + A\right|^p dx\right)^{1/p} &= \left(\int_0^{\delta} \left|\frac{A_0 - A}{a_0 - a}x + A\right|^p dx\right)^{1/p}\\ &\leq \left(\int_0^{\delta} \left|\frac{A_0 - A}{a_0 - a}x\right|^p dx\right)^{1/p} + \left(\int_0^{\delta} |A|^p dx\right)^{1/p}\\ &\leq M\left(\left[\frac{x^{p+1}}{p+1}\right]_0^\delta\right)^{1/p} + \delta^{1/p} A\\ &= M\left(\frac{\delta^{p+1}}{p+1}\right)^{1/p} + \delta^{1/p} A \to 0 \quad \text{as} \quad \delta \to 0,\end{aligned}$$

by a change of variables and Minkowski's inequality. We cannot reasonably expect the converse to hold though, since there are several examples of gaps between $W^{1,p}$ and $W^{1,q}$ for any choice of $p, q \geq 1$. Clarke [Cla86] asks whether a gap between $W^{1,1}$ and $W^{1,\infty}$ is "bridged" by an intermediate function space like $W^{1,p}$, $1 < p < \infty$. This is not true; by [Fos01] and more concretely [Fer07a]. There is also an example reproduced in Mizel's report [Miz02] attributed to A. Siegel's Master thesis (2000), but which is difficult to find.

One might also argue (as suggested by R. Gratwick) that the right "local L^p"-condition should be something like

$$\limsup_{x\downarrow a} ⨍_{[x,a]} (u')^p := \limsup_{x\downarrow a} \frac{1}{x - a}\int_a^x u'(t)^p dt = \lim_{\delta\to 0} \frac{1}{\delta}\int_a^{a+\delta} u'(t)^p dt,$$

the last inequality on the assumption that the limit exists; which is simply a weighted version of our previous condition. Note that by the Lebesgue differentiation theorem ([Whe15], Thm.7.2), $u'(a) = \limsup_{x\downarrow a} \frac{1}{x-a}\int_a^x u(t)dt$ if $u'(a)$ exists, for example under the assumptions of Theorem 2. We easily bound the affine extension:

$$\frac{1}{\delta}\int_a^{a+\delta} \left|\frac{A_0 - A}{a_0 - a}\right|^p dx = \frac{1}{\delta}\left|\frac{A_0 - A}{a_0 - a}\right|^p (a + \delta - a) = \left|\frac{A_0 - A}{a_0 - a}\right|^p.$$

This makes more sense, as this equals the L^p-norm of the derivative of the affine line, which is to be expected since it is "homogeneous", i.e. looks the same on any open set.

2.5 The Lavrentiev set vs. the Lavrentiev phenomenon

Recall that the singular set of a minimiser $u \in \mathrm{AC}(a, b)$ is the set $E(u) \subset [a, b]$ consisting of points x for which $|u'(x)| = \infty$. Note that by Tonelli's partial regularity theorem, (Theorem 2), $E(u)$ is closed.

Question. *Suppose the Lavrentiev phenomenon occurs with boundary data* (a, A, b, B)*, witnessed by a minimiser* $u \in \mathrm{AC}(a, b)$*. Do we have that*

$$\{(x, u(x)) : x \in E(u)\} \cap \mathrm{Lav}(L) \neq \varnothing? \tag{3}$$

In words, this would mean that if the Lavrentiev phenomenon occurs, then it is witnessed by an element in the Lavrentiev set. While this might seem obvious in light of our notation and naming, an affirmative answer is in fact crucial to establish any relevance of our theorems.

There is, however, reason to be sceptical, especially if the singular set is infinite. Suppose for example that $E(u)$ is the Cantor set; then there is no natural point on the graph of u_0 which "contributes more", i.e. in which any other AC-function with finite derivative is bounded away in energy from that of the restriction of u_0 to some small interval. More concretely, the example due to Gratwick [Gra17] reproduced in Section 4 by construction has a gap where each rational point gives an equal contribution, so if the gap is finite, we can not expect any point to be in the Lavrentiev set of the corresponding Lagrangian.

Proposition 13. *Assume that the above holds for minimisers u with $|E(u)| = 1$. Suppose that u witnesses the Lavrentiev phenomenon with boundary data (a, A, b, B). Then*

$$\{(x, u(x)) : x \in E\} \cap \mathrm{Lav}(L) \neq \varnothing.$$

Proof. Let $x_1, \ldots, x_n \in E(u)$, and note that on $[x_k, x_{k+1} - \delta]$ for $\delta > 0$ sufficiently small, the restriction of u is a minimiser with respect to its own boundary conditions, and this restriction has singular set consisting of a single point. Then for at least one k, we have $(x_k, u(x_k)) \in \mathrm{Lav}_{\epsilon/n}(L)$, hence our intersection is non-empty. ■

3 A singular extremal in a field of extremals

3.1 Background

In an upcoming paper, M. Sychev proves the following result:

Proposition 14. *Let u_c be a field of extremals, with the exception of one function u_0, which is a Sobolev function with essentially unbounded derivative. Then u_0 is a singular extremal, and $\mathscr{L}(u) > \mathscr{L}(u_0)$ for each Lipschitz function u with the same boundary data as u_0.*

Here a *field of extremals* is a family of continuous functions $u_c(\cdot)\colon [a, b] \to \mathbb{R}$ parameterised by c covering without intersections a set $G = \{(x, u(x)) \in \mathbb{R}^2 : x \in [a, b], g(x) \leq u(x) \leq f(x)\}$ for some $f, g \in C(a, b)$ and which satisfy the variant of the Euler-Lagrange equation given by

$$u'' = \frac{L_y - L_{zx} - L_{zy}u'}{L_{zz}}.$$

Note that Sychev assumes that $L \in C^3$ in order for the above partial derivatives to be well-defined. A singular extremal is a function which is absolutely continuous and smooth everywhere apart from an exceptional set of measure 0, and which solves the above equation away from this set.

However, the existence of such a singular extremal in a field of extremals is still conjectural. Following an idea outlined by R. Gratwick, we construct here a singular extremal as a special case of a very general construction in the paper [CKO+08].

3.2 The construction

In [CKO+08], Theorem 10 and Lemma 11, applied to the special case where the purely unrectifiable compact set S is $\{(0, 0)\}$, Csörnyei et al. construct a Lagrangian $L(x, y, p) = F(x, y, p) + \omega(p)$, where $\omega \in C^\infty(\mathbb{R})$ is a given strictly convex and superlinear function with $\omega(p) \geq \omega(0) = 0$ for all $p \in \mathbb{R}$, via the construction of a calibrator $\Phi \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\}) \cap C(\mathbb{R}^2)$ which satisfies $\lim_{(x,y)\to(0,0)} \Phi_x/\Phi_y(x, y) = -\infty$. The Lagrangian satisfies the basic conditions for existence of minimisers: it is of class $C^\infty(\mathbb{R}^3)$, and is strictly convex and superlinear in p. Additionally, if $u \in \mathrm{AC}(0, 1)$, then

$$\int_0^1 L(x, u(x), u'(x))dx \geq \Phi(1, u(1)) - \Phi(0, u(0)),$$

with equality iff

$$2\Phi_x(x, u(x)) + \Phi_y(x, u(x))u'(x) = 0 \text{ for a.e. } x \in [0, 1],$$

or equivalently, defining $\psi(x, y) := -2\Phi_x/\Phi_y(x, y)$, iff $u'(x) = \psi(x, u(x))$ for a.e. $x \in [0, 1]$.

Theorem 15. *Let L be as above. There exists a field G of functions $u_c \in \mathrm{AC}(0,1)$ parameterised by $u_c(0) = c$ for $c \in [-1, 0]$ satisfying*

(i) *for each $c < 0$, u_c is a minimiser of*

$$\mathscr{L}(u) = \int_0^1 L(x, u(x), u'(x))dx,$$

with respect to its own boundary conditions $u_c(0) = c$ and $u_c(1)$;

(ii) $\|u_c - u_{c'}\|_\infty$ *depends continuously on $|c - c'|$ for all $c, c' \in [-1, 0]$; and*

(iii) $\lim_{x\to 0} |u_0'(x)| = \infty$.

Proof. By the argument given at the end of Theorem 10 in [CKO+08] considering any $c < 0$ separately, the solutions u_c to

$$\begin{cases} u_c'(x) = \psi(x, u_c(x)) & \text{for a.e. } x \in [0,1], \\ u_c(x_0) = c \end{cases} \tag{4}$$

are all well-defined and absolutely continuous on $[0, 1]$. Moreover, as $\psi \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\})$, if $u_c(x) = u_{c'}(x)$ for some $c, c' \in [-1, 0)$, and $x \in (0, 1)$, then the standard uniqueness result from ODE theory gives us that $c = c'$ (see for example Theorem 3.3 in [Sid13]). In other words, the graphs of (u_c) are disjoint. Since each u_c is smooth as shown in [CKO+08] for $c < 0$, this implies that $\{u_c : c \in [-1, 0)\}$ is a field.

We next aim to define a singular extremal u_0 as the limit of $u_n := u_{-1/n}$. We will show the following:

(α) the sequence (u_n) is uniformly bounded; and

(β) the sequence (u_n') is equiintegrable.

(α) It is sufficient (and necessary) that $\lim_{n\to\infty} u_n(1) < \infty$ since u_n are all non-decreasing by construction, with disjoint graphs and $u_n(0) = -1/n > -1/m = u_m(0)$ if $n > m$. By construction (cf. [CKO+08] p. 409), outside $B_1(0, 0)$—the open ball of radius 1 centred at the origin—the function $\Phi(x, y)$ is linear in each variable, so ψ is constant, $\psi(x) = k$, say. It follows that

$$|u_n(1)| \le 1/n + \int_0^1 |u_n'(x)|dx \le 1/n + 1 + k \le 2 + k,$$

so the sequence is indeed uniformly bounded.

(β) By [BGH98], Theorem 2.12 (i4), it suffices to show that

$$\sup_{n\in\mathbb{N}} \int_0^1 \omega(u_n'(x))dx < \infty.$$

But

$$\int_0^1 \omega(u_n'(x))dx \le \int_0^1 L(x, u_n, u_n') = \Phi(1, u_n(1)) - \Phi(0, u_n(0)),$$

which is bounded since (u_n) are uniformly bounded by (i) and Φ is everywhere continuous.

By Theorem 2.13 in [BGH98], (u_n) has a weakly $W^{1,1}$-convergent subsequence, and we denote the limit by v. By the equiintegrability of (u_n'), (u_n) is equicontinuous, so the Arzela-Ascoli theorem gives, up to passing to a (sub-)subsequence, a uniformly convergent subsequence of (u_n) whose limit we denote by w. Since uniform convergence implies strong L^1-convergence implies weak L^1-convergence, we conclude that $v \equiv w$ almost everywhere. Moreover, from the implied pointwise convergence of $u_n(0)$, we deduce that $w(0) = 0$, so we define $u_0 = w$, the "remaining" member of our field. By the uniform convergence to u_0, and the smoothness of ψ away from the origin, the set $\{u_c : c \in [-1, 0]\}$ (i.e. now including u_0) does indeed form a field.

Finally, we intend to show that u_0 satisfies $u_0'(x) = \psi(x, u_0(x))$ almost everywhere. Note that uniform convergence of (u_n) and continuity of ψ away from the origin gives us pointwise convergence of $u_n'(x) =$

$\psi(x, u_n(x))$ a.e.. This, along with equiintegrability, allows us to apply Vitali's convergence theorem ([RF17] p. 94), yielding

$$\begin{aligned} u_0(x) &= \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \int_0^x \psi(t, u_n(t))dt = \lim_{n\to\infty} \int_0^x u_n'(t)dt \\ &= \int_0^x \lim_{n\to\infty} u_n'(t)dt = \int_0^x \lim_{n\to\infty} \psi(t, u_n(t))dt = \int_0^x \psi(t, u_0(t))dt, \end{aligned}$$

which we recognise as the integral form of the differential equation $u_0'(x) = \psi(x, u_0(x))$ a.e., so $\lim_{x\to 0} u_0'(x) = \infty$, proving (iii). ■

3.3 An interior singular extremal

To demonstrate the usefulness of Proposition 14, it is desirable that the singular extremal be in the interior of the field, which is not the case in the construction of the previous section. A possible approach which has not yet proven fruitful is the following: repeat the construction above with $c > 0$ instead to yield a singular extremal $\widetilde{u}_0$. It suffices to show that $u_0 \equiv \widetilde{u}_0$. This can be achieved through showing that the differential equation (4) has unique solutions, which by the first chapter of [Fil88] follows from showing that ψ satisfies a local Lipschitz condition in the second variable, with Lipschitz constant integrable with respect to the first. Since ψ is smooth away from the origin, it suffices to bound the integral of the Lipschitz constant. This might be achieved by carefully going through the inductive construction in [CKO+08] and choose domains Ω_k narrow enough to ensure that the the rapid growth of ψ^k is counteracted by the narrowness of the domains. However, there are several difficulties; primarily, controlling the Lipschitz constant of $y \mapsto \psi^k(x, y)$ amounts to determining the y-derivatives of the construction functions near the origin, which is very challenging given the absence of regularity required by the original construction.

Thus we have the following open problem:

Question. *Does there exist an interior singular extremal embedded in a field of extremals?*

Interestingly, since the conjecture has resisted naive attempts well so far, it is not inconcievable that the resolution could in fact be in the negative.

4 Table of some occurences of the Lavrentiev phenomenon

Integrand	Boundary conditions	Min	Gap	Lav(L) ([a])	Reference	Comment
$\exp\left(-\frac{2}{(u-\sqrt{x})^2}\right)f(u')$	$u(0)=0,\ u(1)=1$	$\sqrt{x}$	$W^{1,1}-W^{1,\infty}$	$\{(0,0)\}$	[Lav27]	
$(u^3-x)^2(u')^6$	$u(0)=0,\ u(1)=1$	$x^{1/3}$	$W^{1,1}-W^{1,\infty}$	$\{(0,0)\}$	[Man34]	
$(u^6-x^4)^2\|u'\|^s+\epsilon(u')^2,\ s\geq 27$	$u(-1)=-1,\ u(1)=1$	$x^{2/3}$	$W^{1,q}-W^{1,3}, q<3$	$\{(0,0)\}$	[BM87]	Singular set of min. in interior
$(u-x^p)^2(u'-x^q)^2\|u''\|^d, 0<q<1<p<2, d\geq\frac{1+2(p+q)}{1-q}$	$u(0)=0$	$x^p,\ \frac{x^{q+1}}{q+1}$	$W^{2,1}-W^{2,\infty}$	$\{(0,0)\}$	[Bel95]	Higher order Lagrangian, non-unique minimiser
$\Phi(u^l-x^k\varphi(x,u))\Psi(u')$	$u(0)=0,\ u(1)=1$	$x^{k/l}$	$W^{1,1}-W^{1,\infty}$	$\{(0,0)\}$	[Sar97]	Generalises many Manià-type examples
$(u^5-x^3)^2(1+z^{20}),\ x\in[0,1]$	None	$x^{3/5}$	$W^{1,1}-W^{1,\infty}$	$\{(0,0)\}$	[DHM00]	Free endpoints
(Complicated)	$u(0)=0,\ u(1)=1$	$x^{(y-1)/y}$	$W^{1,p_1}-W^{1,p_2}$	?	[Fos01]	$\inf_{u\in W^{1,p}}\mathscr{L}(u)=\mu(p)$ for fixed $\mu\in W^{1,1}_{\text{loc}}([1,\infty];[0,\infty))$
$(u^3-x)^2\|u'\|^s,\ s\geq 9/2$	$u(0)=0,\ u(1)=1$	$x^{1/3}$	$W^{1,q}-W^{1,\infty}, q<\frac{3}{2}$	$\{(0,0)\}$	[Fer07b]	Strong repulsion property iff $s\geq 9/2$
$P(x,u^3-(1-x^2))(E_{c,p}\|u'\|^p+1)$ where P is a polynomial and $E_{c,p}$ a constant	$u(-1)=u(1)=0$	$\sqrt[3]{1-t^2},\ 0$	$W^{1,1}-W^{1,\infty}$	?	[Fer07a]	Infimum is attained in both $W^{1,1}$ and $W^{1,\infty}$. He also presents examples of the separate infima not being attained.
(Complicated)	$v(0)=A,\ v(1)=B$	v	$W^{1,1}-W^{1,\infty}$	$\mathbb{Q}\cap[0,1]$	[Gra17]	$\mathscr{L}(v+u)=\infty$ for any $u\in W^{1,\infty}_0$

[a]Most of these are conjectural

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