Recursion and Sequentiality in Categories of Sheaves

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A model is **fully abstract** if:

Contextual equivalence $=$ Equality in the model

$$t_1 \cong t_2 \iff [t_1] = [t_2]$$

The ⟷ is hard to get.
PCF<sub>v</sub>: A call-by-value language

Types: \( \tau ::= 0 \mid 1 \mid \text{nat} \mid \tau + \tau \mid \tau \times \tau \mid \tau \rightarrow \tau \)

Values: \( v, w ::= \ldots \mid \lambda x. t \mid \text{rec } \text{f } x. t \)

Computations: \( t ::= \ldots \mid v \ w \mid \text{let } x = t \text{ in } t' \)

Typing judgements: \( \Gamma \vdash^v v : \tau \) and \( \Gamma \vdash^c t : \tau \).

An interpretation looks like:

\[
\begin{align*}
\llbracket \text{nat} \rrbracket &= \sum_0^\infty 1 = 1 + 1 + \ldots \\
\llbracket \tau \rightarrow \tau' \rrbracket &= \llbracket \tau \rrbracket \Rightarrow \llbracket \tau' \rrbracket \\
\llbracket \Gamma \vdash^v v : \tau \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket \\
\llbracket \Gamma \vdash^c t : \tau \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket
\end{align*}
\]
The $\omega$cpo model of PCF$_v$:

Types $\leftrightarrow$ posets with sups of $\omega$-chains.

Terms $\leftrightarrow$ continuous functions.

\textbf{Not fully abstract.} E.g. parallel-or not definable.

Need to capture sequentiality

\textbf{O’Hearn and Riecke’s idea} \[OHR’95, \text{Riecke&Sandholm’02}\]

Use logical relations to cut down to sequential functions.

[Plotkin’80], [Jung & Tiuryn’93]: logical relations for $\lambda$-definability.

[Sieber’92]: definability for PCF up to order 2.
The \( \omega \text{cpo} \) model of PCF\(_v\):

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Terms \( \leftrightarrow \) continuous functions.

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  \item \textbf{Not fully abstract.} E.g. parallel-or not definable.
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Need to capture sequentiality

\begin{itemize}
  \item **O’Hearn and Riecke’s idea** [OHR’95, Riecke&Sandholm’02]
    Use logical relations to cut down to sequential functions.
\end{itemize}

\begin{itemize}
  \item **What we did** [MMS’21]
    Describe the OHR model as a sheaf category.
\end{itemize}
Outline

1. Introduction: fully abstract models and PCF$_v$
2. Building a fully abstract model: recursion
3. Building a fully abstract model: sequentiality
4. Summary and future work
Concrete presheaves on the vertical natural numbers

\( V = \{0 < 1 < 2 < \ldots < \infty\} = \) poset of vertical natural numbers

\( V = \) two-object category:

\[ \begin{array}{c}
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\end{array} \]

\( vSet = [V^{op}, \text{Set}] = \) presheaves on \( V \)

Concrete presheaf on \( V \)

- a set \( X(\star) \)
- a set of functions

\[ X(V) \subseteq [V \to X(\star)] \]

\( X(V) \) is a relation with arity \( V \) on \( X(\star) \).
Concrete presheaves on the vertical natural numbers

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A map between concrete presheaves \( X \) and \( Y \) is:

- a function \( f : X(\star) \to Y(\star) \)
- acting by postcomposition: \( g \in X(V) \mapsto f \circ g \in Y(V) \)
  i.e. \( f \) preserves the relation.
If $X$ and $Y$ are **concrete presheaves**, the exponential is also a concrete presheaf:

$$(X \Rightarrow Y)(\star) = \{f : X(\star) \to Y(\star) \mid f \text{ preserves the relation}\}$$

$$(X \Rightarrow Y)(V) \subseteq [V \to (X \Rightarrow Y)(\star)]$$ such that (among other conditions)

if $(f_0, f_1, \ldots) \in (X \Rightarrow Y)(V)$

then $(x_0, x_1, \ldots) \in X(V)$ implies $(f_0(x_0), f_1(x_1), \ldots) \in Y(V)$.

So $(X \Rightarrow Y)(V)$ is a “**logical**” relation.
Partiality monad $L$ on $\nuSet$

For a concrete presheaf $X$:

$$(LX)(\star) = X(\star) + \{\bot\}$$

$$(LX)(V) = \{\bot\} + \sum_{n \in \mathbb{N}} (X(V))_n$$

$(X(V))_n$ contains each chain from $X(V)$ with $n$ $\bot$-elements added at the beginning.
Claim

We can model PCF$_v$ using the concrete presheaves in vSet, starting from:

\[ [\text{nat}] (\star) = \mathbb{N} \]
\[ [\text{nat}] (V) = \{ \text{constant functions } V \to \mathbb{N} \}. \]

The vSet model is actually the \( \omega \text{cpo} \) model.
Modelling fixed points in \( \text{vSet} \)

\[
\begin{align*}
\text{type} & \quad \text{vertical} = \text{Succ of } (\text{unit} \to \text{vertical}) ;; \\
\text{let rec} & \quad \text{top} : \text{vertical} = \text{Succ (fun } () \to \text{top}) ;; \\
\text{let lub} & \quad ((\text{fs}, \text{ax}) : (\text{vertical} \times 'a \to 'b) \times 'a) : 'b = \\
& \quad \text{fs (top, ax)} ;; \\
\text{let rec} & \quad \text{approx} : (\text{vertical} \times (('a \to 'b) \times 'a \to 'b) \times 'a) \to 'b \\
\text{let tarski} & \quad ((('a \to 'b) \times 'a \to 'b) \times 'a) \to 'b \\
\end{align*}
\]

Similarly we can define a fixed point of \( f : (A \Rightarrow LB) \times A \to LB \) in \( \text{vSet} \) if \( LB \) is \textbf{orthogonal} to \( \omega \times X \to yV \times X \) for any \( X \) in \( \text{vSet} \):
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Semidecidable subset of a type $\tau = \text{represented by a program } s : \tau \to 1$.

Category Syn:

- Objects: $(\tau, s)$ type + semidecidable subset
- Morphisms: $f : (\tau, s) \to (\tau', s')$ is a(n equivalence class of) program(s) $x : \tau \vdash f : \tau'$ with domain $s$ and image in $s'$.
Building a fully abstract syntactic model of $\text{PCF}_v$

$[\text{Syn}^{op}, \text{Set}]$ almost a model with full definability $\iff$ full abstraction.

Problems:

1. $y(\text{nat})$ is not $\sum_0^\infty 1$ in presheaves.
2. Recursion.
3. We’d like a non-syntactic model.
Solving 1 and 2: \( \text{nat as } \sum_0^\infty 1 \), and recursion

Use a **sheaf condition** on \( \text{Syn} \) to make \( y(\text{nat}) \) a coproduct.

\[ \exists \text{ there are uncountably many maps } \sum_0^\infty 1 \to \sum_0^\infty 1. \]

- We can’t get full definability.

For each \( n \), consider \( \text{Syn}_n \) such that natural numbers \( > n \) trigger divergence [Milner’77].

Combine the **truncated** sites \( \text{Syn}_n \) and impose a sheaf condition on them.

Solving 2, recursion: add \( \forall \) as one of the sites.

Something like \( \text{Sh}(\forall + \bigvee_n \text{Syn}_n) \) has full definability for truncated types.
Solving 3: Non-syntactic model

Instead of Synₙ use a bigger class of sites.

Given a finite set w:

**A system of partitions** $S^w$ [Streicher’06, Marz’00]

Contains **partial partitions** (=partial equivalence relations) of w s.t.:

1. $\{w\}, \emptyset \in S^w$

2. $P, Q \in S^w$ and $U \in P$ imply that:
   $$(P \setminus \{U\}) \cup (\{U \cap U' \mid U' \in Q\} \setminus \{\emptyset\}) \in S^w.$$

3. $U, U' \in P \in S^w$ implies that
   $$(P \setminus \{U, U'\}) \cup \{U \cup U'\} \in S^w.$$
Systems of partitions

\( w = \) finite set \( S^w \subseteq \{ \text{partial partitions of } w \} \) + axioms

\((w, S^w)\): \( w \) is a finite type

\[ P \in S^w \] is (roughly) a computable function \( w \rightarrow \mathbb{N} \)

The axioms of \( S^w \) imply that the system of functions:

- includes all constant functions
- is closed under postcomposition with any \( f : \mathbb{N} \rightarrow \mathbb{N} \)
- is closed under sequencing of functions from \( S^w \).

For \( P \in S^w \), think of \( \bigcup P \) as the semidecidable subset \( s \)

from \((\tau_n, s : \tau_n \rightarrow 1)\) from \( \text{Syn}_n \).
The systems of partitions form a category SSP:

- **Objects**: 
  
  \( (w, S^w) \)

- **Morphisms**: 
  
  \( f : (v, S^v) \to (w, S^w) \) is a function \( f : v \to w \) s.t. if \( P \in S^w \) then \( f^{-1}(P) \in S^v \).

Partiality monad \( L_{SSP}(w, S^w) = (w \sqcup \{\perp\}, \ldots) \).

A map in \( Syn_n \) is a partial function \( (\tau_n, s) \to (\tau'_n, s') \) with domain \( s \) and image in \( s' \).
Defining sites via systems of partitions

$w$ = finite set $S^w \subseteq \{\text{partial partitions of } w\}$ + axioms
$P \in S^w, \bigcup P =$ semidecidable subset of $w$

SSP$_\perp$ has Kleisli maps $(v, S^v) \to L_{SSP}(w, S^w)$

For a faithful functor $F : C \to SSP_\perp$ define a category $\mathcal{I}_{C,F}$ similar to $\text{Syn}_n$:

- **Objects:** $(c, U), c \in C$ and $U = \bigcup P$ for some $P \in S^{F(c)}$, (and a terminal object).
- **Morphisms:** $f : (c, U) \to (d, W)$ is a function $f : U \to W$
  - either constant
  - or s.t. there is $F(\phi) : F(c) \to L_{SSP}(F(d))$ with domain $U$ and image in $W$.

Each $\mathcal{I}_{C,F}$ is a guess at $\text{Syn}_n$. 

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First attempt at a model using guesses

Candidate model: \[\left((\forall + \bigvee_{F:C \to \text{SSP}_\perp} \mathcal{I}_{C,F})^{op}, \text{Set}\right)\]

If \((c, U)\) is a type, \(S^{F(c)}\) encodes the maps \(U \to \text{nat}\), \(\text{nat}\) needs to be interpreted as the **concrete presheaf**:

\[
\llbracket \text{nat} \rrbracket(\star) = \mathbb{N} \\
\llbracket \text{nat} \rrbracket(c, U) = \{g : U \to \mathbb{N} \mid \{g^{-1}(i) \mid i \in \mathbb{N}\} \in S^{F(c)}\}
\]

But this is not the coproduct \(\sum_0^\infty 1\):

\[
(\sum_0^\infty 1)(\star) = \mathbb{N} \quad (\sum_0^\infty 1)(c, U) = \{f : U \to \mathbb{N} \mid f \text{ constant}\}
\]
From presheaves to sheaves

Final model: \( G = \text{Sh}(\forall + \bigvee_{F:c \to \text{SSP}_\perp} \mathcal{I}_{c,F}) \)

In \( G \) the same \([\text{nat}]\) becomes a coproduct.

Sheaf condition:

- \((c, U)\) covered by \(\{(c, U_i) \to (c, U)\}_{1 \leq i \leq n}\) where
  \[ P = \{U_1, \ldots, U_n\} \in S^{F(c)} \text{ and } \bigcup U_i = U. \]
- A concrete presheaf \(X\) is a sheaf if given a tuple of
  functions \((f_i : U_i \to X(\star) \in X(c, U_i))_{U_i \in P}\) then
  \((f_1 + f_2 + \ldots + f_n) : U \to X(\star) \in X(c, U).\)
- Ensures sum types are interpreted as coproducts.
$\mathcal{G}$ is a model of $\text{PCF}_v$

Partiality monad on $\mathcal{G} = \text{Sh}(\mathbb{V} + \bigvee_{F:C \rightarrow \text{SSP} \bot} \mathcal{I}_{C,F})$:

$$(L_{\mathcal{G}}X)(\star) = X(\star) + \{ \perp \}$$

$$(L_{\mathcal{G}}X)(c, U) = \sum_{W \subseteq U} X(c, W) \text{ s.t. exists } P \in S^F(c), \bigcup P = W.$$

**Theorem**

$\mathcal{G}$, with $L_{\mathcal{G}}$, gives a fully abstract model of $\text{PCF}_v$ such that:

1. nat is interpreted as $\sum_0^\infty 1$
2. we interpret recursion
3. the model is non-syntactic.
The connection between $G$ and logical relations

All types $\tau$ are interpreted as **concrete sheaves** $[\tau]$.

The interpretation of PCF$_v$ can be thought of as a:

**Kripke logical relation of varying arity**

- $[\tau](c, U)$ is a **relation** with arity $U$ (like in $vSet$).
- **logical**: at function types $\tau_1 \rightarrow \tau_2$, a tuple of related functions maps related arguments to related results.
- **Kripke**: the relation $[\tau](c, U)$ is compatible with $[\tau](d, W)$ according to the maps $(d, W) \rightarrow (c, U)$.
- **varying arity**: $[\tau](d, W)$ has arity $W \neq U$. 
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Summary

Fully abstract model of $\text{PCF}_v$:

- Recursion: presheaves on $\forall$
- Definability/Sequentiality: guess the truncated types
- Take sheaves on these guesses to model nat and sum types as coproducts.

- Each partiality monad comes from a dominance, like in synthetic domain theory.
Future work

- Recursive types [Riecke & Sandholm’02]
- Other computational effects
- Non-well-pointed models [Levy’07, Amb breaks well-pointedness...]

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