A unified treatment of concrete sheaf models for higher-order recursion

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Modelling higher-order programs with recursion

Model

- Cartesian closed category (CCC)
- Partiality monad, $L$
- Interpretation: Type $\leftrightarrow$ Object
  Program $\leftrightarrow$ Partial morphism

Examples:

1. **Probabilistic programming**: partial maps that are measurable
   [Heunen et al.’17, Vákár et al.’19]

2. **Automatic differentiation**: partial maps that are smooth
   [Huot et al.’20, Vákár’20]

3. Piecewise differentiable programs [Lew et al.’21]

4. **Full abstraction** for a sequential language: definable partial maps
   [O’Hearn & Riecke’95], [Matache, Moss, Staton, FSCD’21]
Goal of this talk

Main Theorem [Matache, Moss, Staton, in preparation]

The examples

1. Probabilistic programming
2. Automatic differentiation
3. Piecewise differentiation
4. Full abstraction

all model higher-order recursion using the same recipe
- using concrete sheaves
- using ideas from synthetic domain theory for recursion

In each case more domain specific work needs to be done.

Examples of concrete sheaves: subsequential spaces [Johnstone’79], C-spaces [Escardó & Xu’16]
Examples of concrete presheaves: [Rosolini & Streicher’99], finiteness spaces [Ehrhard’07]
Goal of this talk (continued)

Main Theorem [Matache, Moss, Staton, in preparation]

The examples

1. Probabilistic programming
2. Automatic differentiation
3. Piecewise differentiation
4. Full abstraction

- all model higher-order recursion using the same recipe
  - using **concrete sheaves**
  - using ideas from synthetic domain theory for **recursion**

Corollary: conservativity result for (1), (2), (3)

E.g. (2): Programs $\text{real} \rightarrow \text{real}$ are still interpreted as smooth maps even if they use higher-order recursion.
PCF<sub>v</sub>: A call-by-value language

Call-by-value λ-calculus with:
- base types e.g. `nat`, `real`
- function types
- product and sum types
- recursive functions.

An interpretation looks like:

\[
\begin{align*}
    [\text{nat}] &= 1 + 1 + \ldots \\
    [\tau_1 + \tau_2] &= [\tau_1] + [\tau_2] \\
    [\tau_1 \times \tau_2] &= [\tau_1] \times [\tau_2] \\
    [\tau \rightarrow \tau'] &= [\tau] \Rightarrow L[\tau'] \\
    [\Gamma \vdash t : \tau] : [\Gamma] \rightarrow L[\tau]
\end{align*}
\]
Why use categories of concrete sheaves?

Example: **first-order probabilistic computation** can be modelled in $\text{Sbs}$. $\text{Sbs}$ is NOT cartesian closed.

The category of **presheaves** on $\text{Sbs}$ is cartesian closed.

**Yoneda embedding**

\[ y : \text{Sbs} \hookrightarrow \text{PSh}(\text{Sbs}) \]

Full, faithful, preserves limits. Does not preserve colimits.

Restricting to **sheaves** on a site $(\text{Sbs}, J)$ preserves some colimits from $\text{Sbs}$.

**Concrete sheaves** = sets with structure $+$ structure-preserving functions.

\[ \text{ConcSh}(\text{Sbs}, J) \hookrightarrow \text{Sh}(\text{Sbs}, J) \hookrightarrow \text{PSh}(\text{Sbs}) \]
Well-pointed categories and concrete sites

A category $\mathcal{C}$ is **well-pointed** if

- it has a terminal object $\star$
- $\mathcal{C}(\star, -) : \mathcal{C} \to \text{Set}$ is faithful
  i.e. maps $h : d \to c$ are distinguished functions $|h| : |d| \to |c|$ where $|c| = \mathcal{C}(\star, c)$. So $\mathcal{C}$ is a category of sets and certain functions.

Concrete site $(\mathcal{C}, J)$

- A small well-pointed category $\mathcal{C}$
- For every $c \in \mathcal{C}$ a set $J(c)$ of **covering families** $\{f_i : c_i \to c\}_{i \in I}$ of $c$ s.t.
  (C) pullback stability
  (\star) If $\{f_i : c_i \to c\}_{i \in I}$ covers $c$, then $\bigcup_{i \in I} \text{Im}(|f_i|) = |c|$
A concrete sheaf $X : \mathbb{C}^{\text{op}} \to \text{Set}$ is:

- a set $X(\star)$
- $X(c) \subseteq \{ |c| \to X(\star) \}$

$X(h : d \to c)$ is precomposition by $|h|$.

A morphism $\alpha : X \to Y$ is a structure-preserving function $\alpha : X(\star) \to Y(\star)$.
Example: modelling probabilistic programming [Heunen et al.’17, Vákár et al.’19]

A functor $X : \mathbb{C}^{\text{op}} \to \text{Set}$ is a **concrete sheaf** on a **concrete site** $(\mathbb{C}, J)$ if $X(c) \subseteq [\mathbb{C} \to X(\star)]$ and $X$ satisfies the **sheaf condition**.

Quasi-Borel spaces is the category of concrete sheaves on:

- **Sbs**: objects $U$ are Borel subsets of $\mathbb{R}$
  morphisms are measurable functions between these sets.
- $J(U) =$ countable sets of measurable inclusions $\{U_i \hookrightarrow U\}_{i \in I}$ where $U = \bigcup_{i \in I} U_i$ and the $U_i$’s are disjoint.

$X(\mathbb{R}) \subseteq [\mathbb{R} \to X(\star)]$ is the set of “random elements” of $X(\star)$.
Example: modelling probabilistic programming in ConcSh(Sbs, J)

A functor $X : \mathbb{C}^{\text{op}} \to \text{Set}$ is a **concrete sheaf** on a **concrete site** $(\mathbb{C}, J)$ if $X(c) \subseteq [\mathbb{C} \to X(\ast)]$ and $X$ satisfies the **sheaf condition**.

$\text{Sbs} = \text{Borel subsets } U \subseteq \mathbb{R} + \text{measurable functions}$

$J(U) = \text{sets of inclusions } \{U_i \to U\}_{i \in I}$ where $U = \bigcup_{i \in I} U_i$ and the $U_i$’s are disjoint.

In $\text{PSh}(\text{Sbs})$, take $X$ concrete. In $\text{Sbs}$, take $\mathbb{R} = \bigcup_{i \in I} U_i$ and $U_i$’s disjoint:

\[
y\mathbb{R} \xrightarrow{y} \sum_{i \in I} yU_i \xrightarrow{X} X \text{ by Yoneda lemma}
\]

\[
(g : \mathbb{R} \to X(\ast)) \in X(\mathbb{R}) \Downarrow
\]

\[
\{(f_i : U_i \to X(\ast)) \in X(U_i)\}_{i \in I}
\]

**Sheaf condition at $\mathbb{R}$**: for each function $g : \mathbb{R} \to X(\ast)$ and each covering family $\{f_i : U_i \to \mathbb{R}\}_{i \in I} \in J(c)$, if each $g \circ f_i \in X(U_i)$, then $g : \mathbb{R} \to X(\ast) \in X(\mathbb{R})$. 

$U_i \xleftarrow{f_i} \mathbb{R} \xrightarrow{g} X(\ast)$
A functor $X : \mathbb{C}^{\text{op}} \to \text{Set}$ is a **concrete sheaf** on a **concrete site** $(\mathbb{C}, J)$ if $X(c) \subseteq [c \to X(\star)]$ and $X$ satisfies the **sheaf condition**.

Diffeological spaces is the category of concrete sheaves on:

- **Site**: objects are open subsets $U \subseteq \mathbb{R}^n$ for any $n$
  morphisms are smooth maps.

- $J(U) =$ countable sets of open inclusions $\{U_i \hookrightarrow U\}_{i \in I}$ where $U = \bigcup_{i \in I} U_i$.

$X(U) \subseteq [U \to X(\star)]$ is the set of “plots” of $X(\star)$.
1 Introduction

2 Higher-order computation: categories of concrete sheaves

3 Modelling partiality

4 Modelling recursion

5 Putting it all together
In any category, a partial map $X \rightarrow Y$ is a pair $(m, f)$:

where $\mathcal{N}$ is stable class of monos:

- contains all isomorphisms
- closed under composition
- stable under pullback (with arbitrary maps)

(1) **Quasi-Borel spaces**: partial maps that are measurable, with Borel domain

(2) **Diffeological spaces**: partial maps that are smooth, with open domain
From partial maps to a lifting monad

How do we get a monad $L$ with the following property?

For every $X' \xrightarrow{f} Y$,

For every $\exists m \in \mathcal{N}$, where $\mathcal{N}$ is a stable class of monos,

there is exactly one corresponding total map $X \to LY$ such that

and conversely.

$L$ might not exist in general.
In a sheaf category $\text{Sh}(\mathcal{C}, J)$:

**Theorem**

$\mathcal{N}$ has an associated lifting monad $L$ if the class of monos $\mathcal{N}$ “comes from” a class of pre-admissible monos $\mathcal{M}$ in $\mathcal{C}$.

$\mathcal{M}$ is a class of pre-admissible monos in $\mathcal{C}$ if:

- stable class
- $\Delta_\mathcal{M} : \mathcal{C}^{\text{op}} \to \text{Set}$ is a $J$-sheaf, where:
  - $\Delta_\mathcal{M}(c) = \text{iso. classes of } c' \to c \in \mathcal{M}$
  - $\Delta_\mathcal{M}(f : d \to c) = \text{pullback along } f$

$\mathcal{N}$ stable class of monos:
- contains all isomorphisms
- closed under composition
- stable under pullback

Lifting monad:
for every $(m, f) : X \to Y$ with $m \in \mathcal{N}$, there is exactly one total map $X \to LY$.

\[
\begin{array}{cccc}
d' & \overset{\in \mathcal{M}}{\longrightarrow} & d \\
\downarrow & & \downarrow f \\
c' & \overset{\in \mathcal{M}}{\longrightarrow} & c
\end{array}
\]
In a sheaf category $\mathbf{Sh}(\mathcal{C}, J)$:

**Theorem**

$\mathcal{N}$ has an associated lifting monad $L$ if the class of monos $\mathcal{N}$ “comes from” a class of pre-admissible monos $\mathcal{M}$ in $\mathcal{C}$.

$\mathcal{N}$ “comes from” $\mathcal{M}$ if $\mathcal{N} = \text{all pullbacks of } \top : 1 \rightarrow \Delta_{\mathcal{M}}$, where $\top_c = [\text{id}_c]$

$$
\begin{array}{ccc}
X' & \xrightarrow{!} & 1 \\
\downarrow & & \downarrow \top \\
\in \mathcal{N} & \downarrow & \\
X & \xrightarrow{\chi} & \Delta_{\mathcal{M}}
\end{array}
$$

$\mathcal{N}$ stable class of monos:
- contains all isomorphisms
- closed under composition
- stable under pullback

Lifting monad:
for every $(m, f) : X \rightarrow Y$ with $m \in \mathcal{N}$, there is exactly one total map $X \rightarrow LY$.

$\mathcal{M}$ stable class
$\Delta_{\mathcal{M}} : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ a $J$-sheaf
$\Delta_{\mathcal{M}}(c) = \text{iso. classes of } c' \rightarrow c \in \mathcal{M}$
$\Delta_{\mathcal{M}}(f : d \rightarrow c) = \text{pullback along } f$

see e.g. [Rosolini'86] for dominance, [Mulry'94], [Fiore&Plotkin'97] for constructing a lifting monad
Examples: classes of pre-admissible monos

Quasi-Borel spaces:

- Site: objects $U$ are Borel subsets of $\mathbb{R}$, morphisms are measurable functions.
- $\mathcal{M} = \{ \text{for every } U, \text{ the measurable monos with codomain } U \}$

Diffeological spaces:

- Site: objects are open subsets $U \subseteq \mathbb{R}^n$ for any $n$, morphisms are smooth maps.
- $\mathcal{M} = \{ \text{for every } U, \text{ the open inclusion maps into } U \}$

For a concrete sheaf $X$, the lifting monad:

\[
LX(\star) = X(\star) \uplus \{ \bot \}
\]

\[
LX(U) = \{ g : U \to X(\star) \uplus \{ \bot \} \mid \exists U' \to U \in \mathcal{M} \text{ s.t. dom}(g) = U' \text{ and } g|_{U'} \in X(U') \}\]

In general, having the lifting monad is not enough to model recursion.
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The \( \omega \text{cpo} \) model of \( \text{PCF}_v \)

Types = partially ordered sets with least upper bounds of \( \omega \)-chains

Terms = continuous functions

To model recursive functions:

- a lifting monad on \( \omega \text{cpo} \)
- Tarski’s fixed point theorem

We want to recover this model as presheaves with a class of admissible monos in the site.
vSet: A concrete presheaf model of PCF_

\[ V = \{0 < 1 < 2 < \ldots < \infty\} = \text{poset of vertical nat. numbers} \]

\[ \mathbb{V} = \text{two-object category} \]

\[ \text{vSet} = \text{presheaves on } \mathbb{V} \]

Concrete presheaf on \( \mathbb{V} \):
- a set \( X(\star) \)
- a set of functions \( X(V) \subseteq [V \to X(\star)] \)

\( x \in X(V) \) is a completed chain of elements in \( X(\star) \).

Map \( X \to Y = \text{function } X(\star) \to Y(\star) \) that preserves chains.

\( \omega\text{cpo} \) is a full subcategory of \( \text{vSet} \):
\[ D \mapsto (|D|, \omega\text{cpo}(V, D)) \]

See the category \( \mathcal{H} \) from [Fiore & Rosolini'97, '01].
Lifting in $vSet$

$\mathbb{V} = \text{vertical naturals as a two-object category}$
$vSet = \text{presheaves on } \mathbb{V}$

Theorem:
A class of pre-admissible monos $\mathcal{M}$ in $\mathcal{C}$ induces a lifting monad $L$ on the sheaf category $\mathbf{Sh}(\mathcal{C}, J)$.

$\mathbb{V}$ has a class of pre-admissible monos:

$$\mathcal{M}_\mathbb{V} = \{(\lambda x.x + n) \in \mathbb{V}(V, V) \mid n \in \mathbb{N}\} \cup \\{\text{id}_*: \star \to \star\}$$

which induces a lifting monad $L$ on $vSet$, where for a concrete presheaf $X$:

$$(LX)(\star) = X(\star) \uplus \{\bot\} \quad (LX)(V) = \{\bot\} \uplus \biguplus_{n\in\mathbb{N}} (X(V))_n$$

$(X(V))_n \approx \text{chains from } X(V) \text{ with } n \bot \text{'s added at the beginning.}$
Modelling PCF\textsubscript{v} in \textit{vSet}

**Fixed point theorem in \textit{vSet}**

We can construct a fixed point of a map $(A \Rightarrow LB) \rightarrow (A \Rightarrow LB)$ if $LB$ is “complete”.

\[
\omega \times X \xrightarrow{h} LB
\]

\[
yV \times X \xrightarrow{\downarrow} \omega \times X
\]

$\omega = $ greatest subobject of $yV$ without $\infty$

**Theorem**

\textit{vSet} is an adequate model of PCF\textsubscript{v} where types are concrete presheaves.

The interpretation of PCF\textsubscript{v} commutes with the inclusion $\omega_{\text{cpo}} \hookrightarrow \textit{vSet}$. 23/26
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Modelling PCF$_v$ in a category of concrete sheaves

**Main Theorem** [Matache, Moss, Staton, in preparation]

Given a **concrete site** with a **class of admissible monos** ($\mathcal{C}, J, \mathcal{M}$), “combine” it with the site for vSet, ($\mathcal{V}, J_\mathcal{V}, \mathcal{M}_\mathcal{V}$).

The category of **concrete sheaves** on the combined concrete site ($\mathcal{C} + \mathcal{V}, J \cup J_\mathcal{V}, \mathcal{M} \cup \mathcal{M}_\mathcal{V}$) is an adequate model of PCF$_v$.

Example: we recover the $\omega$Qbs model

**Concrete site for Qbs:**

Sbs: objects $U$ are Borel subsets of $\mathbb{R}$, morphisms are measurable functions. $J(U) =$ countable sets of inclusions $\{U_i \hookrightarrow U\}_{i \in I}$ where $U = \bigcup_{i \in I} U_i$ and the $U_i$’s are disjoint.

$\mathcal{M} =$ all monos.
Main Theorem [Matache, Moss, Staton, in preparation]

Given a concrete site with a class of admissible monos \((\mathcal{C}, J, \mathcal{M})\), “combine” it with the site for \(v\text{Set}\), \((\mathcal{V}, J_{\mathcal{V}}, \mathcal{M}_{\mathcal{V}})\).

The category of concrete sheaves on the combined concrete site \((\mathcal{C} + \mathcal{V}, J \cup J_{\mathcal{V}}, \mathcal{M} \cup \mathcal{M}_{\mathcal{V}})\) is an adequate model of PCF\(_v\).

Model higher-order recursion for:

1. Probabilistic programming
2. Automatic differentiation
3. Piecewise differentiation
4. Full abstraction

Using:
- sheaves on a concrete site
- class of admissible monos in the site
- presheaves on the vertical naturals